Rota’s Basis Conjecture and the Wide Partition Conjecture

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Pseudoku, anyone?

			6				5		
	7								2
		9				1			
8				3	6				
4									
				5					
	3								
		1							

						5			2
								1	
10					6				
		5	8			2			
6		1							
	6				1				
					4				
		3							
1									
			3						

A length-ℓ row/column must have each number from 1 to ℓ.

Note: A blank grid is a *partition*; a finished grid is a *tableau*.

Rota's basis conjecture (Rota, 1989)

Let V be an n-dimensional vector space.

Let $B_1, B_2, \ldots, B_n$ be bases of V.

Then there exists an $n \times n$ grid of vectors v_{ij} such that the ith row is B_i and every column is a basis of V.

Example. $V = \mathbb{R}^2$, $B_1 = \{\binom{1}{0}, \binom{0}{1}\}$, $B_2 = \{\binom{1}{0}, \binom{1}{1}\}$.

$$\begin{matrix} \binom{1}{0} & \binom{0}{1} \\ \binom{1}{1} & \binom{1}{0} \end{matrix}$$

Rota on Rota's basis conjecture

"It is probably true for all even integers n. Behind this conjecture lurk certain identities from invariant theory, which remain unproved, and which must be passed over in silence. As a matter of fact, one can rattle off several other conjectures on linear dependence of vectors and tensors, all of them suggested by as yet unproved identities in invariant theory. I would feel crushed if the basis conjecture were to be settled by methods other than some new insight in the algebra of invariant theory."

Gian-Carlo Rota, *Ten mathematics problems I will never solve,* invited address at a joint AMS/MMS meeting, December 6, 1997.

Even and odd Latin squares

A *Latin square* of order n is an $n \times n$ grid in which every row and every column is a permutation of $\{1, 2, \ldots, n\}$. Example:

$$L = \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{matrix}$$

The *sign* of a permutation is 1 if the number of *inversions* is even, and -1 otherwise.

sign(L) := product of the signs of the rows and the columns

In the example, sign(L) = $(1)(-1)(-1)(1)(1)(-1)(-1)(1)$ = 1.

L is *even* if sign(L) = 1 and *odd* if sign(L) = -1.

Alon-Tarsi conjecture

$\mathrm{ELS}(n)$:= number of $n \times n$ even Latin squares
$\mathrm{OLS}(n)$:= number of $n \times n$ odd Latin squares

For n odd, $\mathrm{ELS}(n) = \mathrm{OLS}(n)$ (switch two columns).

Conjecture (Alon-Tarsi '92). $\mathrm{ELS}(n) \neq \mathrm{OLS}(n)$ for even n.

Theorem (Huang-Rota '94). For even n, characteristic 0,

Alon-Tarsi $\Rightarrow$ Rota's basis conjecture

Idea of proof:

$$\sum_{n!^n \text{ configs}} \pm \prod_i \text{det(column } i) \approx \mathrm{ELS}(n) - \mathrm{OLS}(n)$$

Alon-Tarsi partial results

Computationally verified for (even) $n \leq 10$.

Theorem (Drisko '97, '98; Zappa '97). Alon-Tarsi is true for $n = 2^{r+1}p$ and $n = 2^r(p+1)$ for p an odd prime and $r \geq 0$.

Drisko counts isotopy classes of Latin squares mod p^k.

Zappa studies *fixed-diagonal Latin squares* (all-1 diagonal).
Let $Z(n) = \mathrm{FDELS}(n) - \mathrm{FDOLS}(n)$.
If n is even then $n!Z(n) = \mathrm{ELS}(n) - \mathrm{OLS}(n)$.

Theorem (Zappa '97).

(1) $Z(2k) \neq 0 \Rightarrow Z(4k) \neq 0$

(2) $Z(2k-1) \neq 0$ and $Z(2k) \neq 0 \Rightarrow Z(4k-2) \neq 0$

Rota’s conjecture: other results

Theorem (Chan ’95). Rota’s conjecture is true for $n \leq 3$.

Theorem (Ponomarenko ’04). Can ensure that for all i, the first i columns are a disjoint union of i bases.

Theorem (Geelen-Humphries ’06). Rota’s conjecture is true if every subset of $n-1$ vectors is linearly independent.

For matroid theorists: Above are true for all matroids. Also:

Theorem (Wild ’94). Rota’s conjecture is true for strongly base-orderable matroids.

