Mathematica Pannonica
24/1 (2013), 3–14

A PROBLEM ATTRIBUTED TO RADO

from Mirsky's 1971 Monograph *Transversal Theory* and a Conjecture from the 1982 *Proceedings of the American Mathematical Society*

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Received: March 2012

MSC 2010: 05 D 15, 06 E 30, 05 B 35, 90 C 27

Keywords: Transversal, Hall's Marriage Theorem, matching, Boolean polynomial, (free) distributive lattice, function lattice, arithmetic of ordered sets, matroid, Rota's Basis Conjecture.

Abstract: A *restricted Boolean polynomial* $\rho(x_1, \ldots, x_n)$ is a finite expression involving $x_1, \ldots, x_n$ (variables that range over sets) formed by means of unions and intersections. Let $\mathcal{R}$ be a class of pairwise inequivalent restricted Boolean polynomials. It is shown that the conditions

$$|\rho(A_1, \ldots, A_n)| \geq |\rho(B_1, \ldots, B_n)| \quad (\rho \in \mathcal{R})$$

are independent. That is, given $\rho \in \mathcal{R}$, there exist sets $A_1, \ldots, A_n, B_1, \ldots, B_n$ such that the condition

$$|\sigma(A_1, \ldots, A_n)| \geq |\sigma(B_1, \ldots, B_n)|$$

holds for all $\sigma \in \mathcal{R}$ where σ is not equivalent to ρ, but

$$|\rho(A_1, \ldots, A_n)| < |\rho(B_1, \ldots, B_n)|.$$

This solves a problem (attributed to Rado) from Mirsky's 1971 text *Transversal Theory*.
A conjecture of Cohen and Rubin from 1982 is also refuted.
Motivated by Rota's Basis Conjecture, an analogue of the "easy" direction of Edmonds' matroid partitioning theorem is proved for transversals of a family of subsets of a vector space.

1. Introduction and background

Let $n \in \mathbb{N}$. Let $[n] := \{1, \dots, n\}$. Two families of sets $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ are *combinatorially equivalent* if there exists a bijection

$$\phi : \bigcup_{i=1}^{n} A_i \to \bigcup_{i=1}^{n} B_i$$

such that $\phi[A_i] = B_i$ for $i \in [n]$.

A *Boolean polynomial* $\rho(x_1, \dots, x_n)$ is a finite expression involving $x_1, \dots, x_n$ (variables ranging over sets) and formed by means of unions, intersections, and set differences. A *restricted Boolean polynomial* is a Boolean polynomial formed only by means of unions and intersections.

In [9], Rado proved the following

Theorem (Rado; see [9], Th. IV; [8], Th. 5.4.2). *Let $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ be families of finite sets. Then there exist sets $X_i \subseteq A_i$ $(i \in [n])$ such that $(X_1, ..., X_n)$ is combinatorially equivalent to $(B_1, ..., B_n)$ if and only if*

$$(*)_\rho \qquad |\rho(A_1, \dots, A_n)| \geq |\rho(B_1, \dots, B_n)|$$

for every restricted Boolean polynomial ρ. $\Diamond$

This result is discussed by Rota and Harper in [11], §12.3, who write, "A deep minimax theorem has been proved by Rado for distributive lattices." [Actually, Rado's results in [9] are more general – for instance, instead of cardinality he uses a more general notion of "measure" – but this theorem "represents the gist of Rado's conclusions," according to Mirsky, with the exception just noted ([8], p. 89).]

Rado's result generalizes Hall's Marriage Theorem (see [8], pp. 87–88):

Theorem (Hall, [5]). *A family $(A_1, \dots, A_n)$ possesses a system of distinct representatives if and only if, for all non-empty $J \subseteq [n]$,*

$$(**)_J \qquad \Big|\bigcup_{j \in J} A_j\Big| \geq |J|. \qquad \Diamond$$

Inequality $(**)_J$ represents $2^n - 1$ conditions, which are independent in the sense that, for every non-empty $J_0 \subseteq [n]$, there exists a family $(A_1, \dots, A_n)$ such that $(**)_J$ holds for every non-empty $J \subseteq [n]$ other than J_0, but not for $J = J_0$. (See the discussion at the end of [8], §2.1.)

