

## REDUCTION OF ROTA’S BASIS CONJECTURE TO A PROBLEM ON THREE BASES\*

TIMOTHY Y. CHOW†

**Abstract.** It is shown that Rota’s basis conjecture follows from a similar conjecture that involves just three bases instead of  $n$  bases.

**Key words.** common independent sets, non–base-orderable matroid, odd wheel

**AMS subject classifications.** Primary, 05B20; Secondary, 15A03

**DOI.** 10.1137/080723727

**1. Introduction.** In 1989, Rota formulated the following conjecture, which remains open.

**CONJECTURE 1** (Rota’s basis conjecture). *Let  $M$  be a matroid of rank  $n$  on  $n^2$  elements that is a disjoint union of  $n$  bases  $B_1, B_2, \dots, B_n$ . Then there exists an  $n \times n$  grid  $G$  containing each element of  $M$  exactly once, such that for every  $i$  the elements of  $B_i$  appear in the  $i$ th row of  $G$  and such that every column of  $G$  is a basis of  $M$ .*

Partial results toward this conjecture may be found in [1, 2, 3, 4, 5, 6, 7, 8, 12, 14, 15]. Now consider the following conjecture.

**CONJECTURE 2.** *Let  $M$  be a matroid of rank  $n$  on  $3n$  elements that is a disjoint union of 3 bases. Let  $I_1, I_2, \dots, I_n$  be disjoint independent sets of  $M$ , with  $0 \leq |I_i| \leq 3$  for all  $i$ . Then there exists an  $n \times 3$  grid  $G$  containing each element of  $M$  exactly once, such that for every  $i$  the elements of  $I_i$  appear in the  $i$ th row of  $G$  and such that every column of  $G$  is a basis of  $M$ .*

The main purpose of the present note is to make the following observation.

**THEOREM 3.** *Conjecture 2 implies Conjecture 1.*

Our proof is inspired by the proof of Theorem 4 in [10].

*Proof.* Since Conjecture 1 is known if  $n \leq 2$ , we may assume that  $n \geq 3$ . Let  $M$  be given as in the hypothesis of Conjecture 1. Define a *transversal* to be a subset  $\tau \subseteq M$  that contains exactly one element from each  $B_i$ . Define a *double partition* of  $M$  to be a pair  $(\beta, \tau)$  where  $\beta = (\beta_1, \beta_2, \dots, \beta_n)$  is a partition of  $M$  into  $n$  pairwise disjoint bases  $\beta_i$  and  $\tau = (\tau_1, \tau_2, \dots, \tau_n)$  is a partition of  $M$  into  $n$  pairwise disjoint transversals. Given a double partition  $(\beta, \tau)$ , define

$$\mu(\beta, \tau) = \sum_{i \neq j} |\beta_i \cap \tau_j|.$$

Observe that if  $\mu(\beta, \tau) = 0$ , then necessarily  $\beta_i = \tau_i$  for all  $i$ , and then Rota’s basis conjecture follows—just let the  $(i, j)$ th entry of  $G$  be  $B_i \cap \tau_j$ .

So let  $(\beta, \tau)$  be an arbitrary double partition with  $\mu(\beta, \tau) > 0$ . We show how to construct a double partition  $(\beta', \tau')$  with  $\mu(\beta', \tau') < \mu(\beta, \tau)$ ; the proof is then complete, by infinite descent, since by hypothesis there exists at least one double partition. Since  $\mu(\beta, \tau) > 0$ , there exist  $\beta_i$  and  $\tau_j$  with  $i \neq j$  such that  $\beta_i \cap \tau_j \neq \emptyset$ . Since  $n \geq 3$ , there also exists  $k$  such that  $i, j$ , and  $k$  are all distinct. It will simplify

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\*Received by the editors May 8, 2008; accepted for publication (in revised form) September 29, 2008; published electronically January 7, 2009.

<http://www.siam.org/journals/sidma/23-1/72372.html>

†Center for Communications Research, 805 Bunn Drive, Princeton, NJ 08540 (tchow@mit.edu).

notation to assume that  $i = 1$ ,  $j = 2$ , and  $k = 3$ ; no generality is lost, and it will be convenient to be able to reuse the index variables  $i$  and  $j$  below. Let  $S = \beta_1 \cup \beta_2 \cup \beta_3$ , let  $T = \tau_1 \cup \tau_2 \cup \tau_3$ , and let  $M' = M|S$  (i.e.,  $M$  restricted to the ground set  $S$ ).

