

Does Every Money-Game Position Have an Equity?

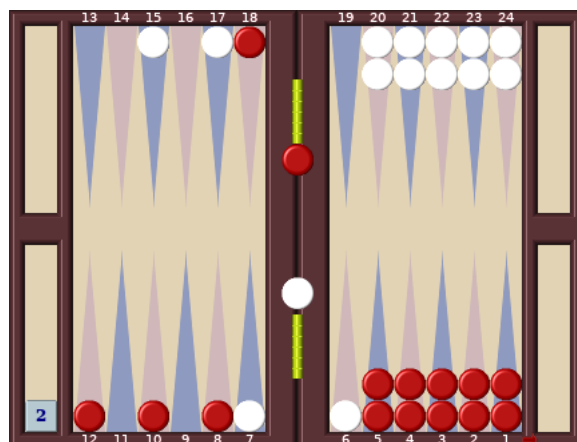
Timothy Y. Chow

The *equity* of a money-game position (without beavers or raccoons) may be thought of as the average amount of money (in dollars) that you would win per game if you were to play the position repeatedly as a proposition for one dollar a point, assuming that both you and your opponent are “perfect players.” Equity is a familiar concept to anyone who has used a bot such as eXtremeGammon (XG), though to be precise, the value that XG displays for a position is its estimate of the *normalized equity*, meaning the equity divided by the current value of the cube. Mathematicians use the term *expected value* instead of *equity*; the technical definition of expected value is the sum, over all possible outcomes, of the probability of that outcome multiplied by your payoff (positive or negative) from that outcome. By something called the law of large numbers, the expected value equals the average payoff per game if you were to play the same position repeatedly—at least, if the expected value is some well-defined finite value. But is the expected value always finite?

We normally take it for granted that every backgammon position has a finite equity. Certainly, in real life, or even in XG, there is a limit to how many times the cube can get turned. Nevertheless, it is fun to think about the theoretical possibility of a doubling cube that is allowed to get arbitrarily high. Under that theoretical assumption, could there be backgammon positions with undefined equity (or as mathematicians prefer to say, undefined expected value)?

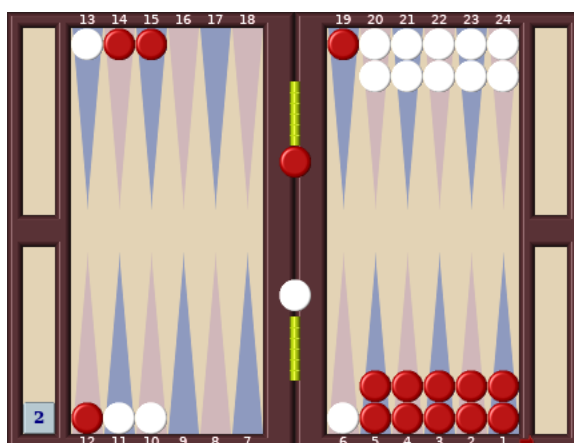
Prior Work

As far as I am aware, the first time the possibility of undefined equity was seriously considered in print was in April 1982, in the *Las Vegas Backgammon Magazine*. In an article entitled “Perpetual Redouble?” Bill Kennedy presented and analyzed the position below. Red is on roll, and the question is whether to redouble. Bob Floyd also analyzed Kennedy’s position in his article, “Riding the Tiger.”



The question resurfaced in 1997, when Paul Tanenbaum posted a very similar position to the USENET newsgroup rec.games.backgammon, generating some lively discussion. In 2004, Douglas Zare wrote an article, “Undefined Equity,” in Gammon Village Magazine, and presented what he called “The Paradox Position,” shown below. (All the articles I have mentioned are available on Tom Keith’s excellent

website, Backgammon Galore!) Though similar to Kennedy's position, Zare's Paradox Position has the additional feature of being symmetrical, which simplifies its analysis. From now on, we restrict our attention to Zare's version.



All of the aforementioned authors quickly zeroed in on the key issue. Rolling any 6 will enter and hit and gain a massive advantage, so it is tempting to redouble from the bar. But if Red redoubles and dances, then by the symmetry of the position, White will be faced with *exactly the same decision*, so if it was right for Red to redouble from the bar, then it must also be right for White to redouble from the bar. So what happens if both players repeatedly dance? A long sequence of repeated dances is exponentially unlikely, but at the same time, if both players are redoubling every time, then the cube value is getting exponentially high. If you do the math, then you will find that the equity seems to be undefined. So is the equity of the Paradox Position undefined? Or maybe it is wrong to redouble, despite the threat of rolling a massive market-losing 6?

