Chess Tableaux and Chess Problems

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20 October 2004

MIT Combinatorics Seminar

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Chess Tableaux

- **A chess tableau is a standard Young tableau in which the parity of the (*i* , *j*) entry equals the parity of *i* + *j* + 1**

1	2	3	6	7	10	15
4	5	8	9	16		
11	12	13	14			

- **First defined by Jonas Sjöstrand in the study of the sign imbalance of certain posets**
- **Problem: Find Chess(λ), the number of chess tableaux of shape λ**

Chess Tableaux with 2 Rows

- **Entry 2*i* + 1 must appear immediately to the right of entry 2*i***

1	2	3	6	7	8	9
4	5	10	11			

- **Finding Chess(*a*, *b*) reduces to enumerating standard Young tableaux with two rows**
- **Chess(2*n* + 1, 2*n* + 1) is the *n*th Catalan number**

Chess Tableaux with 3 Rows

- **No obvious pattern and no known formula in general for Chess(*a*, *b*, *c*)**

- **BUT: Sloane recognizes Chess(*n*, *n*, *n*) for *n* > 1 as the number of Baxter permutations of *n* – 1**

$$\mathrm{Chess}(n,n,n)=\frac{2}{(n-1)n^2}\sum_{k=0}^{n-2}\binom{n}{k}\binom{n}{k+1}\binom{n}{k+2}$$

- **Sloane also reveals that Chess(*n*, *n*, *n*) is the number of 3 x (*n* – 1) nonconsecutive tableaux [Dulucq and Guibert]**

Nonconsecutive Tableaux

- **A nonconsecutive tableau is a standard Young tableau in which *i* and *i* + 1 never appear in the same row**

1	**3**	**5**	**7**	**12**	**15**
2	**6**	**9**	**11**	**13**	
4	**8**	**10**	**14**		

- **$\text{NCon}_i(a, b, c)$ = no. of nonconsecutive tableaux of shape *a*, *b*, *c* whose highest entry is in row *i***

- **Theorem: For all *a*, *b*, and *c*,**

  $\text{NCon}_1(a, b, c) = \text{Chess}(a+b-c,\ a-b+c,\ 1-a+b+c)$

Corollaries

- **NCon(a, b, c) = Chess(a+b–c, a–b+c, 1–a+b+c) + Chess(1+a+b–c, 1+a–b+c, –a+b+c)**

  Proof: By nonconsecutivity,

  $\mathrm{NCon}_1(a+1,b,c) = \mathrm{NCon}_2(a,b,c) + \mathrm{NCon}_3(a,b,c)$

  And it is obvious that

  $\mathrm{NCon}(a,b,c) = \mathrm{NCon}_1(a,b,c) + \mathrm{NCon}_2(a,b,c) + \mathrm{NCon}_3(a,b,c)$

  So $\mathrm{NCon}(a,b,c) = \mathrm{NCon}_1(a,b,c) + \mathrm{NCon}_1(a+1,b,c)$.

  Now apply the theorem.

- **NCon(n–1, n–1, n–1) = Chess(n, n, n)**

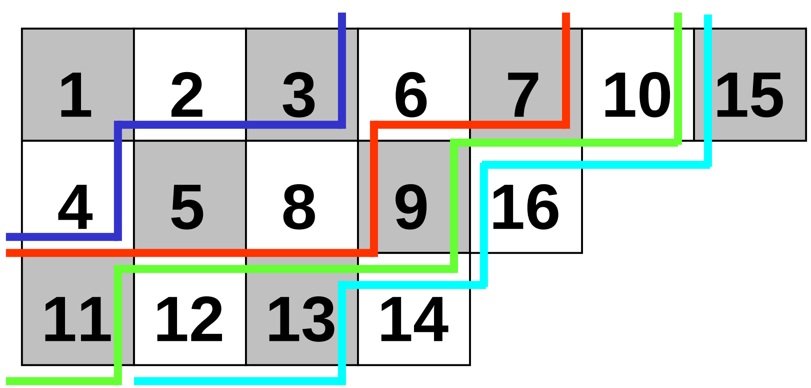
  Proof: The previous corollary implies

  $\mathrm{NCon}(n-1,n-1,n-1) = \mathrm{Chess}(n-1,n-1,n) + \mathrm{Chess}(n,n,n-1)$

  But $\mathrm{Chess}(n-1,n-1,n) = 0$ and $\mathrm{Chess}(n,n,n-1) = \mathrm{Chess}(n,n,n)$

The Bijection (Part 1)

