PROOF TECHNIQUES IN THE THEORY OF FINITE SETS*

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1. INTRODUCTION

This paper is a survey of a class of methods which have proved successful in attacking certain problems in the theory of finite sets. Specifically, we will be concerned with *extremal problems*: i.e., given a family F of finite sets, satisfying certain restrictions, how large can F be? Typically, an answer to this question leads to a classification of the extremal cases, so that many of the results in this paper are ultimately structural as well as numerical.

No attempt has been made to provide a complete survey of the field: many beautiful results which do not fit into our (perhaps rather arbitrary) scheme have been omitted entirely. And although we have included a wide variety of problems, our motivation in choosing them has been at least partially based on how well they illustrate certain techniques. For a more encyclopedic treatment, we recommend the survey articles by Katona [30] and Erdős and Kleitman [16].

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Because our goal is to study techniques, we shall also investigate the extent to which these techniques can be generalized to a wider class of combinatorial objects. Specifically, we will attempt—wherever possible—to extend results about families of sets to more general kinds of partially ordered structures, with a special emphasis on three important combinatorial examples: *multisets* (or *divisors of an integer*), *subspaces of a finite vector space*, and *partitions of a set*. The structures formed by these combinatorial objects are analogous in many ways to Boolean algebras of sets, and provide a rich source of problems. By using them as examples, we hope to illustrate both the power and the limitations of the methods described.

It should be mentioned that our systematic treatment of the analogies between families of finite sets and other combinatorial objects is very much in the spirit of ideas originally suggested by Rota (see, for example, [21], [51], or [63]).

THE SPERNER PROBLEM

If P is a partially ordered set, an *antichain of P* is a subset of P whose elements are totally unrelated (as opposed to a *chain*, whose elements are totally related). One of the earliest results of the kind considered in this paper is a theorem about antichains of sets, due to E. Sperner and published in 1928:

THEOREM 1.1 (Sperner): *Let F be a family of subsets of $\{1, 2, \ldots, n\}$, no member of which contains another. Then $|F| \leq \binom{n}{[\frac{n}{2}]}$. Equality occurs only if all of the sets have the same size.*

Given an arbitrary partially ordered set P, we can ask a similar question: what is the size of the largest antichain in P? In this paper, we will describe a number of ways to approach this problem, and many of the results discussed can be viewed as extensions, refinements, or analogs of Sperner's fundamental theorem.

For arbitrary partially ordered sets, the problem of finding a maximum-sized antichain can be expressed as a network flow problem and solved (efficiently) by standard techniques (see [58] for a description of an elementary algorithm). We will not be concerned with the problem in such generality, but rather with special cases where the answer has a simple form.

Suppose that P is a partially ordered set with a *rank function* r. That is, r is a function defined on the elements of P, taking nonnegative integer values, such that $r(x) = 0$ for every minimal element x, and $r(x_1) = r(x_2) + 1$ whenever x_1 covers x_2. For each integer k, let P_k denote the collection of elements in P having rank k. Clearly each P_k is an antichain. When P consists of all subsets of a finite set, rank coincides with cardinality, and Sperner's theorem asserts than an antichain of maximum size can be obtained by taking all elements of rank $\left[\frac{n}{2}\right]$. In general, we say that P has the *Sperner Property* if the maximum size of an antichain in P is equal to $\max_k |P_k|$.

WHITNEY NUMBERS

The following notation will be useful when considering more general classes of partially ordered sets: if P has a rank function, let $N_k(P)$ denote the number of elements of rank k in P (i.e., $N_k(P) = |P_k|$). Following Crapo and Rota [8], we call $N_k(P)$ the *kth Whitney number of P (of the second kind)*.*

There are four basic families of partially ordered sets considered in this paper: *sets, multisets, subspaces,* and *partitions.* We describe each of them briefly below, including a calculation of the appropriate Whitney numbers for each class:

(1) *Sets*: Let B_n denote the Boolean algebra of all subsets of $\{1, 2, \ldots, n\}$, ordered by inclusion. Then, as noted before, B has a rank function $r(s) = |S|$, and

$$N_k(B_n) = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

(2) *Multisets*: Fix a sequence $\vec{e} = (e_1, e_2, \ldots)$, where each e_i is either a nonnegative integer or $+\infty$. Let $M_{\vec{e}}$ denote the collection of all "finite multisets of integers with multiplicities restricted by $\vec{e}$." By this we mean the family of all finite unordered collections of positive integers, where repetitions are allowed but each i can appear at most e_i times. Each multiset in $M_{\vec{e}}$ can be represented by a sequence $\vec{\sigma} = (\sigma_1, \sigma_2, \ldots,)$ of nonnegative integers σ_i such that $0 \leq \sigma_i \leq e_i$ for each i and also $\Sigma\sigma_i < \infty$. For two multisets $\vec{\sigma}$ and $\vec{\sigma}'$, define $\vec{\sigma} \leq \vec{\sigma}'$ if $\sigma_i \leq \sigma_i'$ for all i. Under this ordering $M_{\vec{e}}$ is a distributive lattice—in fact $M_{\vec{e}}$ is isomorphic to a cartesian product of chains.

It is easy to see that $M_{\vec{e}}$ has a rank function given by $r(\vec{\sigma}) = \Sigma\sigma_i$, and that $N_k(M_{\vec{e}})$ is the coefficient of x^k in the expression

$$\prod_{i=1}^{\infty}(1 + x + \cdots + x^{e_i})$$

(with the obvious convention if $e_i = \infty$). We also introduce the notation

$$N_k(M_{\vec{e}}) = \binom{e_1, e_2, \ldots}{k}.$$

Although these numbers are more difficult to work with than binomial coefficients, one can obtain a trivial analog of the binomial recursion: if $\vec{e}\,'$ is obtained from $\vec{e}$ by replacing e_m by 0, for some m, then

$$\binom{e_1, e_2, \ldots}{k} = \sum_{i=0}^{e_m}\binom{e_1', e_2', \ldots}{k-i}$$

(where, by convention, a term vanishes if $k - i$ is negative).

* When there is no chance of confusion, we will write $N_k(P) = N_k$, and if $x \in P$ we also write $N_{r(x)}(P) = N_x(P) = N_x$.

When $e_1 = e_2 = \cdots = e_n = \infty$, $e_{n+1} = e_{n+2} = \cdots = 0$, we call $M_{\vec{e}}$ the *lattice of unrestricted (finite) multisubsets of* $\{1, 2, \ldots, n\}$. In this case, $N_k(M_{\vec{e}})$ is the coefficient x^k in the expression $(1 - x)^{-n}$, and we obtain the explicit formula

$$N_k(M_{\vec{e}}) = (-1)^k \binom{-n}{k} = \binom{n+k-1}{k}.$$

(2′) *Divisors of an integer*: If N is any positive integer, let D_N denote the lattice of divisors of N, ordered by the relation of divisibility. If $N = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime decomposition of N, the divisors of N can be thought of as multisets of primes, with multiplicities restricted by $e_1, e_2, \ldots, e_k$, and the ordering of multisets coincides with that of divisors. Hence D_N is isomorphic to $M_{\vec{e}}$, where $\vec{e} = (e_1, e_2, \ldots, e_k, 0, 0, \ldots)$.

If we take $e_i = \infty$ for all i, then $M_{\vec{e}}$ is isomorphic to the lattice of all positive integers, ordered by divisibility.

(3) *Subspaces*: Let $L_n(q)$ denote the lattice of subspaces of a vector space of dimension n over a field of q elements, ordered by inclusion. Then $L_n(q)$ has a rank function $r(U) = \dim(U)$, and

$$N_k(L_n(q)) = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)} = \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

These numbers are called *Gaussian coefficients*, and are polynomials in q for fixed n and k, as can be seen from the recursion

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

It is interesting to note that

$$\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

(4) *Partitions*: Let Π_n denote the collection of all partitions of the set $\{1, 2, \ldots, n\}$, ordered by *refinement*. That is, $\sigma \le \tau$ if σ can be obtained by subdividing the blocks of τ. It is well known that Π_n is a lattice, with a rank function r defined by $r(\sigma) = n - |\sigma|$, where $|\sigma|$ denotes the number of blocks of σ. The Whitney numbers of Π_n are given by

$$N_k(\Pi_n) = S(n, n-k),$$

where $S(n, n - k)$ denotes a *Stirling number* (of the second kind). Although no explicit formula for the Stirling numbers is known, it is easy to derive the recursion

$$S(n+1, k) = S(n, k-1) + k\, S(n, k).$$

Among the four classes of partially ordered sets described here, it is known (and we shall prove) that three have the Sperner Property: sets, multisets, and subspaces. For partitions, however, the answer is still not known, and is the subject of a long-standing conjecture of Rota [51]. (**Note added in proof:** Rota's conjecture has recently been settled in the negative by E. R. Canfield.)

2. SYMMETRY

We begin this section by giving one of the shortest known proofs of Sperner's theorem (although it turns out to be one of the least capable of generalization). The proof rests on the following observation, due to Kleitman, Edelberg, and Lubell [43]:

LEMMA 2.0: *Every partially ordered set P contains an antichain of maximum size which is invariant under every order-automorphism of P.*

To see that Sperner's theorem follows from this fact, observe that any invariant antichain in B_n must consist of all sets of some fixed size i. Such a family has $\binom{n}{i}$ members, which is maximized when $i = [n/2]$.

Freese [20] observed that Lemma 2.0 can be proved in the following way: define an ordering on antichains of P by saying $A \leq B$ if every member of A is dominated by some member of B. If A and B are two antichains, define $A \vee B$ to be the antichain of maximal elements in the set $A \cup B$. Trivially $A \leq A \vee B$ and $B \leq A \vee B$. Moreover if A and B are antichains of maximum size, it is easy to see that $A \vee B$ is again of maximum size. Repeating this operation, we obtain a maximum-sized antichain which is "largest" in the sense of the ordering just defined. This antichain must be invariant under every automorphism of P, and the lemma is proved.

Freese's argument is based on ideas of Dilworth [13], who showed that the maximum-sized antichains of a partially ordered set P form a distributive lattice, which is a sublattice of the (distributive) lattice of all antichains in P.

It is clear that the same arguments prove the following:

THEOREM 2.1: *If P is a partially ordered set with rank function whose automorphism group is transitive on each set of elements of fixed rank, then P has the Sperner property.*

As a consequence, we obtain:

COROLLARY 2.2: *For each n and q, $L_n(q)$ has the Sperner property.*

Unfortunately, the hypotheses of Theorem 2.1 do not hold (in general) for either lattices of multisets or lattices of partitions.

A more general class of problems to which these methods apply can be described as follows: a subset $A \subseteq P$ is called a *k-family* of P if A contains no chains of length $k + 1$. Erdös proved the following [15]:

THEOREM 2.3: *Let F be a family of subsets of $\{1, 2, \ldots, n\}$ that contains no chains of length $k + 1$. Then $|F|$ is bounded by the sum of the k largest binomial coefficients.*

Greene and Kleitman [23] showed that Theorem 2.3 could also be proved by an argument of the Dilworth-Kleitman-Freese type. If A and B are k-families of P, define A_i and B_i to be the elements of "depth" i in A and B respectively (the depth of an element $x \in A$ is the length of the longest chain in A whose bottom is x). Define $A \leq B$ if $A_i \leq B_i$ for $i = 1, 2, \ldots, k$. Then the following can be proved:

THEOREM 2.4: *If P is any partially ordered set, then for each k there exists a k-family of maximum size which is largest with respect to the ordering just defined, and hence is invariant under every automorphism of P.*

COROLLARY 2.5: *The analog of Theorem 2.3 holds in $L_n(q)$ for every n and q, and, more generally, in any partially ordered set whose automorphism group is transitive on elements of a given rank.*

The proof of theorem 2.4 is not immediate for $k > 1$. In section 4 we will see that easier proofs of Erdös' theorem and its analogs are available.