Generalizing to partitions

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$.

Let I_i be a linearly independent set of size λ_i.

Does there exist a tableau whose ith row is I_i and whose columns are linearly independent?

Motivation: Rota's invariant-theoretic approach involves partitions, not just squares. Perhaps one can do induction, adding one box at a time?

Special case: Let $I_i = \{e_1, e_2, \ldots, e_{\lambda_i}\}$. Then "linearly independent" means "distinct." Does this case always work?

Not necessarily. If $I_1 = \{e_1\}$ and $I_2 = \{e_1\}$, the only tableau whose ith row is I_i is $\begin{matrix} e_1 \\ e_1 \end{matrix}$

A necessary condition

Say that $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is *Latin* if there exists a tableau for λ whose ith row is $\{1, 2, \ldots, \lambda_i\}$ and whose columns have distinct numbers.

A Latin partition must have at least ℓ columns so that the 1's can go into different columns, i.e., the length of row 1 is at least the height of column 1.

More generally, the sum of the lengths of the first i rows is at least the sum of the heights of the first i columns.

The same must be true for any *subpartition* of λ, obtained by deleting some of the rows.

Wide partitions

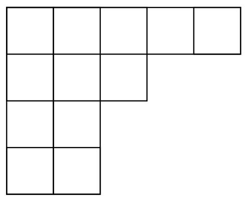
If $\lambda_1 \geq \lambda_2 \geq \cdots$ are the row lengths of a partition λ, then let $\lambda'_1 \geq \lambda'_2 \geq \cdots$ denote its column lengths.

Write $\lambda \succeq \lambda'$ if $\lambda_1 + \cdots + \lambda_i \geq \lambda'_1 + \cdots + \lambda'_i$ for all i.

Write $\mu \subseteq \lambda$ if the rows of μ are a subset of the rows of λ.

Definition. λ is *wide* if $\mu \succeq \mu'$ for every $\mu \subseteq \lambda$.

Example. $\lambda = (5, 3, 2, 2)$ is not wide. Take $\mu = (3, 2, 2)$; then $\mu_1 + \mu_2 = 5 < 6 = \mu'_1 + \mu'_2$.



Wide partition conjecture

Proposition. If λ is Latin then λ is wide.

Wide partition conjecture. If λ is wide then λ is Latin.

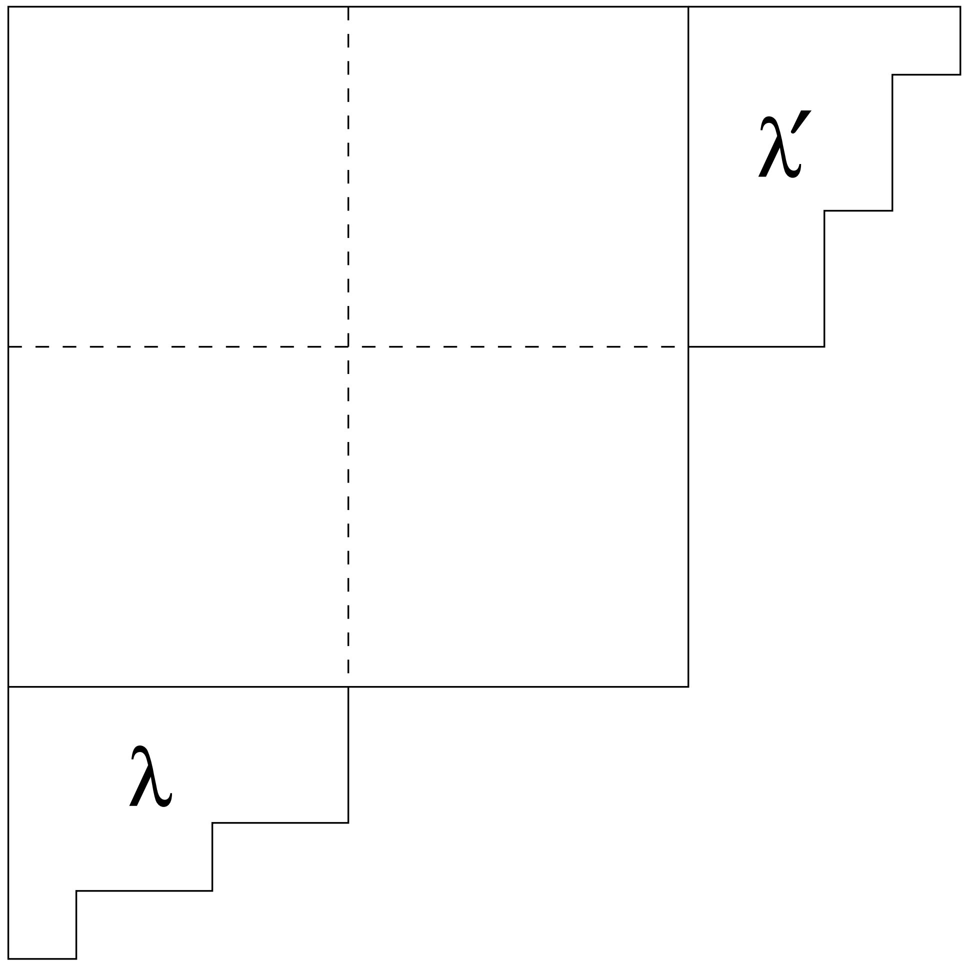
For rectangles, the wide partition conjecture asserts that Latin rectangles exist.