- **Start with a chess tableau T**
- **Assume that T is balanced, i.e., the lengths of rows 2 and 3 have opposite parity**
  - **$a-b+c$ and $1-a+b+c$ have opposite parity**
- **Decompose T as follows:**
  - **Step through the entries until you get an entry in row 2**
  - **Then keep stepping through until you get a total of two more entries in rows 2 and 3 collectively**
  - **Repeat until the chess tableau is exhausted**



The Bijection (Part 2)

- **Create a nonconsecutive tableau *T** section by section**
  - **Roughly speaking, in each section, the elements in row 1 of *T* go into rows 1 and 2 of *T** (alternating between the rows because of nonconsecutivity) with variations depending on the positions of the two elements *x* and *y* of *T* in rows 2 and 3**
- **Four cases:**

1) ***x*** **and** ***y*** **both in row 2**

x–1 → row 3; x → row 1; x+1 to y–1 → rows 1 and 2

2) ***x*** **in row 2,** ***y*** **in row 3**

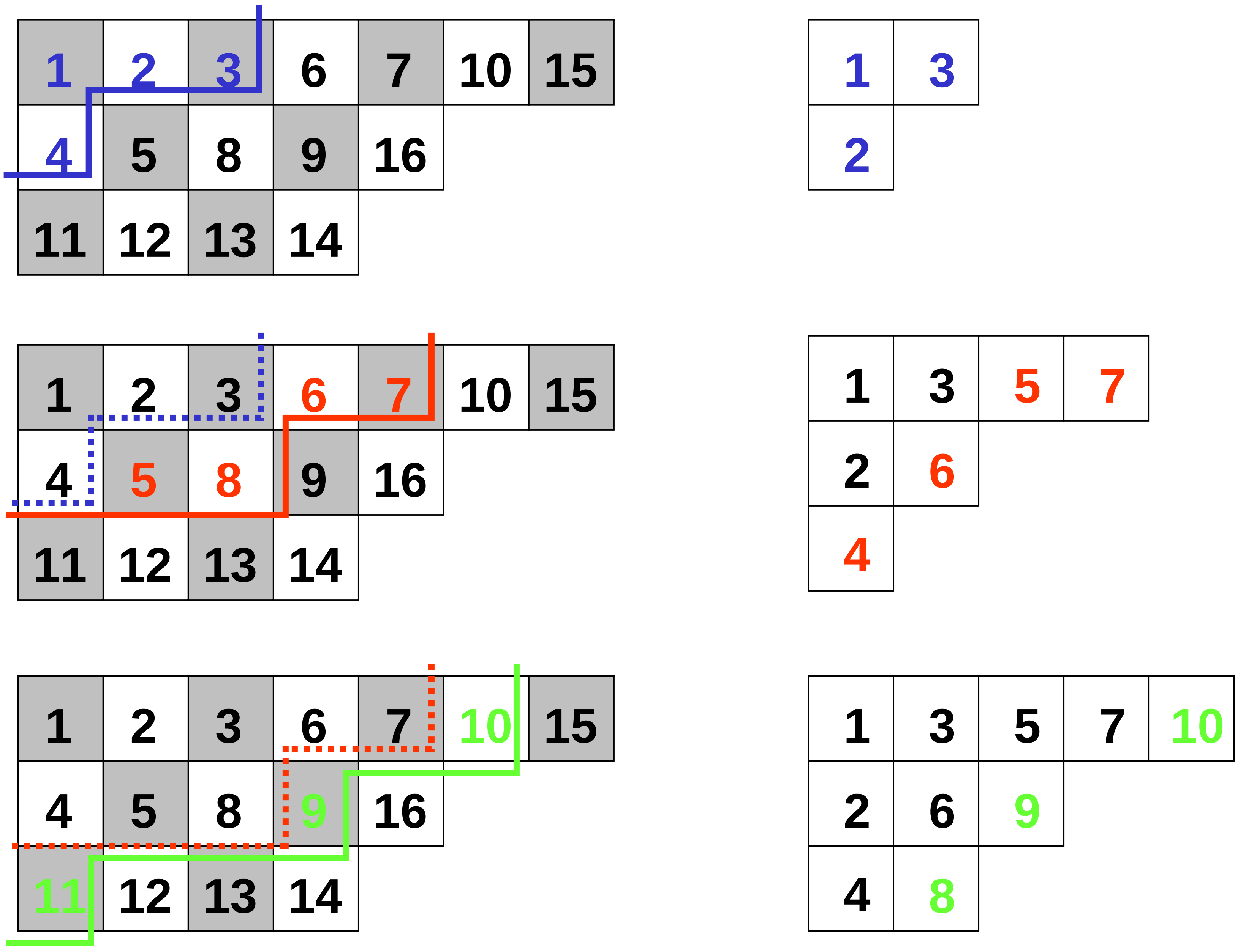
x–1 → row 3; x → row 2; x+1 to y–1 → rows 1 and 2