Strictly speaking, these questions remain unanswered even today. Of all the previously published analyses, Zare's was the most careful, but even he admitted that he did not have a mathematical proof that the Paradox Position has undefined equity. The purpose of the present article is to carry the analysis further, and come very close to mathematically proving that some money-game positions have undefined equity.

Conditional Equity

A key concept in our analysis is the concept of your *normalized equity conditional on rolling a 6*. In the Paradox Position, suppose you do not redouble, and you roll 63 from the bar. We assume that you play this roll perfectly; common sense tells us that the correct way to play a 63 is bar/19 14/11*. We can estimate your normalized equity after bar/19 14/11* by performing an XG rollout. Similarly, we can perform a rollout for each of your rolls that contain a 6. The following table reports the results of 5184-trial 4-ply/XGR+/Large rollouts.

Roll	Play	Normalized Equity
61	Bar/19 12/11*	1.8074 ± 0.0028
62	Bar/19 12/10*	1.8098 ± 0.0032

63	Bar/19 14/11*	1.7722 ± 0.0033
64	Bar/19 15/11*	1.7814 ± 0.0030
65	Bar/19 15/10*	1.7793 ± 0.0027
66	Bar/13*/7 19/13	1.8756 ± 0.0025

Your *normalized equity conditional on rolling a 6*, which we denote by the letter e , is defined to be the average of these normalized equities (remembering of course that a roll of 6x can occur in two different ways if x is not 6). The above rollout results yield the estimate

$$e \approx (2 \times (1.8074 + 1.8098 + 1.7722 + 1.7814 + 1.7793) + 1.8756)/11 = 1.7978.$$

If we make some heuristic assumptions about statistical independence, then we can even estimate the confidence interval as 1.7978 ± 0.0012 .

The rollouts above assumed that you own the cube. We can similarly let f denote your normalized equity conditional on rolling a 6, if your opponent owns the cube. The table below records the results of 5184-trial 4-ply/XGR+/Large rollouts, and yields the estimate $f \approx 1.7280 \pm 0.0011$.

Roll	Play	Normalized Equity
61	Bar/19 12/11*	1.7290 ± 0.0027
62	Bar/19 12/10*	1.7301 ± 0.0027
63	Bar/19 14/11*	1.7054 ± 0.0026
64	Bar/19 15/11*	1.7143 ± 0.0026
65	Bar/19 15/10*	1.7113 ± 0.0025
66	Bar/13*/7 19/13	1.8273 ± 0.0024

The Main Result

Having defined the conditional equities e and f , we can now state a mathematical theorem.

Theorem. Suppose that every money-game position has a finite expected value. Then $86e \geq 97f$.

Now, $97/86 \approx 1.128$, whereas our rollouts yield an estimate of $e/f \approx 1.040$. So if we believe that our rollout results are approximately correct, then the Theorem implies that in fact, not all money-game positions have a finite expected value.

The proof of the theorem proceeds in several stages. Throughout, we assume that equities exist and hence that “perfect play” makes sense.

Redouble/Pass Cannot Be Correct

Let us hypothesize, for the sake of argument, that the correct cube action in the Paradox Position is RD/P. That means that White’s equity is -2 . Suppose, though, that White takes the cube at 4. Then, 25/36 of the time, Red dances, and by the symmetry of the position, White is faced with the Paradox Position. *But by hypothesis, the correct cube action in the Paradox Position is RD/P*, so White’s equity

in this variation is +4. The other 11/36 of the time, Red enters, and it is hard to calculate White's equity exactly, but one thing we can say is that White has the option of adopting the strategy of refusing to redouble again under any circumstances. This may not be White's best strategy, but it is an available option, and if it is adopted, then at worst, White will lose 12 points. Therefore, White's equity if she takes is at worst $(25/36) \times 4 + (11/36) \times (-12) = -8/9$, which is better than -2 . That is, White does better by taking rather than passing, so passing cannot be correct, contrary to our hypothesis that the correct cube action is RD/P. Therefore, RD/P cannot be the correct cube action.