Two restricted Boolean polynomials $\rho(x_1, \ldots, x_n)$ and $\sigma(x_1, \ldots, x_n)$ are considered *equivalent* if $\rho(X_1, \ldots, X_n) = \sigma(X_1, \ldots, X_n)$ for every family of sets $(X_1, \ldots, X_n)$. Of course two formally different polynomials can be equivalent; because of the distributive law, it is clear that there are only finitely many equivalence classes of n-ary restricted Boolean polynomials. Let $\mathcal{R}$ be a class of pairwise inequivalent restricted Boolean polynomials. If $\mathcal{R}$ consists of at most one polynomial from each class, then one may ask if the $|\mathcal{R}|$ conditions $(*)_\rho$ $(\rho \in \mathcal{R})$ are independent.

In a section entitled "Future Research and Open Questions," Mirsky writes (in the 1971 edition of *Transversal Theory*, [8], pp. 220 and 222): "Let us now look in greater detail at what appear to be gaps or inadequacies in current transversal theory. . . ."

> "12. It is clear from the proof of Theorem 5.4.2 that
> the necessary and sufficient conditions appearing in that
> result are equivalent to a *finite* subset of conditions.
> It would be of interest to verify that the conditions
> in this finite set are independent. (Cf. the remarks
> at the end of §2.1.)
>
> [R. Rado]"

[Name in brackets in the original text.] We solve this problem below (Prop. 3.7).[1]

2. Definitions and notation

See [2] for definitions, notation, and basic results.

Let P be a poset. For a subset $M \subseteq P$, let

$$\downarrow M := \{\, p \in P \mid p \leq m \text{ for some } m \in M \,\}.$$

(Dually define $\uparrow M$.) A subset $D \subseteq P$ is a *down-set* (or *order ideal*) if $D = \downarrow D$. The collection of all down-sets of P is denoted $\mathcal{O}(P)$. An element $m \in P$ is *maximal* if, for all $p \in P$, $m \leq p$ implies $m = p$. Let $\max P$ be the set of maximal elements. Dually, let $\min P$ be the set of minimal elements.

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[1] The proof is simple, although we caution the reader against too simple an interpretation. See the Addendum for a discussion of what the theorem does *not* say. Rado may have considered the problem well before 1971, as he considers a similar question in his 1943 paper [10]. Mirsky's text has been cited at least 131 times according to the Science Citation Index, so the problem must have received some exposure.

Let $\mathbf{2}$ be the poset $\{0, 1\}$ with $0 < 1$; $\mathbf{2}^n$ is the set of n-tuples $a = (a_1, \ldots, a_n)$, where $(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n)$ in $\mathbf{2}^n$ if $a_i \leq b_i$ for all $i \in [n]$. Let

$$\hat{a} := \{\, i \in [n] \mid a_i = 0 \,\}.$$

If ρ and σ are n-ary restricted Boolean polynomials, then $\rho \leq \sigma$ if

$$\rho(X_1, \ldots, X_n) \subseteq \sigma(X_1, \ldots, X_n)$$

for every family of sets $(X_1, \ldots, X_n)$. It is well known that the equivalence classes of n-ary restricted Boolean polynomials, so ordered, form the free distributive lattice on n generators ($FD(n)$), which can also be represented as the poset $\mathcal{O}(\mathbf{2}^n) \setminus \{\emptyset, \mathbf{2}^n\}$, ordered by set-inclusion. (See Fig. 3.1 and [12], p. 158.) The correspondence is as follows: For any $C \in \mathcal{O}(\mathbf{2}^n) \setminus \{\emptyset, \mathbf{2}^n\}$ and any $X_1, \ldots, X_n$, let

$$\rho_C(X_1, \ldots, X_n) := \bigcup_{c \in \max C} \bigcap \{\, X_i \mid i \in \hat{c} \,\}.$$

This change of perspective to the arithmetic of ordered sets enables us to refute a conjecture posed in 1982 by Professor Matatyahu Rubin and Dr. Miriam Cohen, Director of the Center for Advanced Studies in Mathematics and former Dean of the Faculty of Natural Sciences at Ben Gurion University of the Negev, as well as a former President of the Israel Mathematical Union. In their *Proceedings of the American Mathematical Society* paper, they conjectured the following:

For $i = 1, 2, 3$, let I_i be a linearly ordered set with the interval topology; let X_i be a compact, Hausdorff partially-ordered topological space such that the partial order relation is closed in $X_i \times X_i$ with the *Tietze extension property* with respect to I_i, meaning that for every closed subset F of X_i and every continuous, order-preserving map $f : F \to I_i$, there is a continuous, order-preserving extension of f to all of X_i. On page 691 of [1], the authors conjecture that if the lattices of continuous order-preserving maps from X_i to I_i are pairwise order-isomorphic for $i = 1, 2, 3$, then two of the X_i are order-homeomorphic.