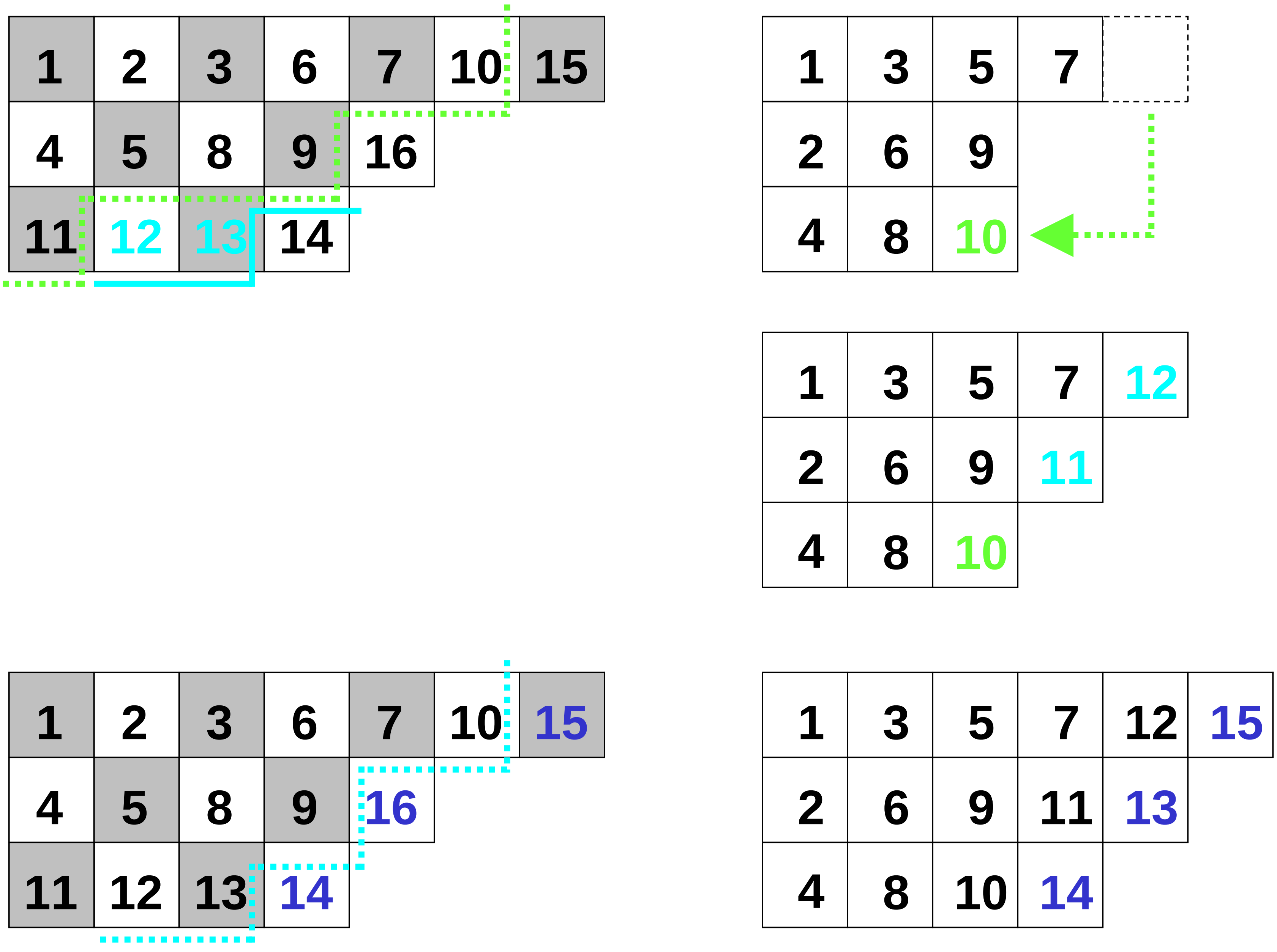
3) ***x*** **in row 3,** ***y*** **in row 2**

x–1 → row 2; x → row 3; x+1 → row 1; x+1 to y–1 → rows 1 and 2

4) ***x*** **and** ***y*** **both in row 3**

move x–2 to row 2 or 3; x–1 → row 2 or 3; x → row 1; x+1 to y–1 → rows 1 and 2


1 2 3 6 7 10 15
4 5 8 9 16
11 12 13 14
1 3
2
1 2 3 6 7 10 15
4 5 8 9 16
11 12 13 14
1 3 5 7
2 6
4
1 2 3 6 7 10 15
4 5 8 9 16
11 12 13 14
1 3 5 7 10
2 6 9
4 8


