Irreducible Character of S_8

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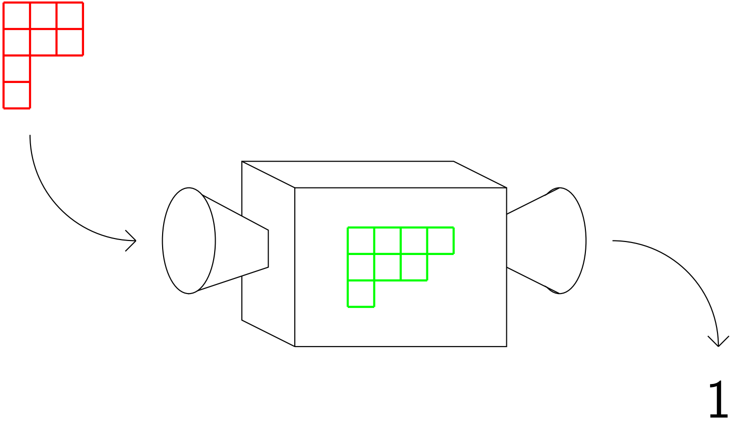
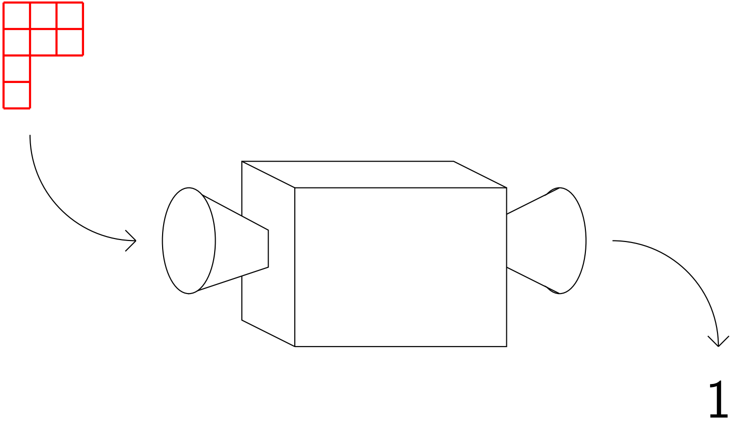


How Do You Reconstruct a Missing Label?



Murnaghan–Nakayama Rule

Border strip: the rightmost box in a row is below the leftmost box of the row above.

Border strip tableau (BST): numbers increase weakly across rows and down columns, and the boxes with a given number form a border strip.

1	1	2	2	2
1	2	2	5	
1	4	5	5	
3				

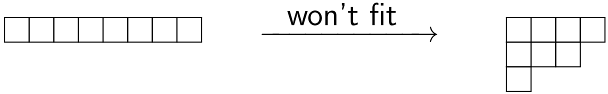
Theorem.

$$\chi_\lambda(\mu) = \sum_{\text{BST } T \,:\, \text{shape}(T)=\lambda,\ \text{type}(T)=\mu} \left(\prod_{\text{border strips } B \text{ of } T} (-1)^{\mathrm{height}(B)} \right)$$

Determining the Longest Hook Length

$$\chi_\lambda(\mu) = \sum_{\text{BST } T\,:\,\text{shape}(T)=\lambda,\ \text{type}(T)=\mu} \left(\prod_{\text{border strips } B \text{ of } T} (-1)^{\text{height}(B)} \right)$$

Suppose $|\lambda| = 8$, and μ is the partition with one part of size 8. A border strip of length 8 simply won't fit into λ unless λ itself is a hook.



If it won't fit, then there are no BST of shape λ and type μ, so $\chi_\lambda(\mu) = 0$.

Proposition. Let μ be the first in the sequence (n), $(n-1,1)$, $(n-2,1,1)$, $(n-3,1,1,1), \ldots$ such that $\chi_\lambda(\mu) \neq 0$. Then μ_1 equals the largest hook length of λ.

Sketch of Algorithm

Proposition. Let μ be the first in the sequence (n), $(n-1,1)$, $(n-2,1,1)$, $(n-3,1,1,1), \ldots$ such that $\chi_\lambda(\mu) \neq 0$. Then μ_1 equals the largest hook length of λ.

Forward Pass:

- The **principal** hook lengths can be iteratively determined from the "outside in" by a process similar to the one on the previous slide.

Backward Pass:

- The **arm lengths** and **leg lengths** can be iteratively determined from the "inside out" via a more complicated sequence of inputs (that are all still easy to compute).

The Backward Pass is difficult, in part because it is tricky to determine whether the arm is longer than the leg, or vice versa.

Results and Open Questions

- Our algorithm requires trying $O(n^{1.5})$ (easy-to-compute) inputs ("Algorithmically distinguishing irreducible characters of the symmetric group," by T. Chow and J. Paulhus, *Electronic J. Combin.* 28(2) (2021), #P2.5).
- In a recent preprint, Alex Miller has improved our result to n (easy-to-compute) inputs ("Character and class parameters from entries of character tables of symmetric groups," arXiv:2312.07267).
- Can the number of queries be reduced to $O(\log n)$ or even $O(1)$?