Redouble/Take Cannot Be Correct

Now let us hypothesize, for the sake of argument, that the correct cube action in the Paradox Position is RD/T, and that NRD is suboptimal. We can calculate your equity by splitting into (infinitely many) cases, depending on how many rolls it takes until someone enters from the bar. In Case 1, after you redouble to 4 and your opponent takes, you enter immediately with probability 11/36; the contribution to your equity from Case 1 is $(11/36) \times 4f$. In Case 2, you dance with probability 25/36, and then your perfect opponent, confronted with the Paradox Position, sends you an 8-cube, which you obligingly take, and then your opponent enters with probability 11/36. The sum of the contributions to your equity from Cases 1 and 2 is

$$(11/36) \times 4f - (25/36) \times (11/36) \times 8f.$$

Continuing in this way, we find that your equity is the sum of the infinite series

$$(11/36) \times 4f - (25/36) \times (11/36) \times 8f + (25/36)^2 \times (11/36) \times 16f - (25/36)^3 \times (11/36) \times 32f + \dots$$

From high-school algebra, we recognize the above infinite series as a geometric series, with the ratio between successive terms being $-50/36$. Since $-50/36 < -1$, the series diverges, which contradicts the assumption that equities are finite. So NRD must be an optimal cube decision for you.

(For the mathematically sophisticated reader, who may worry that if $f = 0$, then the series converges to 0, we make the following remark. If X denotes the terminal payoff of the game, then for the expected value $E[X]$ to be finite, the sum over all outcomes must be *absolutely* convergent; that is, $E[|X|]$ must also be finite. But the value of $|X|$ is the value of the cube, multiplied by 2 or 3 in the event of a gammon or backgammon; therefore, the series for $E[|X|]$ is at least

$$(11/36) \times 4 + (25/36) \times (11/36) \times 8 + (25/36)^2 \times (11/36) \times 16 + (25/36)^3 \times (11/36) \times 32 + \dots$$

and diverges to $+\infty$.)

No Redouble is Correct

The only remaining possibility is that NRD is optimal. Let E denote the equity of the Paradox Position if the player on roll holds a 2-cube, and let F be the equity of the Paradox Position if the opponent of the player on roll holds a 2-cube. Then, holding a 2-cube, you will either enter or dance, so

$$E = (11/36) \times 2e - (25/36) \times F.$$

Since NRD is the correct cube action, you gain nothing (and might lose something) by redoubling, but suppose you redouble anyway. A perfect opponent will take or pass, whichever is worse for you. A pass leaves you with equity 2; a take will leave you with equity $2F$. Could you be too good? That would mean $2F \geq 2$ and $E \geq 2$. But $e \leq 3$, since in the worst case your opponent could commit to passing any redouble and losing at most a backgammon, so

$$E = (11/36) \times 2e - (25/36) \times F \leq (11/36) \times 6 - (25/36) = 41/36,$$

contradicting $E \geq 2$. Therefore, you are not too good, and a perfect opponent will take your redouble, leaving you with equity $2F$. This cannot be better for you than NRD, so $2F \leq E$.

On the other hand, if your opponent holds a 2-cube, then again you either enter or dance, so

$$F = (11/36) \times 2f - (25/36) \times E.$$

Combining this equation with our earlier equation $E = (11/36) \times 2e - (25/36) \times F$, we find that

$$E = (72e - 50f)/61 \quad \text{and} \quad F = (72f - 50e)/61.$$

Therefore, the inequality $2F \leq E$ becomes

$$2 \times (72f - 50e) \leq 72e - 50f,$$

which simplifies to $86e \geq 97f$. This proves the Theorem.

Final Remarks

One might question whether the Paradox Position can ever arise in a perfectly played game of backgammon. It is not hard to show that the Paradox Position can arise in a *legally* played game of backgammon, and it seems plausible that there exists a sequence of rolls which, when played as XG recommends, leads to the Paradox Position. We leave the construction of such a sequence of rolls as a puzzle for the reader.

It seems hard to further extend our results to prove unconditionally that some positions have undefined equity, because it seems necessary to consider something like the Paradox Position, which is too complicated to analyze except with a large approximate computation. However, the XG rollouts provide strong enough evidence to convince almost everyone that if equities exist, then $86e < 97f$.

We should emphasize that even if we accept that $86e < 97f$, it does not quite follow that the Paradox Position itself has undefined equity. Our proof is a proof by contradiction, and hence shows only that the assumption that *all* positions (that arise from playing out the Paradox Position) have finite equity must be rejected. It does not show that any specific position has undefined equity.

Although the text of this article was entirely written by me, some of the arguments were found by ChatGPT 6 Pro. Specifically, I pointed ChatGPT to some of my old BGOnline posts on this topic, asking if it could improve them. ChatGPT cited Zare's article, which I was previously unaware of. Studying Zare's article caused me to notice some flaws in my BGOnline posts, but I suspected that combining my ideas with Zare's could lead to further progress, and ChatGPT confirmed my suspicion, in particular deriving the critical 97/86 ratio.