

# A combinatorial interpretation of Hikita's transition probabilities

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## The Rényi–Foata Fundamental Bijection $\Phi$

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Then  $\Phi$  maps **permutations** of  $S$  (i.e., bijections from  $S$  to  $S$ ) to **linear orderings** of  $S$ .

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1. Given a permutation  $\pi$  of  $S$ , identify its disjoint cycles.
2. In each cycle, write the largest element first.
3. Arrange the cycles in increasing order of said largest elements.
4. Erase the parentheses.
5. Define  $\Phi(\pi)$  to be the resulting linear ordering on  $S$ .

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To construct  $\Phi^{-1}$ , find the **records** or **left-to-right maxima** (in this case, 5, 7, 9).

## A Permutation Statistic: The Watershed

Let  $S$  be a finite set of integers with  $2n$  elements, e.g.,  $\{1, 2, 3, 4, 5, 6\}$ .

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Pick  $0 \leq k \leq n$  and let  $\pi_L$  be the **reverse** of the first  $2k$  elements and let  $\pi_R$  be the remaining  $2n - 2k$  elements. In our example, if we pick  $k = 1$  then  $\pi_L = 5, 4$  and  $\pi_R = 1, 2, 6, 3$ .

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If the cycle lengths of  $\Phi^{-1}(\pi_L)$  and  $\Phi^{-1}(\pi_R)$  are all even, then we say that  $k$  is the **watershed** of  $\pi$ . If  $k = 1$  then  $\Phi^{-1}(5, 4) = (5, 4)$  and  $\Phi^{-1}(1, 2, 6, 3) = (1)(2)(6, 3)$  so 1 is *not* the watershed.

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Is this statistic new? Darij Grinberg noticed it was implicit in “the bulldozer problem.”

## The Bulldozer Problem (IMO 2015 Short List, submitted by Estonia)

In Lineland there are  $n + 1$  towns, arranged along a road running from left to right. Each town has a **left bulldozer** (put to the left of the town and facing left) and a **right bulldozer** (put to the right of the town and facing right). The sizes of the  $2n + 2$  bulldozers are distinct. Every time when a right and a left bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, the bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let  $A$  and  $B$  be two towns, with  $B$  being to the right of  $A$ . We say that town  $A$  can **sweep town  $B$  away** if the right bulldozer of  $A$  can move over to  $B$  pushing off all bulldozers it meets. Similarly,  $B$  can sweep  $A$  away if the left bulldozer of  $B$  can move to  $A$  pushing off all bulldozers of all towns on its way.

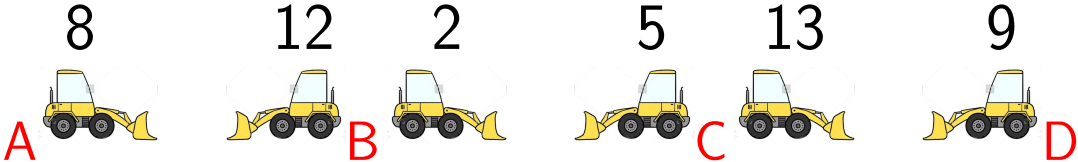
Prove that there is exactly one town which cannot be swept away by any other one.

# Bulldozer Example

The towns A, B, C, D are marked in red.

The leftmost and rightmost bulldozers have been suppressed as they play no role.

The bulldozer sizes are marked in black.

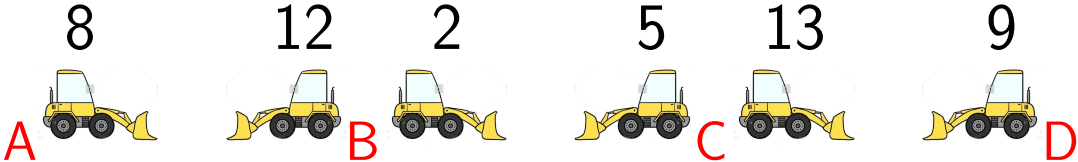


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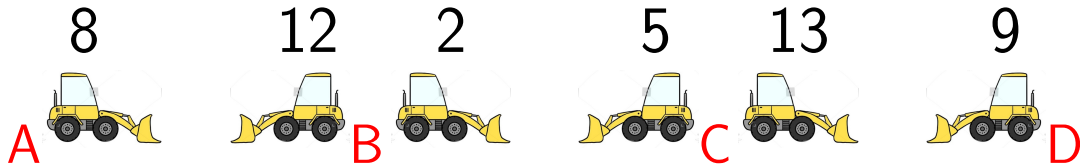
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The safe town is C.