3. SATURATED PARTITIONS

In this section, we describe a method for proving Sperner's Theorem which applies to lattices of sets and lattices of multisets, but (apparently) not to lattices of subspaces or partitions.

Let P be an arbitrary partially ordered set, and let $\mathcal{C} = \{C_1, C_2, \ldots, C_q\}$ be a partition of P into chains C_i. Then $\mathcal{C}$ determines a bound on the size of the largest antichain in P: since chains and antichains have at most one element in common, no antichain in P can have more than q elements. When the bound determined by $\mathcal{C}$ is exact, we call $\mathcal{C}$ a *saturated partition of P*.

By a famous theorem of Dilworth [14], saturated partitions always exist. This means that, if an antichain $A \subseteq P$ is of maximum size, it is always possible to verify this by finding a partition of P into $|A|$ chains.

To prove Sperner's Theorem by this method, we must find a way to partition B_n into $\binom{n}{[\frac{n}{2}]}$ chains, for each n. We shall give an inductive proof that such partitions exist, based on a construction discovered by deBruijn, Tengbergen, and Kruyswijk [5] and rediscovered by a number of others.

The essential feature of this construction is that the chains turn out to be *symmetric*: that is, they stretch from a set of size k to a set of size $n - k$, for some k, meeting every intermediate rank. Trivially, if B_n is partitioned into symmetric chains, the number of chains is exactly $\binom{n}{[\frac{n}{2}]}$.

The construction is as follows: suppose that the subsets of $\{1, 2, \ldots, n - 1\}$ have been partitioned into symmetric chains. For each chain C of length $k \geq 1$, construct a second chain C' by adding n to each set in C. Then construct two new chains in B_n by removing the top of C' and adding it to C. The new chains have lengths $k - 1$ and $k + 1$, and trivially both are symmetric. (If C' becomes empty at this stage it is disregarded.) Applying this procedure to every chain gives a partition of B_n into symmetric chains as desired.

DeBruijn, Tengbergen, and Kruyswijk showed that a minor modification of this procedure works for divisors of an integer as well. That is, the divisors of an integer $N = \prod_{i=1}^{m} p_i^{e_i}$ can be partitioned into $N_k(M_z)$ symmetric chains, where $k = [\Sigma e_i/2]$. (We omit the details.)

Greene and Kleitman [24] and independently Leeb [unpublished] found that the partition constructed above could be described explicitly in the following way:

First associate to each subset $S \subseteq \{1, 2, \ldots, n\}$ a sequence of left and right parentheses, replacing each element of S by a right parenthesis, and each element of the complement of S by a left parenthesis. For example, if $n = 9$ and $S = \{1, 3, 4, 7, 8\}$, we obtain the sequence

```
)  (  )  )  (  (  )  )  (
1  2  3  4  5  6  7  8  9
```


Every sequence of left and right parentheses has a unique "parenthesization" obtained as follows: close all pairs of left and right parentheses which are either adjacent or separated by other such pairs, repeating the process until no further pairing is possible. Note that the remaining unpaired parentheses must necessarily consist of "rights" followed by "lefts".

Now define a partition of B_n by saying that two sets are in the same block if they have the same "parenthesization". From the above remark about unpaired elements, it follows that two sets in the same block must be comparable; hence we have partitioned B_n into chains. In general, a chain in this partition is obtained by starting at the bottom with a set of elements which can be completely paired, and adding the unpaired elements from left to right, one at a time.

For example, the chain in B_9 which contains $S = \{1, 3, 4, 7, 8\}$ consists of the sets $\{3, 7, 8\}$, $\{1, 3, 7, 8\}$. $\{1, 3, 4, 7, 8\}$, $\{1, 3, 4, 7, 8, 9\}$. obtained by adding 1, 4, and 9 in order to $\{3, 7, 8\}$.

It is immediate that the chains constructed in this way are all symmetric, and that the procedure coincides exactly with the one defined inductively by deBruijn, Tengbergen, and Kruyswijk.

Using the above construction, it is possible to obtain the conclusion of Sperner's theorem from slightly weaker hypotheses. It is easy to see that the "unpaired" elements alternate odd-even (since blocks of consecutive paired elements are always even in number); hence the difference of any two members of the same chain is a set which alternates "odd-even". Hence the bound of $\binom{n}{[\frac{n}{2}]}$ remains valid if we exclude only comparable pairs of sets whose difference alternates odd-even. A slightly weaker result (arising from a stronger hypothesis) can be expressed in terms of colorings. If a set X has been colored with two colors—say red and blue—we call the coloring *balanced* if the number of reds differs by at most one from the number of blues.

COROLLARY 3.1: *Suppose that* $\{1, 2, \ldots, n\}$ *has been given a balanced coloring. If F is a family of subsets containing no comparable pairs of sets whose difference is balanced, then* $|F| \le \binom{n}{\left[\frac{n}{2}\right]}$.

Corollary 3.1 follows from our previous observation if we renumber the elements so that the coloring is represented by odds and evens.

Corollary 3.1 contrasts with another extension of Sperner's theorem obtained independently by Kleitman [35] and Katona [31].

THEOREM 3.2: *Suppose that the elements of* $\{1, 2, \ldots, n\}$ *have been given an arbitrary* 2*-coloring. Let F be a family of sets containing no comparable pair whose difference is monocolored. Then* $|F| \le \binom{n}{\left[\frac{n}{2}\right]}$.

The proof of Theorem 3.2 is as follows:

Suppose that the coloring partitions $\{1, 2, \ldots, n\}$ into two parts X and Y. We apply our partitioning procedure separately to the lattices B_X and B_Y, producing a collection of symmetric chains for each. Since B_n is isomorphic to the cartesian product $B_X \times B_Y$, we can partition B_n into "symmetric rectangles" by taking all possible products of pairs of chains, one from B_X and the other from B_Y. If F is a family which satisfies the conditions of Theorem 3.2, it is clear that no rectangle contains two members of F in the same row or column. It follows immediately that *in each rectangle* the number of elements of F is bounded by the number of elements of size $\left[\frac{n}{2}\right]$. Since the rectangles partition B_n, F can have at most $\binom{n}{\left[\frac{n}{2}\right]}$ members altogether.

We show next how the methods of this section can be used to derive an upper bound on the total number of antichains in B_n. This is a problem originally posed by Dedekind, and can be rephrased in several ways: for example, it is equivalent to finding a bound for the size of the free distributive lattice on n generators. The argument presented here is due to Hansel [61].

THEOREM 3.3: *The number of antichains in* B_n *is at most*

$$3^{\binom{n}{\left[\frac{n}{2}\right]}}.$$

Proof: Antichains in B_n are in one-to-one correspondence with monotone Boolean functions on B_n, that is, order preserving maps from B_n to the set $\{0, 1\}$. We can construct such functions in the following way: take the chains in the deBruijn-Tengbergen-Kruyswijk partition, and beginning with the *smallest chains* first, assign values 0 and 1 to the members of each chain, consistent with the requirements of monotonicity. It can be shown easily that a chain always has at most two unassigned elements (which in fact must be adjacent), once values have been given to the members of all smaller chains. These two elements can be assigned values in at most three ways, and the bound follows immediately.

By a much more difficult argument (based on the same ideas) Kleitman [62] has improved this bound to

$$2^{\binom{n}{\left[\frac{n}{2}\right]}(1 + O((\log n)/n))}.$$

This result can be interpreted as saying that "most" antichains in B_n are obtained by taking collections of sets of size $\left[\frac{n}{2}\right]$.

The method of parenthesizations can be extended readily to lattices of multisets. To each element $(\sigma_1, \sigma_2, \ldots, \sigma_n) \in M_{\vec{e}}$ (where $\vec{e} = (e_1, e_2, \ldots, e_n)$) associate a sequence of Σe_i left and right parentheses, as follows: first σ_1 right parentheses, then $e_1 - \sigma_1$ left, then σ_2 right, $e_2 - \sigma_2$ left, and so forth. Now define two elements σ and τ to be equivalent if the corresponding sequences have the same basic parenthesization. Then all of the previous arguments carry over, and we obtain a partition of $M_{\vec{e}}$ into symmetric chains. This is equivalent to the partition obtained (inductively) by deBruijn, Tengbergen, and Kruyswijk [5].

It would be extremely interesting to obtain a similar explicit construction for lattices of subspaces, but none is known.

Our proof that $M_{\vec{e}}$ can be partitioned into symmetric chains actually proves that the analog of Erdös' theorem for k-families holds as well:

THEOREM 3.4: *If F is a family of multisets $\vec{\sigma} \in M_{\vec{e}}$ which contains no chains of length $k + 1$, then $|F|$ is bounded by the largest sum of k Whitney numbers $N_i(M_{\vec{e}})$.*

A similar result holds for every partially ordered set which can be partitioned into symmetric chains—that is, the maximum size of a k-family is equal to the sum of the k largest Whitney numbers. In such cases, the "largest k" will always be the "middle k" (which of course may not be unique).

We conclude this section by mentioning a result of Greene and Kleitman [23] which shows that—in principle—the maximum size of a k-family in any partially ordered set can always be computed by looking at the right partitions of P into chains.

THEOREM 3.5: *Let P be an arbitrary partially ordered set. Then the maximum size of a k-family in P is equal to the minimum, over all partitions $\mathcal{C} = C_1, C_2, \ldots, C_q$ of P into chains C_i, of the expression*

$$\sum_{i=1}^{q} \min \{|C_i|, k\}.$$

This result reduces to Dilworth's theorem if k = 1.

4. THE LYM PROPERTY

We begin this section with a third proof of Sperner's theorem, due independently to Lubell [48], Yamamoto [57], and Meschalkin [50]. This method is much more powerful than those discussed in earlier sections, and a large number of generalizations and extensions are possible. In contrast to previous methods this approach applies simultaneously to lattices of sets, multisets, and subspaces (but again not to partitions).

A *maximal chain* in B_n is a sequence of sets $\Phi = X_0 \subset X_1 \subset \cdots \subset X_n$, where $|X_i| = i$ for each i. There are exactly $n!$ maximal chains in B_n, and exactly $k!(n-k)!$ pass through a given set S of size k. If F is an antichain in B_n, then each maximal chain contains at most one member of F, and so

$$\sum_{S \in F} |S|!(n - |S|)! \le n!.$$

If we denote by f_k the number of sets in F of size k, this inequality becomes

$$(*) \qquad \sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \le 1.$$

But $\binom{n}{k} \le \binom{n}{\left[\frac{n}{2}\right]}$ for all k, and it follows immediately that

$$|F| = \sum_{k=0}^{n} f_k \le \binom{n}{\left[\frac{n}{2}\right]}$$

The inequality (*) is actually a stronger statement than Sperner's theorem. We refer to (*) as the *Lubell-Yamamoto-Meschalkin* (LYM) *inequality*. If a partially ordered set P has a rank function, and if

$$\sum_{x \in F} \frac{1}{N_x} = \sum_{k} \frac{f_k}{N_k} \le 1$$

holds for every antichain $F \subseteq P$, we say that P has the *LYM Property*. Clearly the LYM property always implies the Sperner property.

The above derivation of the LYM inequality uses only the following property of sets: for each set S of size k, the number of maximal chains which pass through S depends only on k and not on S. Hence by a similar argument, we obtain an important sufficient condition for a partially ordered set to have the LYM property.

THEOREM 4.1: *If P is a partially ordered set with a rank function, and if each element of rank k in P is contained in the same number of maximal chains of P (for all k), then P has the LYM property.*

The next observation is due to Baker [2].

COROLLARY 4.2: *If P is "regular" in the sense that for every k, each element of rank k covers the same number of elements of rank $k - 1$ (and dually), then P has the* LYM *property.*

To prove Corollary 4.2, observe that the number of maximal chains through a given element is completely determined by the "covering numbers", and hence depends only on rank.

