Carrying Is a 2-Cocycle

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February 2023

Editor's Note. In 1994, James Dolan posted a series of articles, each with the word "carrying" in the subject line, to the USENET newsgroup `sci.math`. He posted a couple of different versions; the present document is a lightly edited and LaTeXed compilation of the January 19, 1994 version. James Dolan writes, "this is re-posted with my permission, though i'm not sure how differently i might try to explain these things now if i tried it again."

1 Part 1

In another thread, I wrote [in response to Ron Maimon —ed.]:

> Well, I'm impressed. I was well into my thirties before I realized that the "carry digit" function is the premier example of a 2-cocycle.

Some people have asked me to elaborate on this, so I will try to do so, particularly since it ties in somewhat with questions Tim Chow has been asking about homological algebra lately.

First of all, what do I mean by the "carry digit" function here? Well, the binary operation of addition of single-digit numbers gives as output a number with two digits: the "ones" (or "least significant") digit, and the "tens" (or "most significant") digit. Doing the case of base five so as to keep the table small, for example, here is the addition table you get:

	0	1	2	3	4
0	00	01	02	03	04
1	01	02	03	04	10
2	02	03	04	10	11
3	03	04	10	11	12
4	04	10	11	13	13

It turns out to be interesting, from the utmost practical point of view (for example in the implementation of arithmetic on computers), just as well as from the utmost theoretical point of view, to break up this double-digit-number-valued function into two single-digit-number-valued functions. Focusing on just the "ones" digit, we get the well-known operation of "addition of integers modulo the multiples of five":

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	3	3

Focusing on just the "tens" digit, on the other hand, we get the equally important but less remarked upon "carry digit" operation:

	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	1	1
3	0	0	1	1	1
4	0	1	1	1	1

So what is going on here? Let's note the following particular features of the situation:

1. We have a "big" group (the group of two-digit numbers, under the operation of addition modulo the multiples of 100), and two "small" groups (the "tens" group $\{00, 10, 20, 30, 40\}$ and the "ones" group $\{0, 1, 2, 3, 4\}$). The big group can be thought of as "built out of" the two small groups, but in a slightly tricky way that makes crucial use of the carry digit function, and that has the "ones" group and the "tens" group playing sharply contrasting roles. (Be careful to note that the word "group" here is being used in its technical mathematical sense, meaning a set equipped with a certain sort of binary operation into itself, which in all the cases we're interested in here is more or less just the binary operation of addition.)

2. The "tens" group can be thought of as a "subgroup" of the big group. That is, the elements $00, 10, 20, 30, 40$ of the "tens" group add together in the big group just the same way they add together among themselves. The "ones" group, on the other hand, cannot be thought of as a subgroup of the big group. That is, the elements $0, 1, 2, 3, 4$ of the "ones" group add together very differently depending on whether they are thought of as belonging to the big group or as forming a small group all by themselves. For example, $4 + 4 = 13$ using the addition operation in the big group, which is different from the answer $4 + 4 = 3$ that you get in the small "ones" group.

3. The "ones" group can be thought of as a "quotient group" of the big group. That is, the "ones" group can be thought of as the big group, "modulo" the subgroup consisting of the multiples of 10. That is, if I pick a pair x, y of two-digit numbers, but I only tell you the "ones" digit of x and the "ones" digit of y, then you can tell me the "ones" digit of $x + y$. That is, the "ones" digit of the sum $x + y$ depends only upon the "ones"

digits of x and y; the "tens" digits are completely irrelevant if all you care about are the "ones" digits. The "tens" group, on the other hand, cannot be thought of as a quotient group of the big group. That is, if I pick a pair x, y of two-digit numbers, but I only tell you the "tens" digits of x and the "tens" digit of y, then you can't in general tell me the "tens" digit of $x + y$. The best you can do is to ask me, "It all depends—when you added the "ones" digits together, what was the carry digit produced?" For example, $11 + 12 = 23$, whereas $14 + 14 = 33$; you can't figure out the "tens" digit of $x + y$ without having a peek over at the "ones" digits of x and y. That is, the "tens" digit of the sum $x + y$ does not depend only upon the "tens" digits of x and y; it depends also upon the carry digit produced by the "ones" digits of x and y.

