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Theorem 1. There are $\binom{4n}{2n}$ binary sequences of length $4n+1$ with the property that for all j, the jth occurrence of 10 appears in positions $4j+1$ and $4j+2$ or later (if it exists at all).

Problem 1. Find a simple combinatorial proof.

A more general result (Theorem 2 below) is proved in Chapman et al., *JCTA* **116** (2009) 205–214, but not in a way that directly associates a $2n$-element subset of a $4n$-element set with each binary sequence above.

Theorem 2. For $r \le 2n$, let $a(n,s,r)$ be the number of binary sequences of length $(s+2)n+1$ such that for all j, the jth occurrence of 10 (if it exists) appears in positions $(s+2)j+1$ and $(s+2)j+2$ or later, and such that the total number of occurrences of 10 and 01 is at most r. Then

$$a(n,s,r) = 2\binom{(s+2)n-1}{r} - (s-2)\sum_{i=0}^{r-1}\binom{(s+2)n-1}{i}.$$

Problem 2. Can this result be generalized? In general, $a(n,s,r)$ does not factor nicely, stymieing the standard technique of computing lots of specific values, and using the factorizations to guess a formula. Can something like the PSLQ algorithm be used to detect formulas that are simple linear combinations of binomial coefficients?

Given real numbers $x_1, \ldots, x_n$, the PSLQ algorithm can be used to find integers $a_1, \ldots, a_n$, not all zero, such that

$$a_1 x_1 + \cdots + a_n x_n = 0.$$

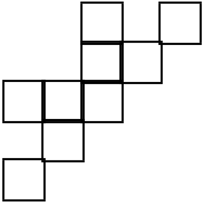
In our problem, we want to express a given large integer as a linear combination of binomial coefficients with "unexpectedly small" coefficients. This has the same flavor but details need to be worked out. For example how do we deal with the fact that every integer is technically a binomial coefficient?

A **stable set** of a graph is a subset of the vertices that are pairwise non-adjacent; i.e., the induced subgraph has no edges.

Given a partition λ, let G_λ be the graph whose vertices are the cells of the Young diagram of λ, and whose edges connect vertices in the same row or column.

Question. Does there always exist a partition of G_λ into disjoint stable sets $S_1, S_2, \ldots$ such that for all i, the cardinality of the union of *any* i stable sets of G_λ is at most $|S_1| + \cdots + |S_i|$?

A positive answer would settle the so-called wide partition conjecture. Note that the condition that λ be a partition cannot be generalized to an arbitrary subset of boxes; one can readily verify that the non-Young diagram on the next page does not work. For more info, see Chow et al., *Advances in Applied Math.* **31** (2003), 334–358.



Backup slides

Restatement in the language of lattice paths:

Definition. Let B_s be the staircase path that starts at $(0, 2)$, then takes $s + 1$ steps east, 2 steps north, s steps east, 2 steps north, s steps east, etc., always alternating between 2 steps north and s steps east except for the initial segment of $s + 1$ steps east.

Let $s = 2k$ be even. Then Theorem 2 implies:

Corollary 1. The number of lattice paths comprising $2(k + 1)n + 1$ east or north unit steps that start at $(0, 0)$ and that avoid touching or crossing B_{2k} is

$$\binom{2(k+1)n}{2n} - (k-1)\sum_{i=0}^{2n-1}\binom{2(k+1)n}{i}.$$