In section 2 we proved that the Sperner property holds if the automorphisms of P are transitive on elements of fixed rank. Corollary 4.2 shows that a stronger conclusion (the LYM property) follows from a much weaker assumption (regularity).

COROLLARY 4.3: *For all n and q. $L_n(q)$ has the* LYM *property (and hence the Sperner property).*

The conditions of theorem 4.1 do not hold for lattices of multisets. However, it turns out that the LYM property remains true, for much deeper reasons to be discussed later (see Corollary 4.12).

Kleitman observed [37] that the hypotheses of Theorem 4.1 can be significantly weakened:

THEOREM 4.4: *Let P be a partially ordered set with a rank function. Suppose that there exists a nonempty collection $\mathcal{C}$ of maximal chains in P (not necessarily distinct) such that, for each k, every element of rank k occurs in the same number of chains in $\mathcal{C}$. Then P has the* LYM *property.*

If the conditions of Theorem 4.1 hold, we can take $\mathcal{C}$ to be the collection of all maximal chains in P. More generally the argument works as before: each element of rank k must occur in exactly $|\mathcal{C}|/N_k$ members of $\mathcal{C}$. Hence if F is an antichain, we have

$$\Sigma f_k \frac{|\mathcal{C}|}{N_k} \leq |\mathcal{C}|$$

and the LYM inequality follows.

A collection of chains $\mathcal{C}$ which satisfies the hypotheses of Theorem 4.4 will be called a *regular covering of P by chains.* Surprisingly, the existence of such coverings turns out to be equivalent to the LYM property. In fact, both are equivalent to a third hypothesis introduced independently by Graham and Harper [22], called the *normalized matching property.* This property can be described as follows:

If A is any subset of elements of rank k in P, let A^* denote the set of elements of rank $k + 1$ which are related to some element of A. If

$$\frac{|A|}{N_k} \leq \frac{|A^*|}{N_{k+1}}$$

for every k and every such A, then P is said to have the normalized matching property.

For lattices of sets, the statement of normalized matching property is equivalent to the following elementary but useful lemma (apparently due originally to Sperner [56]):

LEMMA 4.5: *Let A be a collection of k-subsets of $\{1, 2, \ldots, n\}$ and let A^* be the collection of $(k+1)$-subsets which contain members of A. Then*

$$|A| \leq \frac{k+1}{n-k} |A^*|.$$

Kleitman [37] proved the following:

THEOREM 4.6: *For a partially ordered set P with a rank function the following conditions are equivalent*:

(1) *P has the* LYM *property.*
(2) *P has the normalized matching property.*
(3) *There exists a regular covering of P by chains.*

The implication (3) → (1) has already been observed (Theorem 4.4). We complete the proof of Theorem 4.6 in two steps:

(1) → (2). Suppose that P has the LYM property, and A is a set of elements of rank k. Let P_{k+1} denote the elements in P of rank $k+1$. Then $A \cup (P_{k+1} - A^*)$ is an antichain of P, so that by the LYM inequality we have

$$\frac{|A|}{N_k} + \frac{|P_{k+1} - A^*|}{N_{k+1}} \leq 1.$$

But this inequality trivially implies

$$\frac{|A|}{N_k} \leq \frac{|A^*|}{N_{k+1}}.$$

(2) → (3) Assume that the normalized matching property holds, and define $M = \Pi_i N_i$. Define a new partially ordered set P' as follows: for each i, take M/N_i copies of the elements of rank i in P, with each copy of x less than each copy of y if $x < y$ in P. It is trivial to show that the normalized matching property for P implies

the existence of *ordinary* matchings between successive ranks of P', using P. Hall's matching theorem [26]. Putting these matchings together, we obtain a collection of M maximal chains in P which cover each element of rank k exactly M/N_k times. This completes the proof.

If P has the LYM property, there is an important analog of the LYM inequality which holds for *all* subfamilies of P, and permits arbitrary weighting of the elements of P.

THEOREM 4.7: *Let P be a partially ordered set with the* LYM *property, and let λ be a real-valued function defined on P. For any subset $G \subseteq P$*

$$\sum_{x \in G} \frac{\lambda_x}{N_x} \leq \max_{C \in \mathcal{C}} \sum_{y \in C \cap G} \lambda_y.$$

Here $\mathcal{C}$ denotes a regular covering of P by chains. The proof can be expressed easily in probabilistic language, as follows: choose a chain C at random from $\mathcal{C}$ and record the sum $\sum_{y \in C \cap G} \lambda_y$. This defines a random variable on $\mathcal{C}$ whose expected value and maximum value are given, respectively, by the left and right hand sides of the inequality in Theorem 4.7. This proves the theorem.

By taking $\lambda_x = N_x$ in Theorem 4.7, we obtain the following corollary:

COROLLARY 4.8: *If P has the* LYM *property, and G is any subset of P, then*

$$|G| \leq \max_{C \in \mathcal{C}} \{\sum_{x \in C \cap G} N_x\}.$$

We mention three examples of applications of Corollary 4.8, to give the reader some idea of its usefulness. The results are stated for lattices of sets, but the obvious analogs hold for any partially ordered set with the LYM property.

THEOREM 4.9: *Let G be a family of subsets of $\{1, 2, \ldots, n\}$.*

(i) (Erdös [15].) *If G contains no chains of length $k+1$, then $|G|$ is at most the sum of the k largest binomial coefficients* $\binom{n}{i}$.

(ii) (Erdös [15].) *If G contains no two members $A \supseteq B$ with $|A - B| \geq k$, then $|G|$ is at most the sum of the k largest binomial coefficients* $\binom{n}{i}$.

(iii) (Katona [32].) *If G contains no two members $A \supseteq B$ with $|A - B| < k$, then $|G|$ is bounded by the largest sum of the form*

$$\sum_i \binom{n}{a + ki}.$$

Many similar results can be obtained by considering other restrictions on subfamilies. A bound of the form

$$\sum_{k \in S} N_k(P)$$

for some set S of indices can always be obtained in this way, although it will not always be the best possible bound. A best bound is obtained only when the corresponding union of "levels" obeys the restriction in question.

When P does not possess symmetry (or regularity), it is usually difficult to tell whether or not the LYM property holds. However, under certain conditions it can be shown that the LYM property is preserved under direct products. The following result was obtained first by Harper [27] and independently by Hsieh and Kleitman [29]:

THEOREM 4.10: *Let P_1 and P_2 be partially ordered sets, such that*

(i) *both P_1 and P_2 have the* LYM *property, and*

(ii) *the Whitney numbers of P_1 and P_2 are logarithmically concave (i.e., $N_k^2 \geq N_{k-1} N_{k+1}$ for all k).*

Then both (i) and (ii) hold in the cartesian product $P_1 \times P_2$.

In fact, condition (ii) is preserved under products, independently of (i). Note that the Whitney numbers of $P_1 \times P_2$ are obtained from those of P_1 and P_2 by convolution. That is,

$$N_k(P_1 \times P_2) = \sum_{i+j=k} N_i(P_1) N_j(P_2).$$

LEMMA 4.11: *The convolution of two logarithmically concave sequences is logarithmically concave.*

(The proof is not difficult and is omitted; see [27] or [29].)

We proceed now to the proof of Theorem 4.10. The first step is based on an idea which can be regarded as an "LYM-analog" of Theorem 3.2. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be regular coverings of P_1 and P_2 by chains. By taking all possible products $C_1 \times C_2$ of chains $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$, we can cover $P_1 \times P_2$ by "rectangles", and this covering is "regular" in the sense that the number of rectangles containing an element $(x, y) \in P_1 \times P_2$ depends only on $r(x)$ and $r(y)$. By the same argument as that used to prove Theorem 4.7, we obtain the following inequality, for any subset $G \subseteq P_1 \times P_2$ and any weight function λ:

$$\sum_{\substack{x \in G \\ x=(x_1,x_2)}} \frac{\lambda_x}{N_{x_1} N_{x_2}} \leq \max_{\substack{C_1 \in \mathcal{C}_1 \\ C_2 \in \mathcal{C}_2}} \Big\{ \sum_{\substack{y \in G \\ y \in C_1 \times C_2}} \lambda_y \Big\}.$$

Now suppose that G is an antichain of $P_1 \times P_2$. We wish to prove that the LYM inequality holds for G. For each $x = (x_1, x_2) \in P_1 \times P_2$, take

$$\lambda_x = \frac{N_{x_1} N_{x_2}}{N_x}$$

in the previous inequality. Then

$$\sum_{x \in G} \frac{1}{N_x} \leq 1$$

follows if we can show that

$$\sum_{y \in G \cap (C_1 \times C_2)} \frac{N_{y_1} N_{y_2}}{N_y} \leq 1$$

for every pair of chains $C_1 \in \mathcal{C}_1$, $C_2 \in \mathcal{C}_2$. This inequality can be proved by showing that the elements of smallest rank in $G \cap (C_1 \times C_2)$ can be shifted up one level, without decreasing the sum on the left. Hence repeating this operation allows us to concentrate the elements at a single level, where the inequality is trivial. The contribution of the elements of smallest rank in $G \cap (C_1 \times C_2)$ will consist of "connected" blocks of the form

$$\frac{1}{N_k(P_1 \times P_2)} \sum_{i=a}^{b} N_i(P_1)\, N_{k-i}(P_2).$$

It suffices to show that raising each of these blocks separately does not decrease the sum, since the blocks do not interact with each other. Writing

$$S_k[a, b] = \sum_{i=a}^{b} N_i(P_1)\, N_{k-i}(P_2),$$

we must prove that

$$\frac{S_k[a, b]}{S_k[0, \infty]} \leq \frac{S_{k+1}[a, b+1]}{S_{k+1}[0, \infty]}.$$

But this can be proved by writing each side as a telescoping product and verifying the inequality

$$\frac{S_k[u, v]}{S_k[u, v+1]} \leq \frac{S_{k+1}[u, v+1]}{S_{k+1}[u, v+2]},$$

which is a straightforward consequence of the logarithmic concavity of the N_i's. (We omit the details.) This completes the proof.

The final steps in the above argument amount to proving a form of the LYM inequality for certain "weighted" sums over rectangles. More general weighted inequalities of this type have been obtained in [27], [37], and [46].

COROLLARY 4.12: *Every finite lattice of multisets (or lattice of divisors of an integer) has the* LYM *property.*

Thus, of the four classes of lattices mentioned in section 1, the first three (sets, multisets, and subspaces) have the LYM property. On the other hand, Spencer [54] has shown that for sufficiently large n, the lattice of partitions of $\{1, 2, \ldots, n\}$ does *not* have the LYM property. However, many consequences of the LYM property remain open (and important) questions.

5. GENERALIZED LYM INEQUALITIES

In this section, we will consider several other problems which can be solved by variations of the LYM approach. We will use more general systems of sets instead of maximal chains to derive upper bounds on the size of "antisystems", using arguments similar to those described in the last section. If there is suitable symmetry, one can always obtain inequalities of the LYM type.

An *ordered set-system on* $\{1, 2, \ldots, n\}$ is a sequence of sets $\vec{\alpha} = (A_1, A_2, \ldots, A_p)$, where each A_i is a subset of $\{1, 2, \ldots, n\}$. Let $\mathcal{S}(\vec{\alpha})$ denote the collection of all ordered set systems $\vec{\beta}$ which can be obtained from $\vec{\alpha}$ by permuting the elements $1, 2, \ldots, n$. The arguments in this section are all based on the following general result:

THEOREM 5.1: *Let* $\vec{\alpha} = (A_1, A_2, \ldots, A_p)$ *be an ordered set system on* $\{1, 2, \ldots, n\}$, *with* $|A_i| = \alpha_i$, $1 \leq i \leq p$. *Let F be a family of subsets of* $\{1, 2, \ldots, n\}$ *such that F has at most k members in common with each* $\vec{\beta} \in \mathcal{S}(\vec{\alpha})$. *Let* f_j *denote the number of sets in F of size j. Then*

$$\sum_{i=1}^{p} \frac{f_{\alpha_i}}{\binom{n}{\alpha_i}} \leq k.$$

The proof is by exactly the same argument as that used to prove Theorem 4.7.