Those are the basic ingredients of the situation, which is, for some reason, known as "expressing the big group of double-digit numbers as an extension of the quotient group $\{0, 1, 2, 3, 4\}$ by the subgroup $\{00, 10, 20, 30, 40\}$." Besides the addition tables for the addition operations in the two small groups $Q = \{0, 1, 2, 3, 4\}$ and $S = \{00, 10, 20, 30, 40\}$, the basic data that you need to build the addition table for the big group is the carry digit function, which is a function with two inputs both of type Q and one output of type S, represented in standard "arrow" notation as:

$$Q \times Q \to S$$

Then, thinking of a two-digit number as a pair (s_1, q_1), the formula for the sum of (s_1, q_1) and (s_2, q_2) is

$$(s_1 + s_2 + \text{carry}(q_1, q_2), q_1 + q_2).$$

Essentially this same formula can be used with many different "carry-digit functions." The technical name for such a "carry-digit function" is "2-cocycle" (or more specifically, "2-cocycle on the group Q with coefficients in the group S"). 2-cocycles by definition satisfy certain simple algebraic laws that guarantee that the resulting "big group" is in fact a genuine group extending the quotient group Q by the subgroup S. Different 2-cocycles can result in fundamentally different big groups, but sometimes there are interesting relationships between the resulting big groups.

As long as the situation is viewed as one of pure algebra, however, much about it must appear mysterious and arbitrary. In order to understand the really most important reasons why 2-cocycles (and their relatives the "n-cocycles" (for other natural numbers n) are interesting, you have to learn about what at first may seem like a completely unrelated branch of mathematics: topology.

(Notice that I am essentially repeating here the message conveyed by John Baez in a recent reply to Tim Chow: that the secret weapon to use in understanding even the most algebraic manifestations of "homological algebra" is an understanding of the conceptual origin of homological algebra in problems of topology.)

[This concludes Part 1; in Part 2 I hope to begin to explain how in the world the phenomenon of "extensions of groups" actually has anything to do with topology.]

2 Part 2

[The story so far: the operation of addition (with wrap-around overflow) of double-digit (decimal, or binary, or octal, etc.) numbers can be defined by a formula that uses only:

1. addition (with wrap-around overflow) of single-digit numbers, and:
2. one other primitive binary operation on single-digit numbers, called the "carry-digit" function, which takes the value 0 whenever the single-digit numbers add without overflow, and 1 otherwise.

Representing double-digit numbers as ordered pairs of single-digit numbers, so that for example 17 is represented as $(1, 7)$, this formula is:

$$(a, b) + (c, d) := (a + c + \text{carry}(b, d), b + d).$$

It turns out that this phenomenon is just a special case of a very interesting and important general process for creating a new big "group" (which is a set equipped with a binary "addition" operation of a special type), whose elements are ordered pairs of elements from two smaller groups, and whose addition operation is defined with the help of a "generalized carry-digit function." In the general case, the two small groups may be different from each other; that is, the right digit (the one we call the "ones" digit, and which "sends" the carry) comes from one group, called the "quotient group," while the left digit (the one that we call the "tens" digit, and which "receives" the carry) comes from another group, called the "subgroup." The generalized carry-digit function is called the "2-cocycle" (for reasons which may possibly be within the limits of human understanding, and which I may even get around to trying to explain), and given a quotient group Q, subgroup S, and 2-cocycle c

$$Q \times Q \to S,$$

the group whose underlying set is $S \times Q$ and whose "addition" operation is given by the formula at top is called "the central extension of Q by S, with 2-cocycle c."

(The most general case that I intend to consider here is the case where the quotient group Q may be non-abelian, but the subgroup s is abelian. "Abelian" (also known as "commutative") means that the law "$a + b = b + a$" holds. There are however generalizations of the process that can apply in the case where even S is non-abelian.)

I admit that some people claim to have mastered the art of adding two-digit numbers without any explicit introduction to the concept of 2-cocycle, and I think that Tal Kubo and myself are both resigned to the likelihood that the ordinary carry-digit function will remain the only 2-cocycle that Ron Maimon and millions of other American schoolchildren ever learn, and the only 2-cocycle that is hard-coded into medium-priced personal computers. But I hope that there may be some people struggling with the general concept of 2-cocycle and with other general concepts of homological algebra who will benefit from seeing how the familiar and lowly carry-digit function is an example of a 2-cocycle.]