If we let $\mathbf{n}$ be the n-element linearly ordered set with the discrete topology, where n is a positive integer, and Q^P the poset of order-preserving maps from the poset P to the poset Q ordered pointwise, then $\mathbf{2}^{(\mathbf{4} \times \mathbf{6})}$, $\mathbf{5}^{\mathbf{6}}$ and $\mathbf{7}^4$ refute the conjecture:

For $\mathbf{2}^{\mathbf{n}} \cong \mathbf{n} + \mathbf{1}$ [2] and $P^{(Q \times R)} \cong (P^Q)^R$ ([7], page 87). If Q is a complete lattice, any map $f \in Q^S$ for $S \subseteq P$ can be extended to all of P in an order-preserving way by $g(p) = \sup\{f(s) : s \leq p\}$. If I_i and X_i are finite, then both have the discrete topology. $\Diamond$

3. The solution to the problem from Mirsky's *Transversal Theory*

Fix $D \in \mathcal{O}(\mathbf{2}^n) \setminus \{\emptyset, \mathbf{2}^n\}$; let $M := \max D$ and let $Q := \min(\mathbf{2}^n \setminus D)$. (Note that $M, Q \neq \emptyset$.)

Let $(c_m)_{m \in M}$ be a family of distinct elements. Let X be any set of cardinality $|M| - 1$ and, for each $q \in Q$, let Y_q be a set of cardinality $|M|$ such that X, Y_q $(q \in Q)$ are pairwise disjoint.

For $i \in [n]$, let

$$A_i := \bigcup \{\, X \mid m \in M \text{ and } i \in \hat{m} \,\} \cup \bigcup \{\, Y_q \mid q \in Q \text{ and } i \in \hat{q} \,\},$$

$$B_i := \{\, c_m \mid m \in M \text{ and } i \in \hat{m} \,\}.$$

Note that X is fixed, so that the first union in the definition of A_i is either X or the empty set. Note that $\hat{m} \neq \emptyset$ for all $m \in M$, since $D \neq \mathbf{2}^n$.

Lemma 3.1. *For all $m \in M$, $\bigcap \{\, A_i \mid i \in \hat{m} \,\} = X$.*

Proof. By definition of A_i, $X \subseteq A_i$ when $i \in \hat{m}$. Now let $y \in \bigcap \{A_i \mid i \in$ $\in \hat{m}\}$. If $y \notin X$, then $\hat{m} \neq \emptyset$ implies $y \in Y_r$ for some $r \in Q$ and $j \in \hat{r}$ (just reading the definition of the second union in the definition of A_i). But $r \not\leq m$, so there exists $j' \in \hat{m} \setminus \hat{r}$ and hence $y \in A_{j'}$ –i.e., $y \in Y_{r'}$ for some $r' \in Q$ and $j' \in \hat{r'}$. Since the Y_q $(q \in Q)$ are pairwise disjoint, $r = r'$, and so $j' \in \hat{r}$ after all, a contradiction. $\diamondsuit$

Lemma 3.2. *For all $m \in M$, $\bigcap \{\, B_i \mid i \in \hat{m} \,\} = \{c_m\}$.*

Proof. If $i \in \hat{m}$, then $c_m \in B_i$ by definition. Now suppose that

$$c_l \in \bigcap \{\, B_i \mid i \in \hat{m} \,\},$$

where $l \in M$ but $l \neq m$. Let $j \in \hat{m} \setminus \hat{l}$. Then $c_l \in B_j$, so $j \in \hat{l}$ by definition of B_j, a contradiction. $\diamondsuit$

Lemma 3.3. *For all $a \in \mathbf{2}^n \setminus \uparrow M$, $\bigcap \{\, B_i \mid i \in \hat{a} \,\} = \emptyset$.*