If $\vec{\alpha}$ represents a maximal chain of sets in B_n (that is, if $\Phi = A_1 \subset A_2 \subset \cdots \subset A_{n+1}$) and F is an antichain, then Theorem 5.1 yields the ordinary LYM inequality for families of sets.

By similar reasoning, it is also possible to obtain an analog of Theorem 4.7, which permits weighting of sets and applies to arbitrary subfamilies $G \subseteq B_n$.

Corollary 5.2: *Let G be any family of subsets of* $\{1, 2, \ldots, n\}$, *and let* λ_i *be an arbitrary weight assigned to sets of size i* ($1 \le i \le n$). *If* $\vec{\alpha}$ *is any ordered set system then (with the notation defined above)*

$$\sum_{i=1}^{p} \frac{\lambda_{\alpha_i} g_{\alpha_i}}{\binom{n}{\alpha_i}} \le \max_{\beta \in \mathcal{S}(\vec{\alpha})} \left(\sum_{A \in G \cap \vec{\beta}} \lambda_{|A|} \right).$$

As a first application of Theorem 5.1, we will give a short proof of the following theorem due to Erdös, Ko, and Rado [17]. The proof is a slight modification of one originally given by Katona [33].

Theorem 5.3: *Let* $k \le n/2$ *and let F be a family of k-subsets of* $\{1, 2, \ldots, n\}$, *no two members of which are disjoint. Then* $|F| \le \binom{n-1}{k-1}$.

Proof: Let $\vec{\alpha} = (A_1, A_2, \ldots, A_n)$ be the set system obtained by arranging the numbers $1, 2, \ldots, n$ in a circle and taking the A_i's to be all consecutive segments of length k. We claim that no sequence $\vec{\beta} \in \mathcal{S}(\vec{\alpha})$ contains more than k members of F. By symmetry, this follows if we can prove that $\vec{\alpha}$ contains at most k members of F. But if A_i denotes the segment beginning with i (mod n), and F contains A_1, then F contains at most one set from each of the pairs $\{A_{i-k}, A_i\}$ for $i = 2, 3, \ldots, k$, and no others. Hence F has at most k sets in common with α. By Theorem 5.1,

$$\frac{nf_k}{\binom{n}{k}} = \frac{n\,|F|}{\binom{n}{k}} \le k$$

and the result follows.

An even easier argument, similar to the above, can be obtained in the special case when k divides n. Take $\vec{\alpha}$ to be any partition of $\{1, 2, \ldots, n\}$ into n/k blocks of size k so that $\mathcal{S}(\vec{\alpha})$ is the set of all partitions of this type. Trivially, each partition contains at most one member of F, and Theorem 5.1 yields the inequality

$$\frac{n}{k} \cdot \frac{|F|}{\binom{n}{k}} \le 1$$

from which the result follows as before.

In their original paper, Erdös, Ko, and Rado derived Theorem 5.3 from a more general hypothesis: the sets in F are assumed to have size *at most k*, and the same bound $\binom{n-1}{k-1}$ follows. By a slight change in the above proof, we can obtain this result as a consequence of an even stronger statement, which is an LYM inequality for "Erdös-Ko-Rado families".

Theorem 5.4: *Let F be an antichain of subsets of* $\{1, 2, \ldots, n\}$, *each of size at most n/2, such that no two members of F are disjoint. If* f_i *denotes the number of sets in F of size i, then*

$$\sum_{i=1}^{n/2} \frac{f_i}{\binom{n-1}{i-1}} \le 1.$$

Proof: Instead of taking $\vec{\alpha}$ to be the sequence of consecutive k-segments on a circle, let $\vec{\alpha}$ be the collection of consecutive segments of *all* lengths, in some arbitrary order. In addition, assign a weight $\lambda_j = 1/j$ to sets of size j, and apply Corollary 5.2. Since there are n segments of length α_j for all $\alpha_j > 0$, the left hand side of the inequality in Corollary 5.2 becomes

$$\sum_{j=1}^{n/2} \frac{n}{j} \frac{f_j}{\binom{n}{j}} = \sum_{j=1}^{n/2} \frac{f_j}{\binom{n-1}{j-1}} .$$

Thus to prove the result, we must prove that the right hand side of the inequality in Corollary 5.2 is at most 1. Equivalently, we must show that

$$\sum_{A \in F \cap \vec{\alpha}} \frac{1}{|A|} \leq 1.$$

But this can be proved as follows: suppose that the smallest consecutive segment occurring in $F \cap \vec{\alpha}$ has length k. Then, as in the proof of Theorem 5.3, $F \cap \vec{\alpha}$ has at most k members. Since $1/|A| \leq 1/k$ for all $A \in F \cap \vec{\alpha}$, the inequality follows.

The above argument is essentially due to Bollobas [3], but was discovered independently by Greene, Kleitman, and Katona [25]. Theorem 5.4 suggests that the parameters f_i corresponding to an Erdös-Ko-Rado family behave like those of a Sperner family on a set of size $n - 1$. In [25] the latter authors strengthened Theorem 5.4 by proving that the f_j's are completely characterized by this property. (See Theorem 8.12.)

Bollobas [3] used Theorem 5.4, to derive a result about a special class of antichains:

Corollary 5.5: *Let F be an antichain of subsets of* $\{1, 2, \ldots, n\}$ *such that for every $A \in F$, the complement of A is also in F. Then*

$$\sum_{A \in F} \frac{1}{\mu(A)} \leq 2,$$

where $\mu(A) = \min \left\{ \binom{n-1}{|A|-1} , \binom{n-1}{n-|A|-1} \right\}$.

To prove Corollary 5.5, observe that an antichain F satisfies the given hypothesis if and only if it is obtained by taking the union of an Erdös-Ko-Rado family (whose members all have size $\leq n/2$) and the family of its complements. The inequality now follows from two applications of Theorem 5.4.

Bollobas [3] also observed that Corollary 5.5 can be used to prove the following result due independently to Kleitman and Spencer [44] and Schönheim [53]:

Corollary 5.6: *Let F be an antichain of subsets of* $\{1, 2, \ldots, n\}$ *such that for all $A, B \in F$, $A \cap B \neq \emptyset$ and $A \cup B \neq \{1, 2, \ldots, n\}$. Then*

$$|F| \leq \binom{n-1}{\left\lceil \frac{n}{2} \right\rceil - 1} .$$

To prove Corollary 5.6, let $\bar{F}$ denote the family of all sets which are complements of sets in F. The given conditions imply that $F \cup \bar{F}$ is an antichain, so we may apply Corollary 5.5. Since $\mu(A) \leq \binom{n-1}{\left\lceil \frac{n}{2} \right\rceil - 1}$ for every set A, it follows that

$$\frac{2\,|F|}{\binom{n-1}{\left\lceil \frac{n}{2} \right\rceil - 1}} \leq \sum_{A \in F \cup \bar{F}} \frac{l}{\mu(A)} \leq 2.$$

The inequality in Corollary 5.6 is best possible, since we can construct an extremal family satisfying the desired conditions by taking F to be all $[n/2]$-sets containing a fixed element.

Hsieh [28] proved that an analog of the Erdös-Ko-Rado theorem holds for subspaces of a finite vector space, except that his proof works only when $k < n/2$, leaving the case $k = n/2$ unsettled. We shall show below that Katona's proof can be modified to cover this case. (In fact, it works whenever k divides n.) Combining these results, we can state

Theorem 5.7: *Let F be a family of k-dimensional subspaces of* *a vector space of dimension n over* GF(q), *with* $k \le n/2$. *If no two members of F have intersection* $\{0\}$, *then* $|F| \le \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$.

We will give the proof only for the case $k \mid n$.

LEMMA 5.8: *Let V be an n-dimensional vector space over* GF(q), *and let k be an integer which divides n. Then* $V - \{0\}$ *can be partitioned into sets of the form* $K - \{0\}$, *where K denotes a k-dimensional subspace of V.*

To prove Lemma 5.8 take V to be the finite field of order q^n. If K_0 is the subfield of order q^k, then the cosets of $K_0 - \{0\}$ in the multiplicative group of V have the desired property.

To prove Theorem 5.7 when k divides n, we take a fixed partition of V as guaranteed by Lemma 5.8, and "symmetrize" by taking all partitions obtainable from the first by linear transformations. Then the proof of Theorem 5.3 (or more precisely, the remark following it) remains valid and Theorem 5.7 follows immediately.

We have not been able to show that the analog of a "k-fold covering" exists when $k \nmid n$, which seems to be more difficult for vector spaces than it is for sets. (In this case what is needed is a list of $(q^n - 1)$ k-subspaces which meets an intersecting family at most $q^k - 1$ times.) If such coverings could be constructed another proof of Hsieh's theorem would follow.

As our second example, we will prove the following theorem due to Kleitman [36]:

THEOREM 5.9: *Let* $n = 3k + 1$, *and let G be a family of subsets of* $\{1, 2, \ldots, n\}$ *which contains no two disjoint sets and their union. Then*

$$|G| \le \sum_{i=k+1}^{2k+1} \binom{n}{i}.$$

Thus a family of maximum size is obtained by taking G to be the collection of all subsets A with $k + 1 \le |A| \le 2k + 1$. The proof rests on the following lemma:

LEMMA 5.10: *Let* $\bar{g}_i$ *denote the number of i-sets which are not in G, and suppose* $a + b + c = n$. *Then*

$$\frac{\bar{g}_a}{\binom{n}{a}} + \frac{\bar{g}_b}{\binom{n}{b}} + \frac{\bar{g}_c}{\binom{n}{c}} + \frac{\bar{g}_{a+b}}{\binom{n}{a+b}} + \frac{\bar{g}_{a+c}}{\binom{n}{a+c}} + \frac{\bar{g}_{b+c}}{\binom{n}{b+c}} \ge 2.$$

To prove the Lemma, let $\vec{\alpha}$ be the ordered set system $(A, B, C, A \cup B, A \cup C, B \cup C)$, where $A \cup B \cup C$ is a partition of $\{1, 2, \ldots, n\}$ and $|A| = a$, $|B| = b$, and $|C| = c$. It follows from the conditions on G that each sequence $\vec{\beta} \in \mathcal{S}(\vec{\alpha})$ contains at least two sets which are *not* in G. The lemma follows immediately from Theorem 5.1.

To prove Theorem 5.9, suppose that $a \le b \le c$ and $a + b + c = n$. By Lemma 5.10,

$$\bar{g}_a + \bar{g}_{n-a} + \frac{\binom{n}{a}}{\binom{n}{b}}(\bar{g}_b + \bar{g}_{n-b}) + \frac{\binom{n}{a}}{\binom{n}{c}}(\bar{g}_c + \bar{g}_{n-c}) \ge 2\binom{n}{a}.$$

For each $a < k$, choose $b = \left\lceil \frac{n-a}{2} \right\rceil - 1$ and $c = n - a - b$. Then if we sum these inequalities for all $a < k$, and add the single inequality

$$\bar{g}_k + \bar{g}_{n-k} + \frac{1}{2}\frac{\binom{n}{k}}{\binom{n}{k+1}}(\bar{g}_{k+1} + \bar{g}_{n-k-1}) \ge \binom{n}{k},$$

it can be shown by a simple calculation that the left hand side is at most

$$\sum_{i=0}^{n} \bar{g}_i = 2^n - |G|$$

while the right hand side is exactly

$$\sum_{i \in [k+1, 2k+1]} \binom{n}{i}.$$

Hence $|G| \le \sum_{i=k+1}^{2k+1} \binom{n}{i}$ as desired.

Modifications in the above argument can be made to obtain analogous results for collections of subspaces. We omit the details.