1 2 3 6 7 10 15
4 5 8 9 16
11 12 13 14
1 3 5 7
2 6 9
4 8 10
1 3 5 7 12
2 6 9 11
4 8 10
1 2 3 6 7 10 15
4 5 8 9 16
11 12 13 14
1 3 5 7 12 15
2 6 9 11 13
4 8 10 14

Corollary of Bijective Proof

- **In the formula for Chess(*n*, *n*, *n*), what does *k* mean?**

$$\text{Chess}(n,n,n) = \frac{2}{(n-1)n^2} \sum_{k=0}^{n-2} \binom{n}{k}\binom{n}{k+1}\binom{n}{k+2}$$

- **Dulucq and Guibert have an interpretation for nonconsecutive tableaux; bijecting, we find:**
  - ***k* is the number of sections falling into the first two of the four possible cases (both in row 2, or split with the larger entry in row 3)**

- **Open: Find a bijection to 3 x *k* semistandard Young tableaux with entries between 1 and *n* – *k* + 1**

An Algebraic Approach

- **Recall that before the hook length formula was the determinantal formula $n! \det[1/(\lambda_i + j - i)!]$**
  - **Relax the column constraint on Young tableaux**
  - **Reinterpet as lattice paths or "rat races"**
  - **Intone "Lindström-Gessel-Viennot" and presto!**

- **A similar technique can be applied to enumerate chess tableaux with r rows**
  - **One obtains a rational generating function in r variables**
  - **The diagonal is *P*-recursive [Lipshitz]**
  - **In principle the recurrence can be extracted using the WZ methodology, but even for $r = 3$ the computation is too large to perform naïvely**

