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REDUCTION OF ROTA'S BASIS CONJECTURE TO A PROBLEM ON THREE BASES*

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Abstract. It is shown that Rota's basis conjecture follows from a similar conjecture that involves just three bases instead of n bases.

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1. Introduction.

In 1989, Rota formulated the following conjecture, which remains open.

CONJECTURE 1 (Rota's basis conjecture). *Let M be a matroid of rank n on n^2 elements that is a disjoint union of n bases $B_1, B_2, \ldots, B_n$. Then there exists an $n \times n$ grid G containing each element of M exactly once, such that for every i the elements of B_i appear in the ith row of G and such that every column of G is a basis of M.*

Partial results toward this conjecture may be found in [1, 2, 3, 4, 5, 6, 7, 8, 12, 14, 15]. Now consider the following conjecture.

CONJECTURE 2. *Let M be a matroid of rank n on $3n$ elements that is a disjoint union of 3 bases. Let $I_1, I_2, \ldots, I_n$ be disjoint independent sets of M, with $0 \le |I_i| \le 3$ for all i. Then there exists an $n \times 3$ grid G containing each element of M exactly once, such that for every i the elements of I_i appear in the ith row of G and such that every column of G is a basis of M.*

The main purpose of the present note is to make the following observation.

THEOREM 3. *Conjecture 2 implies Conjecture 1.*

Our proof is inspired by the proof of Theorem 4 in [10].

Proof. Since Conjecture 1 is known if $n \le 2$, we may assume that $n \ge 3$. Let M be given as in the hypothesis of Conjecture 1. Define a *transversal* to be a subset $\tau \subseteq M$ that contains exactly one element from each B_i. Define a *double partition* of M to be a pair (β, τ) where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ is a partition of M into n pairwise disjoint bases β_i and $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$ is a partition of M into n pairwise disjoint transversals. Given a double partition (β, τ), define

$$\mu(\beta, \tau) = \sum_{i \neq j} |\beta_i \cap \tau_j|.$$

Observe that if $\mu(\beta, \tau) = 0$, then necessarily $\beta_i = \tau_i$ for all i, and then Rota's basis conjecture follows—just let the (i, j)th entry of G be $B_i \cap \tau_j$.

So let (β, τ) be an arbitrary double partition with $\mu(\beta, \tau) > 0$. We show how to construct a double partition (β', τ') with $\mu(\beta', \tau') < \mu(\beta, \tau)$; the proof is then complete, by infinite descent, since by hypothesis there exists at least one double partition. Since $\mu(\beta, \tau) > 0$, there exist β_i and τ_j with $i \neq j$ such that $\beta_i \cap \tau_j \neq \emptyset$. Since $n \ge 3$, there also exists k such that i, j, and k are all distinct. It will simplify

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notation to assume that $i = 1$, $j = 2$, and $k = 3$; no generality is lost, and it will be convenient to be able to reuse the index variables i and j below. Let $S = \beta_1 \cup \beta_2 \cup \beta_3$, let $T = \tau_1 \cup \tau_2 \cup \tau_3$, and let $M' = M|S$ (i.e., M restricted to the ground set S).

For each i, let $I_i = B_i \cap T \cap S$. Then I_i is an independent subset of the matroid M', and $|I_i| \le |B_i \cap T| \le 3$. The I_i are pairwise disjoint because the B_i are pairwise disjoint. Therefore, we may apply Conjecture 2 to obtain an $n \times 3$ grid G' whose columns β'_1, β'_2, and β'_3 are disjoint bases of M' (and therefore are bases of M) and whose ith row contains the elements of I_i.

To construct the desired double partition (β', τ'), let $\beta' = \beta$ except with β_1, β_2, and β_3 replaced with β'_1, β'_2, and β'_3, respectively. Similarly, let $\tau' = \tau$ except with τ_1, τ_2, and τ_3 replaced with τ'_1, τ'_2, and τ'_3, which are defined as follows. Let G'' be any $n \times 3$ grid whose ith row contains the elements of $B_i \cap T$ in some order, and whose (i, j)th entry agrees with that of G' whenever that entry is in I_i. Clearly G'' exists (though it may not be unique). Let τ'_j be the jth column of G'' for $j = 1, 2, 3$.

It is easily verified that what we have done is to regroup the elements of M' into three new bases and to regroup the elements of T into three new transversals in such a way that the contribution to $\mu(\beta', \tau')$ from intersections of the new bases with the new transversals is reduced to zero, and such that the total of the other contributions to μ is unchanged. Thus the overall value of μ is reduced, as required. □

Careful inspection of the above proof shows that it is easily adapted to prove a stronger statement than Theorem 3. Let $C(k)$ denote the statement obtained by replacing "3" with "k" throughout Conjecture 2. Then the above argument, mutatis mutandis, yields the following result.

THEOREM 4. *For any* $\ell \ge k \ge 2$, $C(k)$ *implies* $C(\ell)$.

In particular, proving $C(k)$ for any fixed k would prove Rota's basis conjecture (in fact, a stronger statement, namely, $C(n)$) for all n greater than or equal to that fixed k.

It is therefore natural to ask why we have formulated Conjecture 2 as $C(3)$ rather than as $C(2)$. The reason is that $C(2)$ is false. The simplest counterexample is a well-known stumbling block that is partly responsible for the fact that there is no known general "matroid union intersection theorem," i.e., a criterion for determining the minimum number of common independent sets that a set with two matroid structures on it can be partitioned into. Namely, take $M(K_4)$, the graphic matroid of the complete graph on four vertices, and let the I_i be the three pairs of nonincident edges of K_4. Another counterexample arises from a matroid that Oxley [11] calls J. Representing J by vectors in Euclidean 4-space, we can, for example, let

$$\begin{aligned} I_1 &= \{(-2,3,0,1),(0,0,1,1)\}, \\ I_2 &= \{(0,2,0,1),(1,0,3,1)\}, \\ I_3 &= \{(1,0,0,1),(0,1,2,1)\}, \\ I_4 &= \{(0,1,0,1),(4,0,0,1)\}. \end{aligned}$$

It may be possible to construct other examples from non–base-orderable matroids such as those in [9].

Despite these counterexamples to $C(2)$, we believe that Conjecture 2 is plausible. Using a database of matroids with nine elements kindly supplied by Gordon Royle [13], we have computationally verified Conjecture 2 for the case $n = 3$.

In an earlier version of this paper, the formulation of Conjecture 2 did not require the I_i to be independent. A counterexample to that version of the conjecture was found by Colin McDiarmid. Take the complete graph on the vertex set $\{1, 2, 3, 4\}$, and create an extra copy of the three edges incident to vertex 4. Call the edges $12, 13, 14, 23, 24, 34, 14', 24', 34'$, and let $I_1 = \{14, 14', 23\}$, $I_2 = \{24, 24', 13\}$, and $I_3 = \{34, 34', 12\}$. More generally, as pointed out by an anonymous referee, if k is odd, then a wheel with $k - 1$ copies of each of its k spokes yields a counterexample to $C(k)$ if the I_i are not required to be independent.

In closing, we speculate that Conjecture 2 might be provable using the following strategy. First, develop a modified version of $C(2)$ that says that the conclusion holds provided certain "obstructions" (such as $M(K_4)$ and J) are absent. Then use Rado's theorem (12.2.2 of [11]), or a suitable strengthening of it, to construct a first column of G in such a way that the remaining $2n$ elements are obstruction-free. Applying the modified version of $C(2)$ would then yield the desired result. The analysis of obstructions should hopefully be tractable since there are only three columns to consider.

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