6. LINEAR PROGRAMMING TECHNIQUES

In the last two sections, we have described techniques for obtaining certain kinds of linear inequalities which antichains and other families must obey. While these inequalities often have immediate consequences (e.g., Sperner's Theorem, the Erdös-Ko-Rado Theorem), it is sometimes possible to obtain deeper results by using the techniques of linear programming. In this section, we will explore several examples which illustrate this approach.

We begin by proving two closely related theorems, the first due to Kleitman [60], and the second due to Kleitman and Milner [59].

THEOREM 6.1: *Let F be an antichain in B_n, with $|F| \ge \binom{n}{k}$, $k \le n/2$. If F^* denotes the order ideal generated by F (i.e., the family of sets contained in some member of F), then*

(i) $$|F^*| \ge \sum_{i \le k} \binom{n}{i}, \qquad \text{(Kleitman [60])}$$

(ii) $$\frac{1}{|F|} \sum_{A \in F} |A| \ge k. \qquad \text{(Kleitman, Milner [59])}$$

In other words, if $|F| \ge \binom{n}{k}$, then both the size of F^* and the average size of the members of F are at least as big as they would be if F were concentrated at level k.

The method of proof actually applies to any partially ordered set P with the LYM property whose Whitney numbers are logarithmically concave. We shall state and prove the result in this form:

THEOREM 6.2: *Let P be a partially ordered set with the LYM property, such that $(N_i(P))^2 \ge N_{i-1}(P)N_{i+1}(P)$ for each i. Let k be any integer such that $k \le m$, where m denotes the index i for which $N_i(P)$ is maximum. Let $F \subseteq P$ be an antichain satisfying $|F| \ge N_k(P)$, and let F^* denote the order ideal generated by F. Then*

(i) $$|F^*| \ge \sum_{i \le k} N_i(P),$$

(ii) $$\frac{1}{|F|} \sum_{x \in F} r(x) \ge k.$$

Proof: For each i, let $x_i = f_i/N_i$, where f_i denotes the number of elements in F having rank i. Since P has the LYM property, we have

$$\Sigma x_i \le 1.$$

The assumption $|F| \ge N_k$ gives a second linear constraint:

$$\Sigma x_i N_i \ge N_k.$$

Since P has the LYM property, we can assume that $x_i = 0$ for $i > m$, since otherwise the top levels of F can be lowered without reducing $|F|$ (using normalized matching).

To prove (i), we will relate the problem of minimizing $|F^*|$ to the problem (not equivalent, but sufficient for present purposes) of minimizing a certain linear function. To this end, let f_j^* denote the number of elements in F^* of rank i, for each j, and let $x_i^* = f_i^*/N_i$. Observe that for each i we can construct an antichain in P by taking all elements in F of rank $i \geq j$, together with all elements of rank j which are *not* in F^*. Hence, by the LYM inequality,

$$(1 - x_j^*) + \sum_{i \geq j} x_i \leq 1$$

which implies

$$\sum_{i \geq j} x_i \leq x_j^*$$

for each j. Since $|F^*| = \Sigma\, x_j^* N_j$, it follows that

$$|F^*| \geq \sum_j N_j \sum_{i \geq j} x_i = \sum_i x_i \{\sum_{j \leq i} N_j\}.$$

Now consider the linear program

$$\sum_{i=0}^{m} - x_i \geq - 1$$

$$\sum_{i=0}^{m} x_i N_i \geq N_k, \quad x_i \geq 0;$$

$$\textit{minimize}\colon \sum_{i=0}^{m} x_i \overline{N_i},$$

where $\overline{N_i} = \sum_{j \leq i} N_j$. The dual program is:

$$- \alpha + \beta N_i \leq \overline{N_i}, \quad i = 0, 1, 2, \ldots, n,$$
$$\alpha, \beta \geq 0;$$
$$\textit{maximize}\colon - \alpha + \beta N_k.$$

In order to prove (i), it is sufficient to find values of α and β, satisfying the conditions of the dual program, such that $- \alpha + \beta N_k = \overline{N_k}$. By duality, $\overline{N_k}$ must then be a lower bound for the original problem. To find such values, we set $- \alpha + \beta N_k = \overline{N_k}$ and eliminate α, which leads to the conditions

$$\beta N_k \geq \overline{N_k}, \ \beta(N_i - N_k) \leq \overline{N_i} - \overline{N_k},$$

for $i = 0, 1, \ldots, n$. This is equivalent to the conditions

$$\frac{\overline{N_k} - \overline{N_j}}{N_k - N_j} \leq \beta \leq \frac{\overline{N_i} - \overline{N_k}}{N_i - N_k}$$

for all i and j such that $- 1 \leq j < k < i \leq m$. Using the logarithmic concavity of the N_i's, one can easily show that

$$\frac{\overline{N_k} - \overline{N_j}}{N_k - N_j} \leq \frac{N_k}{N_k - N_{k-1}} \leq \frac{N_{k+1}}{N_{k+1} - N_k} \leq \frac{\overline{N_i} - \overline{N_k}}{N_i - N_k},$$

which proves that suitable values for α and β exist. This completes the proof of part (i).*

To prove part (ii), we can assume that $|F| = N_k$ (throwing away the highest elements if necessary). Then the average rank of members of F is bounded from below by the solution to the linear program

$$\sum_{i=0}^{m} - x_i \geq - 1,$$

$$\sum_{i=0}^{m} N_i x_i \geq N_k, \quad x_i \geq 0;$$

$$\textit{minimize}\colon \sum_{i=0}^{m} \frac{iN_i}{N_k} x_i.$$

*This argument was communicated to the authors by A. Odlyzko.

The dual program is

$$-\alpha + \beta N_i \le \frac{iN_i}{N_k},\ i = 0, 1, \ldots, m,$$

$$\alpha, \beta, \ge 0;$$

$$\textit{maximize}: -\alpha + \beta N_k.$$

As before, we seek to find values of α and β satisfying the dual constraints, such that $-\alpha + \beta N_k = k$. Solving and eliminating α leads to the conditions

$$\beta N_k \ge k, \quad \beta(N_i - N_k) \le \frac{iN_i}{N_k} - k,$$

for $i = 0, 1, \ldots, m$. These conditions can be satisfied if and only if

$$\frac{kN_k - jN_j}{N_k - N_j} \le \frac{iN_i - kN_k}{N_i - N_k}$$

for all j, k such that $j < k < i \le m$, which in turn can be written as

$$\frac{1}{N_k} \le \frac{(i-k)}{(i-j)}\frac{1}{N_j} + \frac{(k-j)}{(i-j)}\frac{1}{N_i}$$

for $j < k < i \le m$. The proof of part (ii) can now be completed by the application of the following easy lemma, whose proof we omit:

LEMMA 6.3: *If $\alpha_0, \alpha_1, \alpha_2, \ldots,$ is any sequence which is logarithmically concave, then the sequence $\alpha_0^{-1}, \alpha_1^{-1}, \alpha_2^{-1}, \ldots,$ is convex (in the usual sense).*

As our last example, we will prove the following theorem due to Kleitman [41]:

THEOREM 6.4: *Let F be a family of sets in B_n which contains no three distinct members A, B, C, with $A \cup B = C$. Then*

$$|F| \le \binom{n}{\left[\frac{n}{2}\right]}\left(1 + \frac{1}{n^{1/2}}\right).$$

Proof: Let f_k denote the number of elements in F of rank k. Fix an integer $j \le n/2$ and consider the collection $\mathcal{C}_{(j)}$ of maximal chains between rank j and rank $2j$. Let S be a member of F of rank $k \ge j$. We claim the following: among all of the chains in $\mathcal{C}_{(j)}$ passing through S, at least a proportion j/k contain no smaller member of F. This follows immediately from the Erdös-Ko-Rado Theorem (viewed upside down): if we add all of the rank j elements below S which are dominated by members of F, the proportion of chains which meet smaller members of F can only increase. But these sets form a (dual) Erdös-Ko-Rado family of j-subsets of S, whose size must be bounded by $\binom{k-1}{k-j-1}$. Hence the proportion of chains through S which meet smaller members of F must be at least

$$\binom{k-1}{k-j-1}\Big/\binom{k}{j} = \frac{k-j}{k}$$

as desired. If we count, for each $S \in F$, the number of chains through S which contain no smaller elements of F, the result is clearly bounded by the total number of chains in $\mathcal{C}_{(j)}$. This implies

$$\sum_{k=j}^{2j} \frac{f_k}{\binom{n}{k}} \frac{j}{k} \le 1$$

for each $j = 1, 2, \ldots, [n/2]$.

Next consider the linear program whose object is to maximize the linear function

$$\sum_{k=0}^{n} f_k$$

subject to these constraints, together with the conditions $f_k \geq 0$, $k = 0, 1, \ldots, n$. The dual program is to maximize

$$\sum_{k=1}^{[n/2]} \alpha_k$$

subject to the constraints

$$\sum_{k=j/2}^{\min\{j, [\frac{n}{2}]\}} \frac{k\,\alpha_k}{j\binom{n}{j}} \geq 1,$$

$j = 1, 2, \ldots, n$, with $\alpha_k \geq 0$, $k = 1, 2, \ldots, \left[\frac{n}{2}\right]$.

The values

$$\alpha_{2k+1} = \binom{n}{2k+1} - \binom{n}{2k} + \frac{1}{2k+1}\binom{n}{2k} + \alpha_k.$$

$$\alpha_{2k} = \binom{n}{2k} - \binom{n}{2k-1} + \frac{1}{2k}\binom{n}{2k}.$$

satisfy the dual constraints and yield as objective function

$$\binom{n}{[\frac{n}{2}]} + \frac{2^n}{n+1} + o\left(\binom{n}{[\frac{n}{4}]}\right)$$

which implies that the latter is an upper bound for the solution to the original problem. This bound is asymptotic to the one stated in Theorem 6.4.

We note that this bound is only asymptotic and may not be best possible. One can construct examples which satisfy the conditions of the theorem and attain the value $\binom{n}{[\frac{n}{2}]}\left(1 + \frac{1}{n}\right)$ in the following way: let $n = 2^\alpha - 1$, and consider the Hamming code H_α of order α. We can think of H_α as a family of n-sets with the property that every pair of sets differ in at least three places. Moreover each of the cosets of H_α can also be thought of as a family with this property. Since $|H_\alpha| = 2^{2^\alpha - \alpha - 1} \sim 2^n/n$, at least one of the cosets must contain a proportion $1/n$ of the $\left[\frac{n}{2}\right]$-sets. If we take these $\left[\frac{n}{2}\right]$-sets, together with *all* of the $\left[\frac{n}{2}\right]+1$-sets, we obtain a family of size at least $\binom{n}{[\frac{n}{2}]}\left(1 + \frac{1}{n}\right)$ with the desired properties.

7. INTERSECTIONS OF ORDER IDEALS

If P is any partially ordered set, an *order ideal* of P is a subset $K \subseteq P$ such that whenever $x \in K$ and $y \leq x$, then $y \in K$. If $P = B_n$, an order ideal is sometimes called a *simplicial complex.* This section is based on the following elementary but useful fact:

THEOREM 7.1: (Kleitman [38]) *Suppose that F and G are order ideals in B_n. Then*

$$\frac{|F|}{2^n} \cdot \frac{|G|}{2^n} \leq \frac{|F \cap G|}{2^n}.$$

In other words, the proportion of sets in $F \cap G$ is at least as large as the product of the proportions of sets in F and G.†

We can turn one of the order ideals upside down by taking complements and derive the following from Theorem 7.1 as an immediate corollary:

†Or in probabilistic language, if A is a set chosen at random from B_n, *then Prob* $\{A \in F\} \leq$ Prob $\{A \in F | A \in G\}$.