Generating Function for 3 Rows

$F(x, y, z) = N/D$

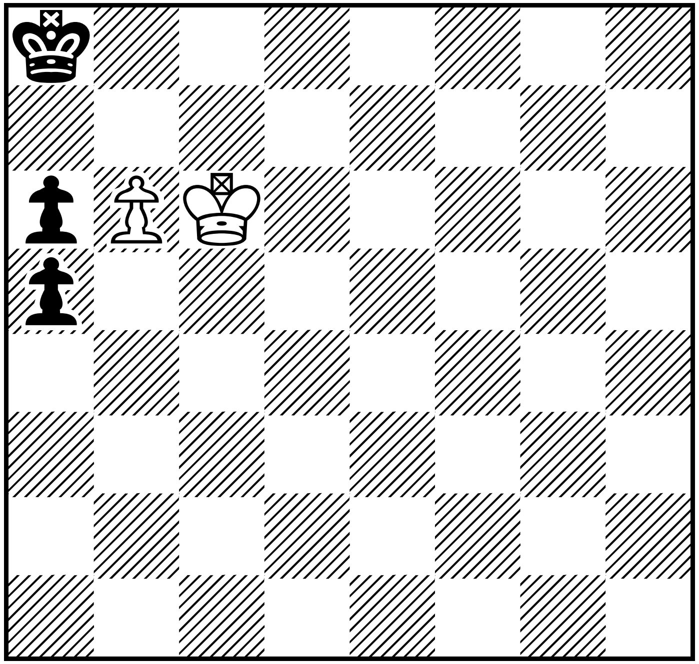
$$
\begin{aligned}
N = {} & 8x^2y^4z^5 - 8x^2y^3z^6 + 8y^4z^7 - 8y^3z^8 + 4x^3y^3z^4 - 4x^3y^2z^5 + 8x^2y^3z^5 + 4x^2y^2z^6 + 4x^2z^8 - \\
& 4xy^4z^5 + 4xy^3z^6 - 12xy^2z^7 - 4xz^9 - 4y^4z^6 + 8y^3z^7 - 4y^2z^8 + 2x^3y^5z + 2x^3y^3z^3 + 4x^3y^2z^4 \\
& - 2x^2y^6z + 2x^2y^5z^2 - 10x^2y^4z^3 + 8x^2y^3z^4 - 6x^2y^2z^5 + 2x^2yz^6 - 2x^2z^7 + 4xy^5z^3 + 4xy^4z^4 \\
& + 6xy^3z^5 + 12xy^2z^6 + 2xyz^7 + 4xz^8 - 2y^6z^3 - 18y^4z^5 + 12y^3z^6 - 6y^2z^7 - 2z^9 - 2x^3y^5 + \\
& 2x^3y^4z - 5x^3y^3z^2 + 5x^3y^2z^3 - x^3yz^4 + x^3z^5 + 2x^2y^6 - 2x^2y^5z + 5x^2y^4z^2 - 8x^2y^3z^3 - \\
& 2x^2y^2z^4 - 2x^2yz^5 - 9x^2z^6 - 2xy^5z^2 + 4xy^4z^3 - 9xy^3z^4 + 21xy^2z^5 - xyz^6 + 11xz^7 + 2y^6z^2 \\
& + 11y^4z^4 - 12y^3z^5 + 14y^2z^6 + z^8 - 2x^3y^4 - 3x^3y^3z - 5x^3y^2z^2 - x^3yz^3 - x^3z^4 + 4x^2y^4z - \\
& 3x^2y^3z^2 + 9x^2y^2z^3 - 3x^2yz^4 + 5x^2z^5 - 3xy^5z - 4xy^4z^2 - 11xy^3z^3 - 21xy^2z^4 - 6xyz^5 - \\
& 11xz^6 + 2y^6z - y^5z^2 + 14y^4z^3 - 5y^3z^4 + 15y^2z^5 + 7z^7 + 3x^3y^3 - 3x^3y^2z + 2x^3yz^2 - 2x^3z^3 \\
& - 5x^2y^4 + 3x^2y^3z - 6x^2y^2z^2 + 3x^2yz^3 + 5x^2z^4 + 2xy^5 - 2xy^4z + 8xy^3z^2 - 12xy^2z^3 + 3xyz^4 \\
& - 11xz^5 - 2y^6 + y^5z - 12y^4z^2 + 5y^3z^3 - 20y^2z^4 - 4z^6 + 3x^3y^2 + x^3yz + 2x^3z^2 - 3x^2y^2z + \\
& x^2yz^2 - 4x^2z^3 + 2xy^4 + 5xy^3z + 12xy^2z^2 + 6xyz^3 + 11xz^4 - 4y^4z + y^3z^2 - 12y^2z^3 - 9z^5 - \\
& x^3y + x^3z + 4x^2y^2 - x^2yz + x^2z^2 - 3xy^3 + 3xy^2z - 3xyz^2 + 5xz^3 + 5y^4 - y^3z + 14y^2z^2 + \\
& 6z^4 - x^3 + x^2z - 3xy^2 - 2xyz - 5xz^2 + 3y^2z + 5z^3 - x^2 + xy - xz - 4y^2 - 4z^2 + x - z + 1
\end{aligned}
$$

$$D = (2xyz + x^2 + y^2 + z^2 - 1)\,(y^2 + z^2 - 1)^2(x^2 + z^2 - 1)(1 - z)$$

- **Coefficient of $x^a y^b z^c$ is Chess(a, b, c) provided $a \geq b \geq c > 0$; otherwise the coefficient is "junk"**

Queue Problems in Chess

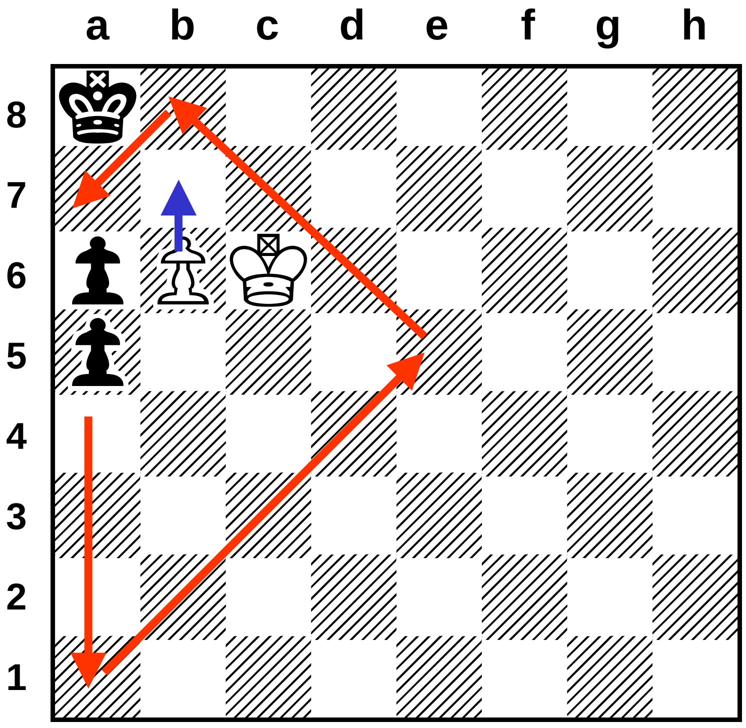
Serieshelpmate in 14:
How many solutions?
(E. Bonsdorff and K. Väisänen)