Let's return to the example of the carry-digit function for base five numerals. In this case, both the quotient group Q and the subgroup S are the integers modulo five, and the 2-cocycle $c\colon Q \times Q \to s$ is:

	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	1	1
3	0	0	1	1	1
4	0	1	1	1	1

But let's consider now also another 2-cocycle $d\colon Q \times Q \to s$, as follows:

	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	4	2
2	0	0	4	1	2
3	0	4	1	1	2
4	0	2	2	2	3

You can check for yourself that this is in fact a 2-cocycle; that is that if you use it as a "generalized carry-digit function," it actually makes $S \times Q$ into a group with S as subgroup and Q as quotient group. However, there is a sense in which the 2-cocycle d is "equivalent" to the 2-cocycle c that we have already seen. This equivalence can be expressed in a number of different ways. One way of expressing it is to say that the groups $(S \times Q, c+)$ and $(S \times Q, d+)$ (where "$g+$" denotes the addition operation defined using the 2-cocycle g as generalized carry-digit function) are isomorphic, and moreover isomorphic in such a way that both the inclusion map from the subgroup S and the projection map onto the quotient group Q are preserved.

Another way of expressing it is to say that the 2-cocycles c and d are "cohomologous" to each other. This means that if you subtract the function c from the function d (which makes sense since c and d are functions taking values in an abelian group), the result is a "coboundary" 2-cocycle. A coboundary 2-cocycle is a 2-cocycle that's in the image of the map "coboundary"

$$\text{1-cochains} \longrightarrow \text{2-cocycles},$$

where the "1-cochains" are the arbitrary functions from Q to S, and where the coboundary of a 1-cochain

$$Q \xrightarrow{f} S$$

is the 2-cocycle

$$Q \times Q \xrightarrow{g} S$$

given by the formula:

$$g(a,b) := f(a) + f(b) - f(a+b).$$

You can check for yourself that the 2-cocycles c and d given above are cohomologous to each other because when you subtract c from d you get:

	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	4	1
2	0	0	4	0	1
3	0	4	0	0	1
4	0	1	1	1	2

which is the coboundary of the 1-cochain $f\colon Q \to S$ given by:

a	0	1	2	3	4
$f(a)$	0	0	0	0	1

It is then an important but straightforward theorem that these two equivalence relations on the class of 2-cocycles, namely isomorphicness of the associated central extensions, and cohomologousness, are in fact the same.

[This concludes Part 2. In Part 3 I hope to begin to show how, by reinterpreting the situation from the viewpoint of topology, much of what is going on and which appears mysterious and arbitrary from the viewpoint of pure algebra becomes much more clearly motivated. Of course that's the same thing I said about Part 2, but hope springs eternal.]

3 Part 3

[So far I have introduced a concept of "2-cocycle" which is a sort of "generalized carry-digit function" from $Q \times Q$ to S, where Q is some group and S is some abelian group; and I have discussed an equivalence relation "a is cohomologous to b" on the set of 2-cocycles, for which the equivalence classes (called "cohomology classes") correspond to isomorphism classes of "central extensions of Q by S."]

There is a very close relationship between the "cohomology of groups" that we have been considering so far, and the older "cohomology of spaces." One way of trying to understand this relationship is to associate to each group Q a topological space (called "$K(Q,1)$" [an Eilenberg–Mac Lane space —ed.] for some (pretty good) reason), constructed in stages as follows:

- Stage 0: Start with a single point p, called the "basepoint" of the space.
- Stage 1: For each element $a \in Q$, sew in a path "P_a" from the basepoint p to itself.

- Stage 2: For each pair (a, b) of elements in Q, sew in a solid triangle, filling in the hollow triangle formed by traversing the paths P_a and then P_b in the forwards direction, followed by the path $P_{[ab]}$ in the backwards direction.

Already after Stage 2 we have achieved an important goal: the group Q can now be recovered from the space as its "fundamental group." This is the group whose elements are equivalence classes of paths from the basepoint of the space to itself, under the equivalence relation of "homotopy," with the group operation being concatenation of paths.

Two paths are "homotopic" if you can gradually deform one of them into the other, keeping the ends of the path fixed at the basepoint. For example, notice that the solid triangles that we sew in make the concatenation of P_a and P_b homotopic to $P_{[ab]}$, which is as it should be.

(One way to verify that Q is the fundamental group of the space after Stage 2 is to apply the "Seifert–Van Kampen theorem," which is a general theorem about how to compute the fundamental group of a space sewn together from a bunch of little pieces. This theorem and its proof are not that hard to understand.)