COROLLARY 7.2: *Suppose that F is an order ideal in B_n, and G is an order ideal in the dual of B_n. Then*

$$\frac{|F|}{2^n} \cdot \frac{|G|}{2^n} \geq \frac{|F \cap G|}{2^n}.$$

Before proving theorem 7.1, we mention a typical application: Suppose that $F \subseteq B_n$ is a family of sets with the property that no two members are disjoint (no restriction on rank or comparability is assumed). It is trivial to see that F has at most 2^{n-1} members, since no set can occur together with its complement. The same bound holds for families $G \subseteq B_n$ with the property that no two sets cover all of the points. Daykin and Lovasz [11] proved that a family which satisfies *both* of these conditions can have at most 2^{n-2} members:

THEOREM 7.3: *Let H be a family of subsets of $\{1, 2, \ldots, n\}$ with the property that for all $A, B \in H$, $A \cap B \neq \emptyset$ and $A \cup B \neq \{1, 2, \ldots, n\}$. Then $|H| \leq 2^{n-2}$.*

The upper bound is achieved by taking all subsets which miss one point and contain another.

We can derive Theorem 7.3 from Corollary 7.2 as follows: Let H satisfy the given conditions, and define F to be the order ideal generated by H, and G to be the dual order ideal generated by H. Then no two members of G are disjoint, and no two members of F have union equal to $\{1, 2, \ldots, n\}$. Hence by Corollary 7.2,

$$|H| \leq |F \cap G| \leq \frac{|F| \cdot |G|}{2^n} \leq 2^{n-2}.$$

The statement of Theorem 7.1 is also valid for divisors of any integer, and we shall give a proof of the result in this form:

THEOREM 7.4: *Let $N = \prod_{i=0}^{m} p_i^{e_i}$ be a positive integer, whose prime*

decomposition is as shown. Let $\delta = \prod_{i=0}^{m} (1 + e_i)$ denote the total number of divisors of N. If F and G are order ideals of divisors of N, then

$$\frac{|F|}{\delta} \cdot \frac{|G|}{\delta} \leq \frac{|F \cap G|}{\delta}.$$

The proof is by induction on m, the number of distinct prime divisors of N. Fix a prime p_0, and let F_i denote the subset of F consisting of those members in which p_0 occurs to the ith power $(0 \leq i \leq e_0)$. Define G_i and $(F \cap G)_i$ similarly. Let $\delta' = \prod_{i=1}^{m} (1 + e_i)$ denote the total number of divisors of $N/p_0^{e_0}$. If $0 \leq i \leq j \leq e_0$ it is not difficult to show that $|F_i|\,|G_j| \leq \delta' \cdot |(F \cap G)_i|$ (by removing all occurrences of p_0 and applying the inductive hypothesis). The result now follows immediately from the following lemma (we omit the details):

LEMMA 7.5: *Let $(x_0, x_1, \ldots, x_q)$, $(y_0, y_1, \ldots, y_q)$ and $(z_0, z_1, \ldots, z_q)$ be sequences of nonnegative real numbers, such that for every $i, j \leq q$, $x_i y_j \leq z_{\min(i,j)}$.* Then

$$(\Sigma x_i)(\Sigma y_j) \leq (q + 1)(\Sigma z_k).$$

Lemma 7.5 in turn can be derived from the following elementary fact, which is sometimes known as Chebyschev's inequality:

LEMMA 7.6: *Let $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_q$ and $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_q$ be sequences of nonnegative real numbers. Then*

$$(\Sigma \alpha_i)(\Sigma \beta_i) \leq (q + 1)(\Sigma \alpha_i \beta_i).$$

Surprisingly, the analog of Theorem 7.4 fails to hold for subspaces of a finite vector space. Consider a vector space V of dimension 4 over GF(q). Let F consist of all subspaces of dimension 0 and 1, together with half of the subspaces of dimension 2.

Let G consist of all subspaces of dimension 0 and 1, plus the other half of the 2-spaces. If δ denotes the total number of subspaces, then

$$\frac{|F|}{\delta} = \frac{|G|}{\delta} = \frac{1}{2}.$$

On the other hand, $|F \cap G| = 1 + (q^4 - 1)/(q - 1)$, while $\delta = 2\,(1 + (q^4 - 1)/(q - 1)) + (q^2 + 1)\,(q^2 + q + 1)$. Hence

$$\frac{|F \cap G|}{\delta} \to 0$$

as $q \to \infty$, which shows that the inequality in Theorem 7.4 cannot be valid.

We conclude this section by giving another application of Corollary 7.2.

THEOREM 7.7: (Kleitman [38]). *If $F_1, F_2, \ldots, F_k$ are disjoint collections of subsets of $\{1, 2, \ldots, n\}$ such that $A \cap B \neq \emptyset$ for all $A, B \in F_i$, $i = 1, 2, \ldots, k$, then*

$$|\bigcup_{i=1}^{k} F_i| \le 2^n - 2^{n-k}.$$

The bound is achieved by taking F_1 to be the collection of all sets which contain 1, F_2 to be sets which contain 2 but not 1, F_3 to be the sets which contain 3 but not 2 or 1, and so forth.

The proof of Theorem 7.7 is by induction on k. For $k = 1$, the result is trivial. Moreover, it is not hard to see that if F_i is a *maximal* family satisfying $A \cap B \neq \emptyset$ for all $A, B \in F_i$, then $|F_i| = 2^{n-1}$. If $k > 1$, let U denote the union of the families $F_1, F_2, \ldots, F_{k-1}$. Assuming that the union of all F_i's is as large as possible it follows that U must be a dual order ideal. By the inductive hypothesis, $|U| = 2^n - 2^{n-k+1}$. Extend F_k to a maximal family F_k' (satisfying the above condition with $k = 1$), and let L denote the collection of sets which are not in F_k'. Then L is an order ideal, and $|L| = 2^{n-1}$. We have

$$|\bigcup_{i=1}^{k} F_i| = |U \cup F_k| \le |U \cap L| + |F_k'|$$

$$\le \frac{|U|\,|L|}{2^n} + |F_k'| \quad \text{(by Corollary 7.2)}$$

$$\le 2^n - 2^{n-k}.$$

8. CANONICAL FORMS

One of the most versatile theorems of extremal set theory was proved by Kruskal [45] and later independently by Katona [34] (although related results were obtained much earlier by Macaulay [49]). The Kruskal-Katona theorem answers the following question:

If F is a family of k-sets, let ∂F denote the family of $(k - 1)$-sets which are subsets of members of F. How small can $|\partial F|$ be, given $|F|$?

A crude lower bound on $|\partial F|$ can be obtained from the trivial Lemma 4.5 proved earlier (which is essentially the normalized matching property for sets):

$$|\partial F| \ge \frac{k}{n - k + 1}\,|F|.$$

Kruskal and Katona obtained a much more precise statement, which gives a best possible lower bound on $|\partial F|$:

THEOREM 8.1: *Let F be a family of k-sets. If*

$$(**) \qquad |F| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i},$$

$$a_k > a_{k-1} > \cdots > a_i \ge i > 0,$$

then

$$|\partial F| \geq \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_i}{i-1}.$$

It is well known that every positive integer can be expressed uniquely in the form (**). This representation is known as the *k-binomial expansion* of an integer, and its existence can be proved easily from elementary properties of binomial coefficients (see [45]). To find such a representation of $|F|$, choose a_k to be as large as possible with $\binom{a_k}{k} \leq |F|$. Then subtract and repeat with $k-1$, $k-2$, and so forth.

We will defer a proof of Theorem 8.1 until the end of this section.

If the numbers associated with the Kruskal-Katona Theorem seem mysterious at first glance the following observation should help to explain their significance. Consider the set S_k of all infinite sequences $(x_0, x_1, \ldots)$ of zeros and ones, with exactly k ones in each sequence. We introduce a linear ordering on S_k by ordering sequences with respect to the *last* position in which they differ (reverse lexicographic ordering). Denote the elements of S_k by σ_0, $\sigma_1, \sigma_2, \ldots$.

LEMMA 8.2: *Let $\sigma_m = (x_0, x_1, \ldots)$ denote the m-th member of S_k, and suppose that σ_m has ones in positions with indices $a_1 < a_2 < \cdots < a_k$. Then*

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_1}{1}$$

(where by convention a term is zero if its numerator is less than its denominator).

For example, (0, 0, 1, 0, 1, 1, 0, ...,) is the 18th member of S_3, since $\binom{5}{3} + \binom{4}{2} + \binom{2}{1} = 18$. In general, the k-binomial expansion of a number provides a direct way of writing down the m^{th} sequence in S_k.

k-sequences in S_k can be associated with k-sets of positive integers in the obvious way, and we will regard these two notions as freely interchangeable. In the lexicographic ordering of S_k the initial segment of length m forms a "cascade" of sets (Kruskal [45]) which can be described as follows. If

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_1}{1},$$

then the first m sets in S_k consist of:

all k-sets in $[0, a_k - 1]$,

all k-sets formed by adding a_k to a $(k-1)$-set in $[0, a_{k-1} - 1]$,

all k-sets formed by adding $\{a_k, a_{k-1}\}$ to a $(k-2)$-set in $[0, a_{k-2} - 1]$,

.

.

.

and so forth.

The next lemma explains the operation of "lowering denominators" in a k-binomial expansion.

LEMMA 8.3: *Let $F = \{\sigma_0, \sigma_1, \ldots, \sigma_{m-1}\}$ be the initial segment of length m in S_k, and let ∂F denote the collection of $(k-1)$-sets which are subsets of members of F. If*

$$m = |F| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i},$$

then

$$|\partial F| = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_i}{i-1}.$$

The proof is an immediate consequence of Lemma 8.2. We can thus restate the Kruskal-Katona Theorem as follows:

For $F \subseteq S_k$, $|F| = m$, the cardinality of ∂F is minimized by taking F to be the first m sets in S_k, in lexicographic order.

Another useful reformulation of Theorem 8.1 is based on notation introduced by Clements and Lindstrom [7]: given a family F of k-sets, define the *compression of* F (denoted CF) to be the family consisting of the first $|F|$ sets in S_k. We say that a family of F is *compressed* if $CF = F$.

It is trivial to verify that if F is compressed then ∂F is also compressed. Using Lemma 8.3, we can interpret the Kruskal-Katona Theorem as a statement about how the operators ∂ and C commute: for any family F of k-sets,

$$\partial CF \subseteq C\partial F.$$

Equivalently, if $F \subseteq S_k$ and $G \subseteq S_{k-1}$ and $\partial F \subseteq G$, then $\partial CF \subseteq CG$.

If F is a family of sets of varying size, we can interpret the operators C and ∂ as acting on each rank of F separately. A family K is called a *simplicial complex* (or *order ideal of sets*) if $\partial K \subseteq K$. By the above remarks, the following is also equivalent to Theorem 8.1:

THEOREM 8.4: *If K is a finite simplicial complex then so is its compression CK.*

Hence we can regard CK as a *canonical form* for simplicial complexes having a specified number of faces of each size.

If K is any simplicial complex, we define the *f-sequence* of K to be the sequence $\vec{f}(K) = (f_0, f_1, f_2, \ldots,)$ where f_i denotes the number of i-sets (or $(i-1)$-faces) in K. Theorem 8.1 permits a complete characterization of those sequences of integers which arise as f-sequences. It is convenient to introduce the following notation: if

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i},$$
$$a_k > a_{k-1} > \cdots > a_i \geq i > 0,$$

define

$$\partial_k(m) = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_i}{i-1}.$$

THEOREM 8.5: *A sequence of integers $\vec{f} = (f_0, f_1, f_2, \ldots)$ is the f-sequence of some finite simplicial complex if and only if for each $k \geq 1$, $\partial_k(f_k) \leq f_{k-1}$.*

The proof is immediate: to construct a complex K with $\vec{f}(K) = (f_0, f_1, f_2, \ldots)$ take the first f_i sets of size i for each i. By Lemma 8.3, these sets form a simplicial complex.