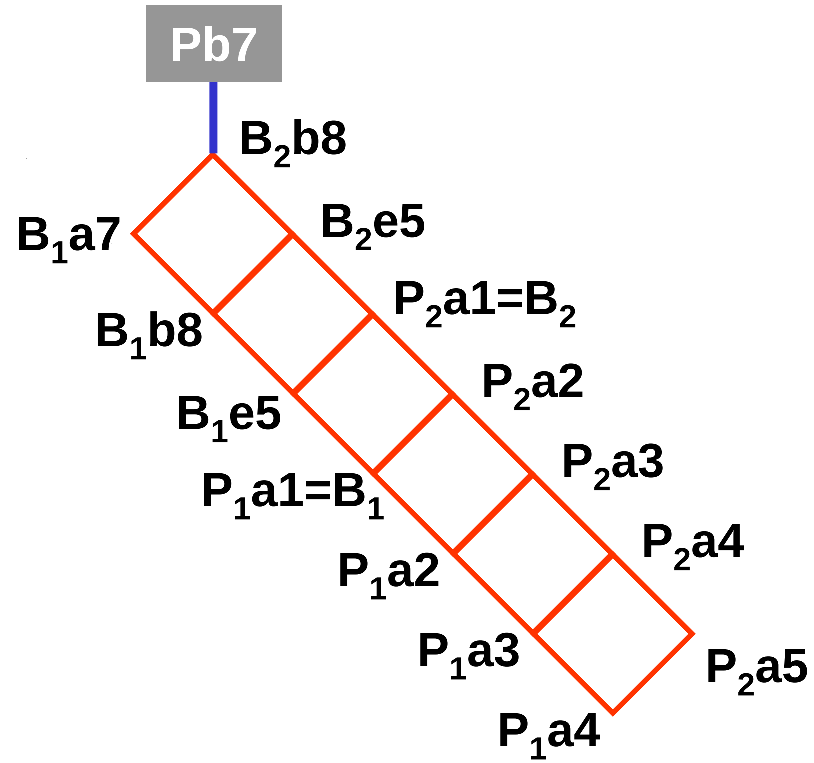
- **“Serieshelpmate in 14” means Black makes 14 consecutive moves while White does nothing, and then White makes a single move to checkmate Black**
- **Black and White cooperate to checkmate Black**
- **None of the 14 moves except the last may cause either player to be in check**
- **This is a queue problem because it turns out that there is a fixed set of moves that Black must make; only the order of the moves varies**

Solution to Bonsdorff-Väisänen

Serieshelpmate in 14:
How many solutions?



Promote to bishops



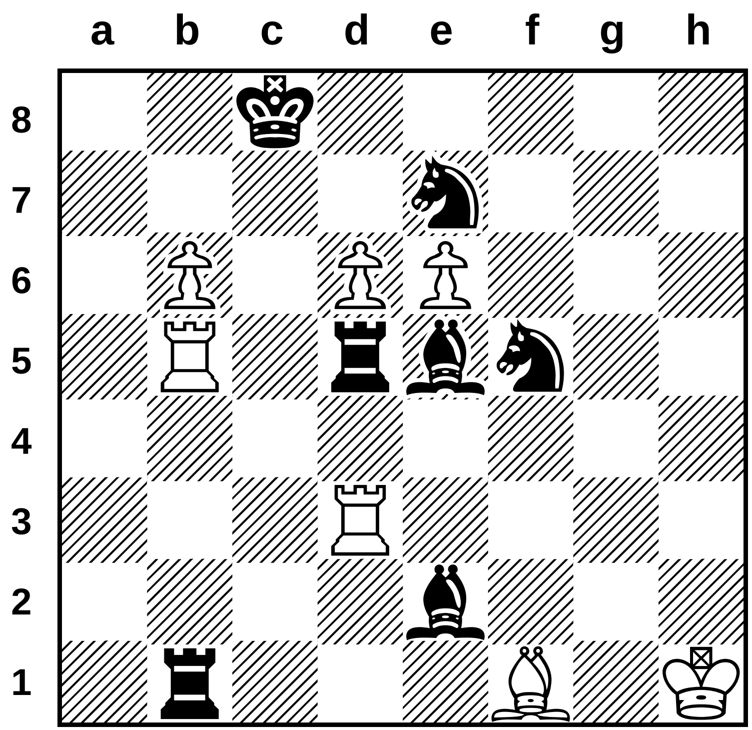
Solutions are in bijection with linear extensions of this poset, i.e., 2x7 standard Young tableaux