It holds for all partitions fitting inside a 10×10 square, and all partitions having at most 65 boxes.

We conjecture that wideness is the right condition even for the linearly-independent-set version of the conjecture.

Symmetric shapes suffice

Theorem. If λ is wide, then so is the partition below.



Partitions with few distinct parts

Theorem. If λ is wide and has only 2 distinct row lengths, then λ is Latin.

Theorem. If λ is wide and symmetric and has only 3 distinct row lengths, then λ is Latin.

Theorem. If λ is wide and has only 3 distinct row lengths and either the 2nd or 3rd row lengths occurs with multiplicity one, then λ is Latin.

Theorem. If λ is wide and has only 4 distinct row lengths and both the 2nd and 4th row lengths occurs with multiplicity one, then λ is Latin.

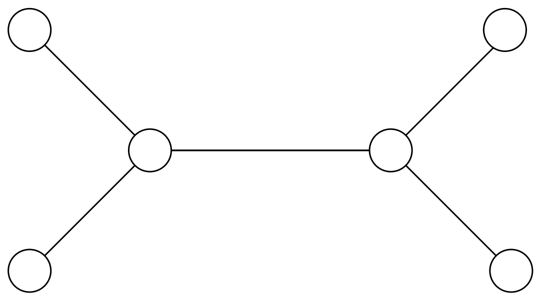
Graphs

A *stable set* in a graph is a set of vertices with no edges between them.

A *clique* is a set of vertices such that every possible edge between them is present.

A *k-stable set* is a disjoint union of *k* stable sets.

A *k-clique* is a disjoint union of *k* cliques.

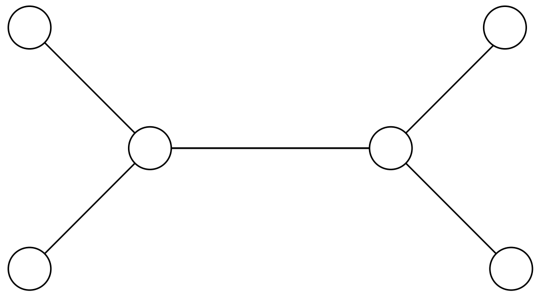


Greene-Kleitman theorem

α_k: max # vertices of a k-stable set
ω_k: max # vertices of a k-clique

$\Delta\alpha_k := \alpha_k - \alpha_{k-1}$
$\Delta\omega_k := \omega_k - \omega_{k-1}$

Theorem (Greene-Kleitman). If G is a comparability graph, then $\Delta\alpha$ and $\Delta\omega$ are partitions and $(\Delta\alpha)' = \Delta\omega$.



Example: $\Delta\alpha = (4, 2)$, $\Delta\omega = (2, 2, 1, 1)$

Uniform k-stable sets

Definition. A k-stable set is *uniform* if for all i, the ith largest stable set has size $\Delta\alpha_i$.

The example on the previous slide was *not* uniform.

Definition. G_λ is the graph whose edges join boxes in the same row or column of λ.

Theorem. If λ is wide and there is a uniform k-stable set covering G_λ, then λ is Latin.

Remark. It is open whether *every* partition can be covered by a uniform k-stable set.