By similar reasoning, it is possible to obtain canonical forms for *antichains* having a fixed number of sets of each size. If F is an antichain in B_n, let f_i again denote the number of i-sets in F, and define $\vec{f}(F) = (f_0, f_1, f_2, \ldots)$. The following theorem is due to Clements [6] and independently to Daykin, Godfrey, and Hilton [12]:

THEOREM 8.6: *Let $\vec{f} = (f_0, f_1, f_2, \ldots, f_n)$ be a sequence of nonnegative integers, and let k and l be the smallest and largest indices i for which $f_i \neq 0$. Then $\vec{f} = \vec{f}(F)$ for some antichain $F \subseteq B_n$ if and only if*

$$(***) \qquad f_k + \partial_{k+1}(f_{k+1} + \partial_{k+2}(f_{k+2} + \cdots + \partial_{l-1}(f_{l-1} + \partial_l(f_l))\cdots) \leq \binom{n}{k}.$$

Theorem 8.6 essentially states that $\vec{f} = \vec{f}(F)$ for some antichain $F \subseteq B_n$ if and only if it is possible to construct F in the following canonical way: take the first f_l l-sets in lexicographic order; then take the next f_{l-1} $(l-1)$-sets which are available (i.e., which are not subsets of sets already chosen); then the next f_{l-2} $(l-2)$-sets, and so forth.

To prove Theorem 8.6, observe that for any antichain F, $\vec{f}(F)$ must satisfy (***) since the left hand side represents the smallest possible number of k-sets which are contained in members of F. Conversely, if $\vec{f}$ satisfies (***), it is easy to see that the canonical construction described above can always be carried out.

By analogy with simplicial complexes, we define the *compression* CF of an antichain F to be the canonical antichain whose f-sequence is $\vec{f}(F)$.

Every antichain F determines a unique simplicial complex (denoted by $K(F)$) whose maximal elements are the members of F. ($K(F)$ is sometimes called the *order ideal generated by* F.) It follows immediately from the definitions that F is compressed (as an antichain) if and only if $K(F)$ is compressed (as a simplicial complex). Equivalently, F is compressed if and only if $F = K - \partial K$ for some compressed simplicial complex K.

We can restate these observations in the form of a slight generalization of the Kruskal-Katona Theorem:

COROLLARY 8.7: *Let F be an antichain of sets in B_n, with $\vec{f}(F) = (f_0, f_1, f_2, \ldots)$. Let k be the smallest index for which $f_k \neq 0$, and let $l \leq k$. Then F dominates the smallest possible number of l-sets (among all antichains in B_n with the same f-sequence) if F is compressed.*

The inequality (***) in Theorem 8.6 can be viewed as a refinement of the fundamental LYM inequality (Section 4), if we divide both sides by $\binom{n}{k}$. For example, if F consists only of sets of size k and $k + 1$, then (***) becomes

$$\frac{f_k}{\binom{n}{k}} + \frac{\partial_{k+1}(f_{k+1})}{\binom{n}{k}} \leq 1.$$

This is stronger than the LYM inequality since

$$\frac{\partial_{k+1}(f_{k+1})}{\binom{n}{k}} \geq \frac{f_{k+1}}{\binom{n}{k+1}},$$

as can be seen from Lemma 4.5 (the normalized matching property for sets).

Daykin [9] observed that the Kruskal-Katona Theorem can be used to give a short proof of the Erdös-Ko-Rado Theorem. We will discuss this argument next, as well as a number of extensions and refinements. In fact, we will prove a more general result, due to Kleitman [42], from which the Erdös-Ko-Rado Theorem follows immediately.

THEOREM 8.8: *Let F be a family of k-subsets of $\{1, 2, \ldots, n\}$, and let G be a family of l-subsets. Suppose that $k + l \leq n$, and that no member of F is disjoint from a member of G. If $|F| \geq \binom{n-1}{k-1}$ then $|G| \leq \binom{n-1}{l-1}$.*

To prove Theorem 8.8, let $\bar{F}$ denote the family of $(n - k)$-sets which are complements of members of F. Then $|\bar{F}| = |F|$, and the conditions on F and G imply that no member of $\bar{F}$ contains a member of G—that is, $\bar{F} \cup G$ is an antichain. By Theorem 8.6,

$$|G| + \partial_{l+1}(\partial_{l+2}(\cdots \partial_{n-k}(|F|)\cdots) \leq \binom{n}{l},$$

which implies $|G| \leq \binom{n}{l} - \binom{n-1}{l} = \binom{n-1}{l-1}$, since $|F| \geq \binom{n-1}{n-k}$.

We have actually proved more:

COROLLARY 8.9: *If F and G are as in the statement of Theorem 8.8, except that $|F|$ is arbitrary, then*

$$|G| \leq \binom{n}{l} - \partial_{l+1}(\partial_{l+2}(\cdots \partial_{n-k}(|F|)\cdots).$$

As a consequence of the proof of Theorem 8.8 we obtain the following corollary: If F is any collection of k-sets in B_n, let C^*F denote the *reverse compression of F in* B_n. That is, C^*F consists of the *last* $|F|$ sets in the lexicographic ordering of k-subsets of $\{1, 2, \ldots, n\}$. (Equivalently, $C^*F = \overline{C\overline{F}}$, where a bar denotes taking all sets which are complements of sets in the given family.)

COROLLARY 8.10: *If F and G are families of k-sets and l-sets which satisfy the conditions of Theorem 8.8, then* C^*F *and* C^*G *also satisfy these conditions.*

Thus the pairs C^*F, C^*G of families which are "reverse-compressed" form a set of *canonical forms* for pairs of families satisfying the conditions of Theorem 8.8. Not every pair of families which are reverse-compressed has the "pairwise nondisjointness property," however. For this is to be true, it is necessary that the inequality in Corollary 8.9 hold.

When $k = l$ and $F = G$, Theorem 8.8 reduces to the Erdös-Ko-Rado Theorem, and Corollary 8.10 can be restated as follows:

COROLLARY 8.11: *Let F be a family of k-sets in* B_n, $k \le n/2$, *which satisfies the conditions of the Erdös-Ko-Rado Theorem. Then* C^*F *also satisfies these conditions. In fact, this is true because every member of* C^*F *contains the element n.*

Using the above arguments, it is possible to obtain considerable refinements of the Erdös-Ko-Rado Theorem for arbitrary antichains (i.e., when the condition of uniform size is removed). In fact, the next theorem completely characterizes the "f-sequences" of antichains with the Erdös-Ko-Rado property, improving an earlier partial result of the "LYM" type (Theorem 5.4).

THEOREM 8.12: *Let* $\vec{f} = (f_0, f_1, f_2, \ldots)$ *with* $f_0 = 0$ *and* $f_i \neq 0$ *only if* $i \le n/2$. *Then there exists an antichain* $F \subseteq B_n$ *whose members are pairwise nondisjoint, with* $\vec{f}(F) = \vec{f}$ *if and only if there exists an antichain* $F' \subseteq B_{n-1}$ *with* $\vec{f}(F') = (f_1, f_2, f_3, \ldots)$.

In other words, the f_i's corresponding to Erdös-Ko-Rado families on a set of size n behave exactly like those of Sperner families on a set of size $n - 1$. Necessary and sufficient conditions for sequences of the latter type are provided by Corollary 8.6.

To prove Theorem 8.12, we argue as follows: suppose that $F \subseteq B_n$ is an antichain which satisfies the given conditions. Let $C^*F = \overline{C\overline{F}}$. Intuitively, C^*F is obtained by "reverse-compressing" each rank of F, beginning with the smallest ranks first. Clearly C^*F is an antichain with $\vec{f}(F) = \vec{f}(C^*F)$. The proof will be complete if we can show that each member of C^*F contains the element n, for then the required $F' \subseteq B_{n-1}$ can be obtained from C^*F by removing n from each set. Let l denote the largest index for which $f_l \neq 0$, and let $[F]^l$ denote the result of "projecting F up to rank l", i.e., $[F]^l$ consists of all l-sets which contain members of F. Similarly, define $[C^*F]^l$ to be the projection of C^*F up to rank l. Clearly $[F]^l$ is an Erdös-Ko-Rado family, since F itself is. Moreover, by Corollary 8.11 the reverse compression of $[F]^l$ has the property that each member contains n. But it is immediate from the definition of C^*F that $[C^*F]^l$ is reverse-compressed. Moreover $|[F]^l| \ge |[C^*F]^l|$ by Corollary 8.8 (applied upside down), and hence each member of $[C^*F]^l$ contains n. It follows easily that each member of C^*F contains n, and the proof is complete.

Reverse-compressed antichains, all of whose members contain the element n, can be thought of as canonical forms for Erdös-Ko-Rado families having a specified f-sequence.

Next we consider extensions of the Kruskal-Katona Theorem to lattices of multisets $M_{\vec{e}}$, when $\vec{e} = (e_0, e_1, \ldots)$, $e_i \in Z^+ \cup \{\infty\}$. We extend the notation introduced earlier: if $F \subseteq M_{\vec{e}}$, let ∂F denote the family of multisets of the form $(\sigma_0, \sigma_1, \ldots, \sigma_i - 1, \ldots)$, where $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_i, \ldots)$ is a member of F. Let S_k denote the set of elements of rank k in $M_{\vec{e}}$. Again order the elements of S_k in reverse lexicographic order (i.e., by the rightmost position in which they differ), and define the *compression* of a set $F \subseteq S_k$ (denoted CF) to be the initial segment of length $|F|$ in S_k.

Clements and Lindstrom [7] proved that for any family $F \subseteq S_k$, the identity

$$\partial CF \subseteq C\partial F$$

holds, provided that $e_0 \ge e_1 \ge e_2 \ge \cdots$. This identity leads to an analog of the Kruskal-Katona Theorem for lattices of multisets, which we restate as follows:

THEOREM 8.13: *Let F denote a family of multisets of rank k in* $M_{\vec{e}}$, *where* $\vec{e} = (e_0, e_1, \ldots)$ *and* $e_0 \ge e_1 \ge e_2 \ge \cdots$. *If F is fixed, then* $|\partial F|$ *is minimized by taking F to be compressed.*

Theorem 8.13 can be used to characterize the f-vectors of generalized "simplicial complexes" in lattices of the form $M_{\vec{e}}$. The characterization is analogous to that already obtained for sets (Theorem 8.5). The case $\vec{e} = (\infty, \infty, \infty, \ldots)$ of Theorem 8.13 is due to Macaulay [49].* (Another proof was given by Sperner [56].)

The lower bound on $|\partial F|$ can also be expressed in numerical form, using the notation introduced in section 1: one can show that $|F|$ has a unique decomposition

$$|F| = \binom{e_0, \ldots, e_{a_k}}{k} + \binom{e_0, \ldots, e_{a_{k-1}}}{k-1} + \cdots + \binom{e_0, \ldots, e_{a_i}}{i},$$

($i > 0$) where each term is nonzero, and $a_k \ge a_{k-1} \ge \cdots \ge a_i$ with no integer j repeated more than e_j times. The conclusion of Theorem 8.13 is that

$$|\partial F| \ge \binom{e_0, \ldots, e_{a_k}}{k-1} + \binom{e_0, \ldots, e_{a_{k-1}}}{k-2} + \cdots + \binom{e_0, \ldots, e_{a_i}}{i-1}.$$

If $\vec{e} = (1, 1, 1, \ldots)$, then $a_k > a_{k-1} > \cdots > a_i$, and these expressions reduce to the standard binomial decompositions. If $\vec{e} = (\infty, \infty, \infty, \ldots)$, there is no restriction on the a's, and we get a "negative binomial decomposition":

* Macaulay's object in studying this question was to obtain a characterization of *Hilbert functions* of certain kinds of modules. He considered R-modules of the form R/I, where polynomial ring in finitely many variables over a field, an I is a homogeneous ideal in R. A function $H: Z^+ \to Z^+$ is the Hilbert function of such a module if and only if there exists a generalized simplicial complex $F \subseteq M_{\vec{e}}$, where $\vec{e} = (\infty, \infty, \infty, \ldots)$ such that $\vec{f}(F) = (H(0), H(1), H(2), \ldots)$.

$$|F| = \left|\binom{-a_k}{k}\right| + \left|\binom{-a_{k-1}}{k-1}\right| + \cdots + \left|\binom{-a_i}{i}\right|$$

$$= \binom{a_k + k - 1}{k} + \binom{a_{k-1} + k - 2}{k-1} + \cdots + \binom{a_i + i - 1}{i}$$

with $a_k \ge a_{k-1} \ge \cdots \ge a_i > 0$.