However, although we have (after Stage 2) succeeded in sculpting the fundamental group of the space to be exactly what we wanted it to be (namely Q), we have in the process played havoc with the "higher homotopy groups" of the space. The "first homotopy group" of a space is just its fundamental group, whose elements are homotopy classes of "loops"; that is, of paths from the basepoint to itself; that is, of "figures shaped like the circle." The elements of the second homotopy group of the space are, similarly, homotopy classes of "2-loops"; that is, of figures shaped like the sphere; and so on for the other higher homotopy groups. To see how we have managed to play havoc with these higher homotopy groups, think of all of the solid triangles that we sewed into the space. From four solid triangles you can make a hollow tetrahedron, which is topologically just a sphere; and indeed, in general we will have in this way introduced some hollow spheres into our space.

These higher dimensional holes that have wormed their way into our space detract from the purity with which the space acts as a "geometric realization" of the group Q. It would be nice if we could get rid of all of the higher-dimensional holes without disturbing the fundamental group. That is, it would be nice if we could end up with a space such that not only is its fundamental group equal to Q, but also all of its higher homotopy groups are trivial. And this is in fact the defining property that a space needs to satisfy in order to qualify as being "the" space $K(Q, 1)$. So the remaining stages of the construction are devoted to the purpose of filling in all of the higher-dimensional holes without disturbing the fundamental group. And this is not so hard to do, because, in general, filling in an "$[n+1]$-loop" with an "$[n+2]$-cell" has absolutely no effect on the homotopy classes of n-loops in the space; the $[n+2]$-cells are in a sense "too big for the n-loops to notice." Thus, whatever damage is caused by tinkering with the n-loops propagates only in the upwards direction; so that if we take care at stage n of our construction to get the $[n-1]$th homotopy group of the space correct, then after an infinite number of stages all of the homotopy groups will be correct, all of the upwards-propagated damage having been repaired at the appropriate stage. Thus the construction continues:

- Stage 3: Here we want to insure that the second homotopy group is correct; that is, we want to kill it off entirely, by filling in all of the hollow tetrahedrons formed by appropriately adjoining quadruples of the solid triangles that we laid down in Stage 2. In fact, these hollow tetrahedrons correspond to triples (a, b, c) of elements of Q, as follows:
  - triangle #1: sides $P_a, P_b, P_{[ab]}$
  - triangle #2: sides $P_b, P_c, P_{[bc]}$
  - triangle #3: sides $P_{[ab]}, P_c, P_{[abc]}$
  - triangle #4: sides $P_a, P_{[bc]}, P_{[abc]}$

  You can check for yourself that these four triangles in fact touch one another so as to form a hollow tetrahedron; then filling in each such hollow tetrahedron with a solid tetrahedron completes Stage 3.

- . . .
- Stage n: The pattern established in stages 0, 1, 2, and 3 continues: at Stage n, the "solid n-simplexes" that need to be sewn in correspond to the n-tuples of elements of Q.
- . . .

Thus, after all of the stages have been completed, we have a space "$K(Q, 1)$," with fundamental group equal to Q and with all other homotopy groups trivial; and this space is equipped with a combinatorial decomposition into "cells" of all dimensions, which leads to a particular combinatorial scheme for computing the cohomology of the space. (I think that this particular scheme is called something like "cellular cohomology.") And it is easy (if somewhat laborious) to see that the cellular cohomology of the space $K(Q, 1)$ is in fact precisely just the "group cohomology" of the group Q.

Thus, to focus on the example that started this discussion off, consider some 2-cocycle $c\colon Q \times Q \to S$, where s is some abelian group. From the algebraic point of view this acts as a "generalized carry-digit function," but from the topological point of view it is just a "cellular 2-cocycle" on the space $K(Q, 1)$. Each pair (a, b) of elements of Q corresponds to a solid triangle ("2-cell") sewn into the space at Stage 2 of the construction, and the 2-cocycle c assigns to each such 2-cell the element $c(a, b)$ of the "coefficients" group S.

Furthermore, the equivalence relation "c is cohomologous to d" on the set of cellular 2-cocycles, defined in terms of the cellular 1-cochains, is, on the one hand, just the same as I defined it to be in the purely group-theoretical setting; while on the other hand, the set of equivalence classes with respect to it ("cohomology classes") is a topological invariant of the space $K(Q, 1)$.

[This concludes Part 3. I have now given a fairly thorough explanation of my original smart-alecky remark about the "carry-digit function" being a 2-cocycle, but in future installments I would like to try to explain some more important general ideas about the conceptual interaction between group theory and algebraic topology. What I have described so far is only a very small taste of the thorough inter-mixture of ideas from these two areas that occurs.]