**Exercise:** Show that the safe town is located at the watershed.

## Hikita's Probability Distribution

In Hikita's remarkable preprint, "A proof of the Stanley–Stembridge conjecture," he defines, for all positive integers  $a_1, b_1, \dots, a_n, b_n$ , and a parameter  $q$ ,

$$\varphi_k := \prod_{i=1}^k \frac{q^{a_i} \left[ b_i + \sum_{j=i+1}^k (a_j + b_j) \right]_q}{\left[ \sum_{j=i}^k (a_j + b_j) \right]_q} \prod_{i=k+1}^n \frac{\left[ a_i + \sum_{j=k+1}^{i-1} (a_j + b_j) \right]_q}{\left[ \sum_{j=k+1}^i (a_j + b_j) \right]_q},$$

and shows  $\sum_k \varphi_k = 1$ . Even if  $q = 1$  and  $n = 2$ , we obtain the nontrivial identity

$$\left( \frac{a_1}{a_1 + b_1} \cdot \frac{a_1 + b_1 + a_2}{a_1 + b_1 + a_2 + b_2} \right) + \left( \frac{b_1}{a_1 + b_1} \cdot \frac{a_2}{a_2 + b_2} \right) + \left( \frac{b_2}{a_2 + b_2} \cdot \frac{b_1 + a_2 + b_2}{a_1 + b_1 + a_2 + b_2} \right) = 1.$$

We now show how to interpret these probabilities combinatorially in terms of the watershed.

## Hikita's Probability Distribution: An Interpretation

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As an example, let  $a_1, b_1, a_2, b_2$  be the integers 4, 2, 5, 3.

1. Let  $M = a_1 + b_1 + a_2 + b_2 = 14$ , and generate a uniformly random permutation  $\pi$  of  $\{1, 2, \dots, M\}$  (e.g., 8, 11, 12, 10, 2, 1, 4, 14, 5, 13, 7, 3, 9, 6).

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2. Let  $\sigma_1$  (respectively,  $\sigma_2, \sigma_3, \sigma_4$ ) be the max of the first  $a_1$  elements (respectively, the next  $b_1$  elements, the next  $a_2$  elements, the last  $b_2$  elements).

$$\underbrace{8, 11, 12, 10}_{\sigma_1=12} \quad \underbrace{2, 1}_{\sigma_2=2} \quad \underbrace{4, 14, 5, 13, 7}_{\sigma_3=14} \quad \underbrace{3, 9, 6}_{\sigma_4=9}$$

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3. The 1st (respectively, 2nd, 3rd) summand is the probability that the watershed of  $\sigma$  is 0 (respectively, 1, 2); e.g.,  $a_1/(a_1 + b_1)$  is the probability that  $\sigma_2$  is not a record, and  $(a_1 + b_1 + a_2)/(a_1 + b_1 + a_2 + b_2)$  is the probability that  $\sigma_4$  is not a record.

## Concluding remarks

1. In that last calculation, the probabilities seem dependent, so can we just multiply them? Doug Jungreis observed that being a record is a fact about the *relative* sizes of the numbers up to that point. So the probability that  $\sigma_{2k}$  is a record is independent of any information we have about whether  $\sigma_{2j}$  is a record for  $j < k$ .

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2. If  $|S| = 2n$  then the number of permutations of  $S$  whose cycle lengths are all even equals the number of permutations of  $S$  whose cycle lengths are all odd (both equal  $(2n - 1)!!^2$ ). Adin, Hegedűs, and Roichman (arXiv:1909.04460) refined this equality to “respect descents” and Elizalde (arXiv:2503.09972) gave a bijective proof. What does this mean about watersheds and Hikita’s proof?

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3. For more details, including a simple recursive algorithm to compute the watershed, see arXiv:2603.23879 (corrected version will appear Monday, March 30).