Proof. Suppose $c_m \in \bigcap \{\, B_i \mid i \in \hat{a} \,\}$ for some $m \in M$. As $m \not\leq a$, there exists $j \in \hat{a} \setminus \hat{m}$. Hence $c_m \in B_j$, contradicting the definition of B_j. $\diamondsuit$

Lemma 3.4. *For all $q \in Q \setminus \{(1, \ldots, 1)\}$, $\bigcap \{\, A_i \mid i \in \hat{q} \,\} \supseteq Y_q$.*

Proof. By definition of A_i. $\diamondsuit$

Lemma 3.5. *For all $C \in \mathcal{O}(\mathbf{2}^n) \setminus \{\emptyset, D, \mathbf{2}^n\}$,*

$$|\rho_C(A_1, \ldots, A_n)| \geq |\rho_C(B_1, \ldots, B_n)|.$$

Proof. *Case 1.* $C \cap Q \neq \emptyset$.

Let $c \in \max C$, $q \in Q$ be such that $q \leq c$. Then

$$\begin{aligned}\rho_C(A_1,\ldots,A_n) &\supseteq \bigcap\{\, A_i \mid i\in\hat{c}\,\} \supseteq \\ &\supseteq \bigcap\{\, A_i \mid i\in\hat{q}\,\} \supseteq \\ &\supseteq Y_q\end{aligned}$$

by Lemma 3.4, so

$$|\rho_C(A_1,\ldots,A_n)| \geq |M|.$$

But

$$|\rho_C(B_1,\ldots,B_n)| \leq |M|.$$

Case 2. $C\cap Q=\emptyset$.

In this case, $C\subseteq D$. Thus $M\nsubseteq C$ since $D={\downarrow}M$ and $C\neq D$.

Case 2a. $C\cap M\neq\emptyset$.

Let $l\in C\cap M$. Clearly $l\in\max C$ as $C\subseteq D$. Thus

$$\rho_C(A_1,\ldots,A_n) \supseteq \bigcap\{\, A_i \mid i\in\hat{l}\,\} = X$$

by Lemma 3.1, and

$$|\rho_C(A_1,\ldots,A_n)| \geq |M|-1.$$

But

$$\begin{aligned}\rho_C(B_1,...,B_n) &= \bigcup_{m\in(\max C)\cap M}\bigcap\{B_i \mid i\in\hat{m}\}\cup \bigcup_{c\in(\max C)\setminus M}\bigcap\{B_i \mid i\in\hat{c}\} = \\ &= \left\{c_m \mid m\in(\max C)\cap M\right\}\cup\emptyset = \\ &= \left\{c_m \mid m\in(\max C)\cap M\right\}\end{aligned}$$

by Lemmas 3.2 and 3.3. Thus

$$|\rho_C(B_1,\ldots,B_n)| = |(\max C)\cap M| < |M|$$

since $M\nsubseteq C$.

Case 2b. $C\cap M=\emptyset$.

Then $C\cap{\uparrow}M=\emptyset$ and

$$\begin{aligned}\rho_C(B_1,\ldots,B_n) &= \bigcup_{c\in\max C}\bigcap\{\, B_i \mid i\in\hat{c}\,\} = \\ &= \bigcup_{c\in\max C}\emptyset = \\ &= \emptyset\end{aligned}$$

by Lemma 3.3. $\Diamond$

Lemma 3.6. *The inequality*

$$|\rho_D(A_1,\ldots,A_n)| < |\rho_D(B_1,\ldots,B_n)|$$

holds.