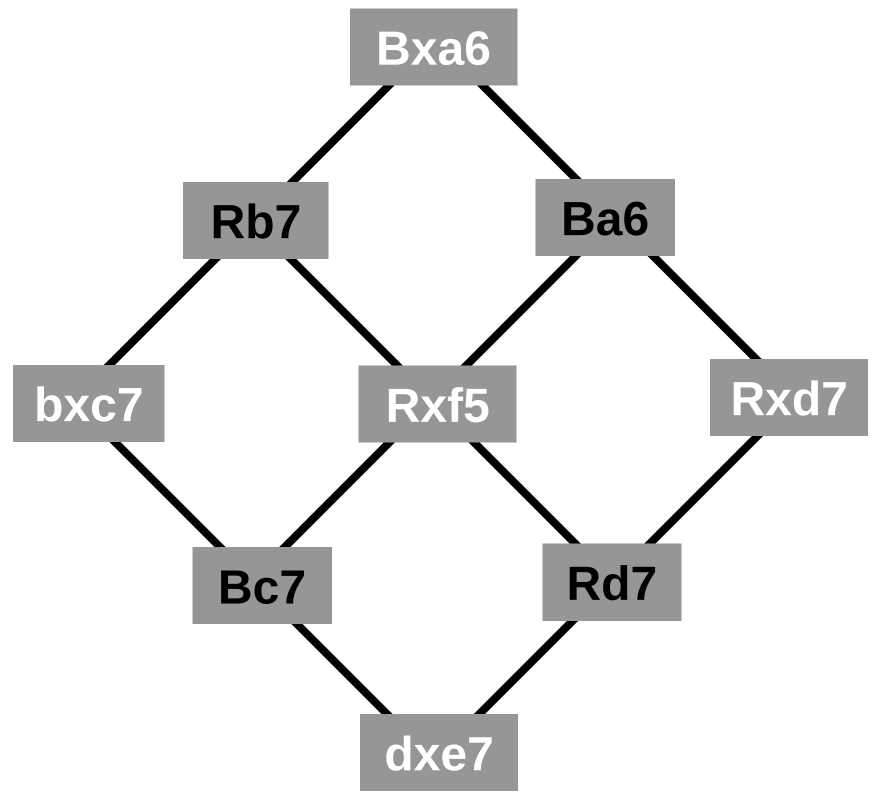
ANSWER: C_7 = 429

From Serieshelpmates to Helpmates

- **Until recently all queue problems were serieshelpmates (or serieshelpstalemates)**
- **What if we want helpmates (or helpstalemates), in which Black and White alternate moves?**
- **We are led to consider posets whose elements are colored either black or white, and to enumerate their alternating linear extensions, i.e., linear extensions in which black and white elements alternate**
  - **For example, chess tableaux!**