Many of the results mentioned earlier for lattices of sets can be extended to lattices of multisets using Theorem 8.13. We mention one example, which extends the Erdös-Ko-Rado Theorem in a direction which is not obvious at first glance. Before stating the general result, we begin with a special case, which is most conveniently stated in integer-divisor form. Recall that the "rank" of an integer is the total number of prime factors which appear in it.

THEOREM 8.14: *Let* $N = \prod_{i=1}^{m} p_i^{e_i}$, *where* $e_0 \ge e_1 \ge \cdots \ge e_m$ *and* e_m *is odd. Let F be an antichain of divisors of N, each having rank* k (with $k \le \Sigma e_i/2$). *Suppose that for all a, b,* $\in F$, *ab does not divide N. Then*

$$|F| \le \binom{e_0, e_1, \ldots, e_m - \alpha}{k - \alpha}$$

where $\alpha = [e_m/2] + 1$.

Theorem 8.14 states that $|F|$ is maximized by taking all rank k divisors of n which have p_m^{α} as a factor, provided that e_m is odd. If n is square free (i.e., $e_0 = e_1 = \cdots = e_m = 1$) this result is equivalent to the Erdös-Ko-Rado Theorem since $ab \mid n$ if and only if the sets of primes occurring in a and b are disjoint.

If e_m is even, the statement must be modified as follows:

THEOREM 8.15: *Let N and F be as in Theorem 8.14, except that* e_m *is arbitrary. Let* G_0 *denote the set of divisors of N of the form* $p_0^{f_0} p_1^{f_1} \cdots p_m^{f_m}$, *where* $f_i > e_i/2$ *for some index i and* $f_j = e_j/2$ *for* *all $j > i$. Let F_0 denote the elements of G_0 which have rank k. Then $|F| \le |F_0|$.*

It is easy to verify that $ab \not| N$ for all $a, b \in G_0$, and hence F_0 is a family of the desired type. If e_m is odd, then G_0 consists of all $a|N$ such that $p^\alpha|a$, and Theorem 8.14 follows as a special case.

To prove Theorem 8.15, let $\overline{F}$ be the set of all divisors of N of the form N/a, $a \in F$, and define $\overline{F_0}$ similarly. Let $\overline{F}^*$ and $\overline{F_0}^*$ denote the "descendants" of $\overline{F}$ and $\overline{F_0}$ at level k. Clearly both $F \cup \overline{F}$ and $F_0 \cup \overline{F_0}$ are antichains. Moreover it is trivial to check that $\overline{F_0}^* = S_k - F_0$. Note that $\overline{F_0}$ is an "initial segment" in the reverse lexicographic ordering of $S_{\Sigma e_i - k}$, and hence so is $\overline{F_0}^*$. Hence by Theorem 8.13, if $|F| > |F_0|$, then $|\overline{F}^*| \ge |\overline{F_0}^*| = |S_k - F_0| > |S_k - F|$, which is impossible since $\overline{F}^* \subseteq S_k - F$.

In the statement of theorem 8.15, it is not necessary to assume that all of the members of F have rank k. The same result holds if F is any antichain whose members all have rank $\le k$.

If we remove the conditions that F be an antichain from the hypotheses of Theorem 8.15, and also the restriction on ranks, we obtain the following:

THEOREM 8.16: *Let $N = \prod_{i=0}^{m} p_i^{e_i}$, with $e_0 \ge e_1 \ge \cdots \ge e_m$. Let G be an arbitrary set of divisors of N, such that $ab \not| N$ for all $a, b \in G$. Then $|G| \le |G_0|$, where G_0 is as defined in Theorem 8.15.*

The proof is immediate: for each $a \in G_0$, at most one of $\{a, N/a\}$ can be in G, and every divisor is either a or N/a for some $a \in G_0$. Trivially,

$$|G_0| = \begin{cases} \frac{1}{2}\,|M_r|\,, & \text{if } N \text{ is not a square,} \\ \frac{1}{2}\,(|M_r| - 1), & \text{if } N \text{ is a square.} \end{cases}$$

It is interesting to consider what happens when, in the lattice of divisors of an integer, the condition $ab \not| N$ is replaced by the more natural condition $(a, b) = 1$. Erdös and Schönheim [18] obtained an analog of Theorem 8.16 for this case, but the analog of Theorem 8.15 seems more difficult, and little is known about it.

We conclude this section with a brief proof of Kruskal's theorem which is due to Clements and Lindstrom [7]. Their proof is actually more general, and applies to multisets as well (Theorem 8.13). We will give only the set-theoretic version, which is somewhat simpler:

Proof of Theorem 8.1: Our object is to prove that if $F \subseteq S_k$, $G \subseteq S_{k-1}$, and $\partial F \subseteq G$, then $\partial CF \subseteq CG$. Let n denote the largest index for which n is a member of some set in F or G. For each index, i, $0 \le i \le n$, define a new operator C_i (called *i-compression*) as follows: write $S_k(i) = \{A \in S_k \mid i \in A\}$ and $\overline{S_k(i)} = \{A \in S_k \mid i \notin A\}$. Then both $S_k(i)$ and $\overline{S_k(i)}$ inherit a linear ordering from the lexicographic ordering defined on S_k. For any family F of k-sets, define the i-compression of F (denoted C_iF) to be the union of the first $|F \cap S_k(i)|$ members of $S_k(i)$ and the first $|F \cap \overline{S_k(i)}|$ members of $\overline{S_k(i)}$. Clearly $|C_iF| = |F|$.

The proof of Theorem 8.1 is by induction on n, and is based on four elementary observations.

(1) If $F \subseteq S_k$, $G \subseteq S_{k-1}$, and $\partial F \subseteq G$, then $\partial C_iF \subseteq C_iG$, for all $i \le n$. (The i-compression of a simplicial complex is again a simplicial complex.)

(2) After repeated application of C_i for various i, F and G are transformed into sets F' and G' (with $|F| = |F'|$ and $|G| = |G'|$) which are i-compressed for all $i \le n$.

(3) If $2k \ne n + 1$, then F' is compressed (that is, $F' = CF$) and the proof is complete.

(4) In the special case when $2k = n + 1$, it is possible for F' to be i-compressed for all i but not compressed. However, F' can be made compressed by exchanging its *last* member for its immediate predecessor, an operation which preserves the relation $\partial F' \subseteq G'$. Hence the theorem is true in every case.

Statement (1) follows from the inductive hypothesis. The argument is straightforward. Statement (2) is trivial. To see that (3) holds, let x be the last member of F' (in ordering of S_k), and let y be any element of S_k which precedes x. If $y \notin F'$, then x and y cannot agree in position i for any $i \leq n$, since F' is i-compressed. Hence x and y must be complements, which implies $n + 1 = 2k$. In this case, it is easy to see that $F' - x + y$ is compressed and satisfies $\partial(F' - x + y) \subseteq G'$, which proves statement (4), and completes the proof of Theorem 8.1.

Apparently nothing is known about analogs of the Kruskal-Katona Theorem for subspaces of a finite vector space. On the other hand several consequences of it are known to be valid (Hsieh's Theorem, the LYM inequality) so there is reason to hope that such a result might exist.

9. APPLICATION: THE LITTLEWOOD-OFFORD PROBLEM

We conclude by showing how some of the ideas, methods, and results presented earlier can be applied to a geometric problem concerning distributions of linear combinations of vectors. The question was first raised by Littlewood and Offord [47]:

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be vectors in a Hilbert space V, each of magnitude at least one. What is the maximum number of linear combinations of the form

$$\sum_{i=1}^{n} \epsilon_i \vec{v}_i \; (\epsilon_i = 0 \text{ or } 1)$$

which can lie in a sphere of diameter 1?

The answer is $\binom{n}{\left[\frac{n}{2}\right]}$, independently of the dimension of V, and this number can be achieved by taking all of the vectors to be the same. The solution to the Littlewood-Offord problem was conjectured by Erdös [15] and proved in several stages by Erdös, Katona, and Kleitman. Since the earlier arguments are elegant and

elementary, we have included them below, together with the final dimension-free proof due to Kleitman.

Proof (when dim $V = 1$, (Erdös [15])): We can assume that all of the vectors are positive, since changing the sign of a vector only translates the set of linear combinations. Now to each linear combination associate the set of indices for which $\epsilon_i = 1$. Clearly, if a collection of linear combinations lies in a unit interval, the corresponding sets must form an antichain. Hence, by Sperner's Theorem, the number of linear combinations can be at most $\binom{n}{\left[\frac{n}{2}\right]}$.

Proof (when dim $V = 2$, (Katona [31], Kleitman [35])): As above, we can change the direction of vectors if necessary and assume that the vectors all lie in two quadrants (say the first and second). Now we associate two sets of indices to each linear combination—one corresponding to vectors in the first quadrant and the other corresponding to vectors in the second quadrant. Since the sum of two unit vectors in a quadrant has magnitude at least $\sqrt{2}$ and lies in the same quadrant, we can deduce that two linear combinations lying within a unit diameter circle cannot have sets of indices which agree in one quadrant and are comparable in the other. The conditions on the pairs of index sets are therefore precisely those of Theorem 4.5, from which the bound of $\binom{n}{\left[\frac{n}{2}\right]}$ follows as before.

Proof (when dim V is arbitrary, (Kleitman [39])): The idea of this proof is to construct "saturated partitions" for linear combinations of vectors, imitating the methods of section 3. We will show that the collection of all linear combinations can be partitioned into $\binom{n}{\left[\frac{n}{2}\right]}$ blocks, such that no two linear combinations in a block can lie in the same unit diameter sphere. Trivially, this implies a bound of $\binom{n}{\left[\frac{n}{2}\right]}$.

The construction is analogous to the inductive procedure of deBruijn, Tengbergen, and Kruyswijk, in which a k-chain in B_{n-1} produces two new chains in B_n, one of length $k + 1$ and the other of length $k + 1$. We proceed in exactly the same way: suppose that the linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1}$ have been appropriately partitioned into blocks. Consider one such block U of size k, whose members we denote by $\vec{\gamma}_1, \vec{\gamma}_2, \ldots, \vec{\gamma}_k$. We can obtain two new blocks from U by taking all linear combinations with and without $\vec{v}_n$, denoting these blocks by $U + \vec{v}_n$ and U, respectively. Next take that linear combination $\vec{\gamma}_i \in U$ which has maximal component in the direction of $\vec{v}_n$, and transfer $\vec{\gamma}_i + \vec{v}_n$ from $U + \vec{v}_n$ to U. This gives new blocks of size $k - 1$ and $k + 1$, and one can easily check that no two members of either block lie in a sphere of diameter one. Repeating this construction for each block gives a partition of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Since the number of blocks of each size propagates in the same way as the number of chains in a partition of B_n, there must be $\binom{n}{\left[\frac{n}{2}\right]}$ blocks altogether, and the proof is complete.

Final remark: Because of the large number of papers in the literature on this subject, and because some of the results described here have been rediscovered several times, there may have been cases where our attribution of results has been incomplete. We apologize in advance for any such occurrences, and hope that the authors involved will inform us of any errors in the references. Also, requirements of space have forced us to leave unmentioned many interesting results which are closely related to the ones we have chosen to include. Again we offer apologies but suggest that the only remedy would be a much longer treatise on the subject.

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