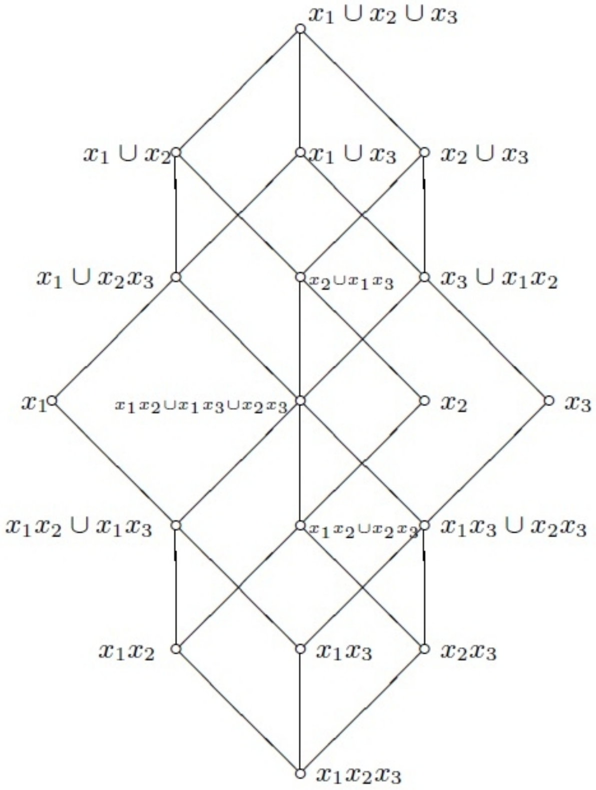

Figure 3.1. The free distributive lattice on 3 generators

Proof. We have

$$\rho_D(A_1, \dots, A_n) = \bigcup_{m \in M} \bigcap \{ A_i \mid i \in \hat{m} \} = X$$

(by Lemma 3.1), which has cardinality $|M| - 1$. But

$$\begin{aligned}\rho_D(B_1, \dots, B_n) &= \bigcup_{m \in M} \bigcap \{ B_i \mid i \in \hat{m} \} = \\ &= \{ c_m \mid m \in M \}\end{aligned}$$

(by Lemma 3.2), which has cardinality $|M|$. $\Diamond$

Proposition 3.7. *Let $n \in \mathbb{N}$. For every n-ary restricted Boolean polynomial ρ, there exist finite sets $A_1, \dots, A_n, B_1, \dots, B_n$ such that*

$$|\sigma(A_1, \dots, A_n)| \geq |\sigma(B_1, \dots, B_n)|$$

for every n-ary restricted Boolean polynomial σ not equivalent to ρ, but

$$|\rho(A_1, \dots, A_n)| < |\rho(B_1, \dots, B_n)|.$$

Proof. The result follows from Lemmas 3.5 and 3.6. $\Diamond$

As an example, Fig. 3.1 shows $FD(3)$. (We use concatenation instead of "$\cap$", so x_1x_2 means $x_1 \cap x_2$.)

Suppose $n = 3$ and
$$\rho(x_1, x_2, x_3) = x_1 \cup (x_2 \cap x_3).$$
Then $D = \{011, 100, 010, 001, 000\}$. (See $\mathbf{2}^3$ in Fig. 3.2.)

Clearly $M = \{011, 100\}$ and $Q = \{101, 110\}$.

Thus,
$$\begin{aligned} X &= \{x\}, \\ Y_{101} &= \{y_1, y_2\}, \\ Y_{110} &= \{y_1', y_2'\}, \\ A_1 &= \{x\}, \\ A_2 &= \{x, y_1, y_2\}, \\ A_3 &= \{x, y_1', y_2'\}, \\ B_1 &= \{c_{011}\}, \\ B_2 &= \{c_{100}\}, \\ B_3 &= \{c_{100}\}. \end{aligned}$$

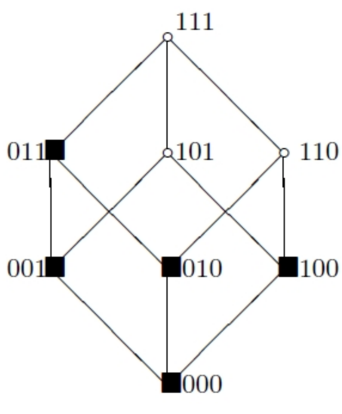

Figure 3.2. A down-set D in $\mathbf{2}^3$

For a related result, see [10], Th. 7.

Thus the problem (attributed to Rado) from Mirsky's 1971 text *Transversal Theory* is solved.