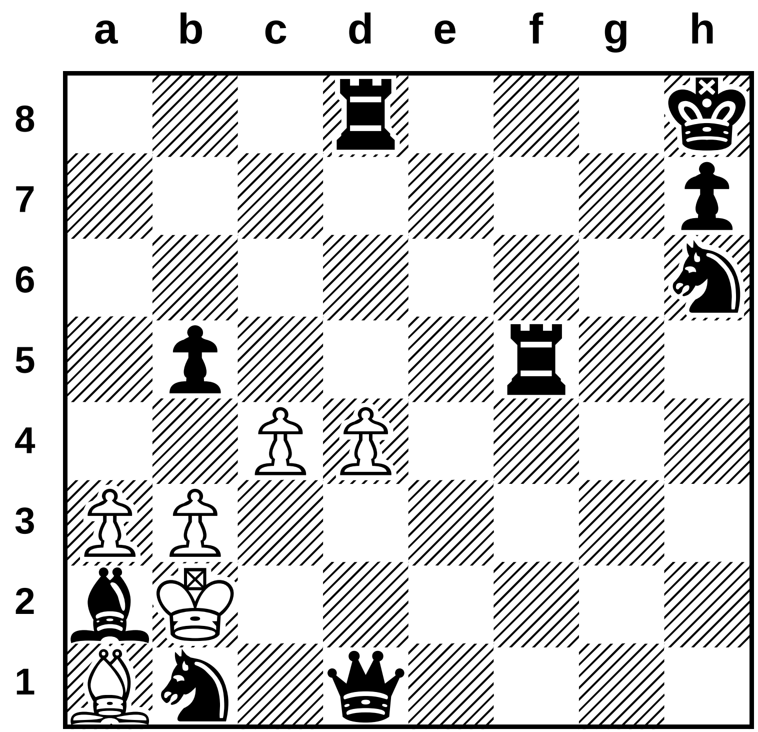
Helpstalemate in 4.5: Two Solutions

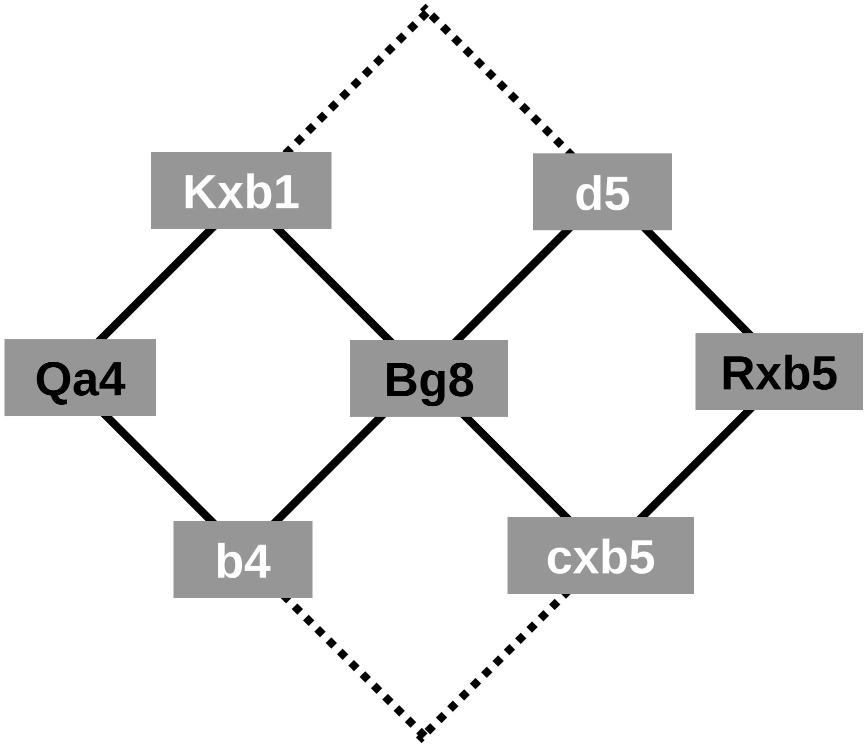




Helpmate in 3.5: Two Solutions

N. Elkies





Open Problems

- **The Charney-Davis statistic CD(λ) = Σ_T $(-1)^{d(T)}$**
  - **Sum is over all standard Young tableaux *T* of shape λ**
  - **$d(T)$ = # { i : i + 1 is in a lower-numbered row }**
  - **Studied by Reiner, Stanton, and Welker**
  - **Equals Chess(λ) for 2x*n* and 3x*n* rectangles (up to sign), but not for 4x*n* rectangles or most other shapes**
  - **No combinatorial proof for the 3x*n* "Baxter" case**

- **Enumerate chess tableaux with more than 3 rows**
  - **Chess($2n + 1, 2n + 1$) = hypergeom($-n, -(n - 1); 2; 1$)**
  - **Chess(n, n, n) = hypergeom($-n, -(n - 1), -(n - 2); 2, 3; -1$)**
  - **But the obvious conjecture fails for 4 and 5 rows**
    - **Currently, no candidate formulas even for rectangles**

Open Problems (cont'd)

- **If we compute $\sum_{\lambda \vdash n}$ Chess$(\lambda)^2$ then we get:**
  - **$1, 2, 2, 2^2, 2^3, 2^4, 2^4\cdot 3, 2^5\cdot 5, 2^6\cdot 7, 2^{11}, 2^8\cdot 5^2, 2^9\cdot 61, 2^{10}\cdot 3\cdot 41, 2^{11}\cdot 5\cdot 59, 2^{11}\cdot 1523, 2^{13}\cdot 23\cdot 83, 2^{13}\cdot 11411, 2^{15}\cdot 103\cdot 163, \ldots$**
  - **Why such high powers of 2?**

- **Feigin and Loktev (math.QA/0212001) define "Weyl modules" for $\mathfrak{sl}_2$ that conjecturally have dimensions equal to the number of Baxter permutations**
  - **Is there a connection to chess tableaux?**

- **Find new classes of bicolored posets with an interesting number of alternating linear extensions and compose corresponding queue problems**