σ	$\sigma(A_1, A_2, A_3)$	$\lvert\sigma(A_1, A_2, A_3)\rvert$	$\sigma(B_1, B_2, B_3)$	$\lvert\sigma(B_1, B_2, B_3)\rvert$
$x_1x_2x_3$	x	1	$\emptyset$	0
x_1x_2	x	1	$\emptyset$	0
x_1x_3	x	1	$\emptyset$	0
x_2x_3	x	1	c_{100}	1
$x_1x_2 \cup x_1x_3$	x	1	$\emptyset$	0
$x_1x_2 \cup x_2x_3$	x	1	c_{100}	1
$x_1x_3 \cup x_2x_3$	x	1	c_{100}	1
$x_1x_2 \cup x_1x_3 \cup x_2x_3$	x	1	c_{100}	1
x_1	x	1	c_{011}	1
x_2	x, y_1, y_2	3	c_{100}	1
x_3	x, y_1', y_2'	3	c_{100}	1
$\rho = x_1 \cup x_2x_3$	x	1	c_{011}, c_{100}	2
$x_2 \cup x_1x_3$	x, y_1, y_2	3	c_{100}	1
$x_3 \cup x_1x_2$	x, y_1', y_2'	3	c_{100}	1
$x_1 \cup x_2$	x, y_1, y_2	3	c_{011}, c_{100}	2
$x_1 \cup x_3$	x, y_1', y_2'	3	c_{011}, c_{100}	2
$x_2 \cup x_3$	x, y_1, y_2, y_1', y_2'	5	c_{100}	1
$x_1 \cup x_2 \cup x_3$	x, y_1, y_2, y_1', y_2'	5	c_{011}, c_{100}	2

Table 3.1. An illustration of Prop. 3.7

4. A "converse" to a counterexample of de Sousa

In 1989, Gian-Carlo Rota formulated his famous Basis Conjecture ([6], Conj. 4):

Rota's Basis Conjecture. *Let V be a vector space of dimension n over an infinite field. Let $B_1, \ldots, B_n$ be bases of V. Then there exists an $n \times n$ table (v_{ij}) such that the i-th row $\{v_{i1}, \ldots, v_{in}\}$ equals B_i for $i = 1, \ldots, n$, and the j-th column $\{v_{1j}, \ldots, v_{nj}\}$ is an independent set for $j = 1, \ldots, n$.*

Since the field is infinite, we can assume the bases are pairwise disjoint by taking scalar multiples if needed. Then the result we want is equivalent to asserting that the set $\bigcup_{i=1}^{n} B_i$ can be partitioned into n pairwise disjoint transversals of the family $(B_1, \ldots, B_n)$, each of which is an independent set.

A result of Edmonds is the following:

Theorem 4.1 ([4], Th. 1)**.** *Let $n \geq 1$. A finite matroid E with rank function ρ can be partitioned into n pairwise disjoint independent sets if and only if, for all subsets F of E,*

$$|F| \leq n\rho(F). \qquad \lozenge$$

This implies ([8], Cor. 3.3.4):

Theorem 4.2. *Let $m, n \geq 1$. Let $\mathcal{B} = (B_1, \ldots, B_m)$ be a family of subsets of a finite set E. For $F \subseteq E$, let $\rho(F)$ be the size of the largest partial transversal of $\mathcal{B}$ contained in F.*

Then E can be partitioned into n pairwise disjoint partial transversals if and only if, for all subsets F of E,

$$|F| \leq n\rho(F). \qquad \Diamond$$

The natural "join" of Theorems 4.1 and 4.2 to partial *independent* transversals is known to be false ([3], pp. 93–94). But we do have the following:

Proposition 4.3. *Let $n \geq 1$. Let V be an n-dimensional vector space. Let $B_1, \ldots, B_n$ be bases for V that are pairwise disjoint. Let $E := \cup_{i=1}^{n} B_i$.*

Then for all $F \subseteq E$,

$$|F| \leq n\rho(F),$$

where $\rho(F)$ is the size of the largest linearly independent partial transversal of $\mathcal{B} = (B_1, \ldots, B_n)$ that is a subset of F.

Proof (by induction). The result is trivial if $F = \emptyset$.

Now let $F \subseteq E$ with $k := |F| \geq 1$ and suppose that, for all $G \subset E$ with $|G| < k$ we have $|G| \leq n\rho(G)$. Let $l := \rho(F)$. Pick any $f_1 \in F$ and let $F_1 := F \setminus \{f_1\}$. Let $k_1 := |F_1|$ and $l_1 := \rho(F_1)$. Then $l_1 \leq l \leq l_1 + 1$.

If $l = l_1 + 1$, then $k_1 \leq nl_1$, so $|F| = k = k_1 + 1 \leq nl_1 + 1 \leq \leq nl_1 + n = n(l_1 + 1) = nl = n\rho(F)$.

So assume $l = l_1$. We know $l \geq 1$.

If $k_1 < nl_1 = nl$, then $|F| = k = k_1 + 1 \leq nl = n\rho(F)$.

So assume $k_1 = nl_1 = nl$ (so that $k = nl + 1$).

Claim 1. *We may assume that the number of $i \in \{1, \ldots, n\}$ such that $F \cap B_i \neq \emptyset$ is at least $l + 1$.*

Proof of Claim. Without loss of generality, suppose

$$F \cap B_i = \emptyset$$

for $i = l + 1, \ldots, n$. Then $F \subseteq \cup_{i=1}^{l} B_i$; but then

$$|F| = k \leq |\cup_{i=1}^{l} B_i| \leq nl,$$

and we would be done. $\Diamond$

Claim 2. *We may assume there exists $i \in \{1, \ldots, n\}$ such that $|F \cap B_i| \geq \geq l + 1$.*

Proof of Claim. If $|F \cap B_i| \leq l$ for $i = 1, \ldots, n$, then

$$k = |F| = |F \cap \cup_{i=1}^n B_i| = | \cup_{i=1}^n (F \cap B_i)| \leq nl,$$

and we would be done. $\diamondsuit$

Let $i_F \in \{1, \ldots, n\}$ be such that $|F \cap B_{i_F}| \geq l+1$. Let $F_2 := F \setminus B_{i_F}$ and let $k_2 := |F_2|$ and $l_2 := \rho(F_2)$. Note that $k - n \leq k_2 \leq k - 1$, so $nl + 1 - n \leq k_2 \leq nl$ or $n(l-1) + 1 \leq k_2 \leq nl$.

Of course $l_2 \leq l$. But if $l_2 \leq l - 1$, then

$$k_2 = |F_2| \leq n\rho(F_2) = nl_2 \leq n(l-1),$$

a contradiction. Thus $l_2 = l$, so there exists an independent partial transversal of $(B_1, \ldots, B_n)$ in F_2 of size l, call it $(v_{i_1}, \ldots, v_{i_l})$, where $1 \leq i_1 < \cdots < i_l \leq n$ and $v_{i_j} \in B_{i_j}$ for $j = 1, \ldots, l$.

Clearly $i_F \notin \{i_1, \ldots, i_l\}$ and the dimension of W, the span of $\{v_{i_1}, \ldots, v_{i_l}\}$, is l.

As $|F \cap B_{i_F}| \geq l + 1$, there exists $v_F \in F \cap B_{i_F} \setminus W$. Thus $\{v_{i_1}, ..., v_{i_l}, v_F\} \subseteq F$ is independent and a partial transversal of $(B_1, ..., B_n)$; hence $\rho(F) = l + 1$, a contradiction to the fact that $\rho(F) = l$. $\diamondsuit$

Addendum

A reader of an earlier version of this manuscript, trying to interpret Prop. 3.7 in the context of voting theory, made the following claims.

Claim A.1. *Given any non-trivial up-set W in a finite Boolean algebra B, there are subsets X and Y of B such that W is the only non-trivial up-set of B containing more elements of X than of Y.*

Claim A.2. *Given any non-trivial up-set W in a finite poset P with 1, there are subsets X and Y of P such that W is the only non-trivial up-set of P containing more elements of X than of Y.*

Claim A.3. *Claim A.1 is equivalent to Prop. 3.7.*

We have proven that the three claims are false, no matter what reasonable interpretation is given to "non-trivial" and "more."

Observation A.4. *Claim A.1 is false, if "non-trivial" means either "non-empty" or "non-empty and proper" and if "more" means either "strictly more" or "at least as many."*

Observation A.5. *Claim A.2 is false, if "non-trivial" means either "non-empty" or "non-empty and proper" and if "more" means either "strictly more" or "at least as many."* $\diamondsuit$

Observation A.6. *Claim A.3 is false.*

Acknowledgement. The author would like to thank Dr. Hazel Perfect for comments regarding this paper.

The author gives his heartfelt thanks to the Research Institute for Mathematics for supporting this work.

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