For each  $i$ , let  $I_i = B_i \cap T \cap S$ . Then  $I_i$  is an independent subset of the matroid  $M'$ , and  $|I_i| \leq |B_i \cap T| \leq 3$ . The  $I_i$  are pairwise disjoint because the  $B_i$  are pairwise disjoint. Therefore, we may apply Conjecture 2 to obtain an  $n \times 3$  grid  $G'$  whose columns  $\beta'_1$ ,  $\beta'_2$ , and  $\beta'_3$  are disjoint bases of  $M'$  (and therefore are bases of  $M$ ) and whose  $i$ th row contains the elements of  $I_i$ .

To construct the desired double partition  $(\beta', \tau')$ , let  $\beta' = \beta$  except with  $\beta_1$ ,  $\beta_2$ , and  $\beta_3$  replaced with  $\beta'_1$ ,  $\beta'_2$ , and  $\beta'_3$ , respectively. Similarly, let  $\tau' = \tau$  except with  $\tau_1$ ,  $\tau_2$ , and  $\tau_3$  replaced with  $\tau'_1$ ,  $\tau'_2$ , and  $\tau'_3$ , which are defined as follows. Let  $G''$  be any  $n \times 3$  grid whose  $i$ th row contains the elements of  $B_i \cap T$  in some order, and whose  $(i, j)$ th entry agrees with that of  $G'$  whenever that entry is in  $I_i$ . Clearly  $G''$  exists (though it may not be unique). Let  $\tau'_j$  be the  $j$ th column of  $G''$  for  $j = 1, 2, 3$ .

It is easily verified that what we have done is to regroup the elements of  $M'$  into three new bases and to regroup the elements of  $T$  into three new transversals in such a way that the contribution to  $\mu(\beta', \tau')$  from intersections of the new bases with the new transversals is reduced to zero, and such that the total of the other contributions to  $\mu$  is unchanged. Thus the overall value of  $\mu$  is reduced, as required.  $\square$

Careful inspection of the above proof shows that it is easily adapted to prove a stronger statement than Theorem 3. Let  $C(k)$  denote the statement obtained by replacing “3” with “ $k$ ” throughout Conjecture 2. Then the above argument, mutatis mutandis, yields the following result.

**THEOREM 4.** *For any  $\ell \geq k \geq 2$ ,  $C(k)$  implies  $C(\ell)$ .*

In particular, proving  $C(k)$  for any fixed  $k$  would prove Rota’s basis conjecture (in fact, a stronger statement, namely,  $C(n)$ ) for all  $n$  greater than or equal to that fixed  $k$ .

It is therefore natural to ask why we have formulated Conjecture 2 as  $C(3)$  rather than as  $C(2)$ . The reason is that  $C(2)$  is false. The simplest counterexample is a well-known stumbling block that is partly responsible for the fact that there is no known general “matroid union intersection theorem,” i.e., a criterion for determining the minimum number of common independent sets that a set with two matroid structures on it can be partitioned into. Namely, take  $M(K_4)$ , the graphic matroid of the complete graph on four vertices, and let the  $I_i$  be the three pairs of nonincident edges of  $K_4$ . Another counterexample arises from a matroid that Oxley [11] calls  $J$ . Representing  $J$  by vectors in Euclidean 4-space, we can, for example, let

$$\begin{aligned} I_1 &= \{(-2, 3, 0, 1), (0, 0, 1, 1)\}, \\ I_2 &= \{(0, 2, 0, 1), (1, 0, 3, 1)\}, \\ I_3 &= \{(1, 0, 0, 1), (0, 1, 2, 1)\}, \\ I_4 &= \{(0, 1, 0, 1), (4, 0, 0, 1)\}. \end{aligned}$$

It may be possible to construct other examples from non-base-orderable matroids such as those in [9].

Despite these counterexamples to  $C(2)$ , we believe that Conjecture 2 is plausible. Using a database of matroids with nine elements kindly supplied by Gordon Royle [13], we have computationally verified Conjecture 2 for the case  $n = 3$ .

In an earlier version of this paper, the formulation of Conjecture 2 did not require the  $I_i$  to be independent. A counterexample to that version of the conjecture was

found by Colin McDiarmid. Take the complete graph on the vertex set  $\{1, 2, 3, 4\}$ , and create an extra copy of the three edges incident to vertex 4. Call the edges  $12, 13, 14, 23, 24, 34, 14', 24', 34'$ , and let  $I_1 = \{14, 14', 23\}$ ,  $I_2 = \{24, 24', 13\}$ , and  $I_3 = \{34, 34', 12\}$ . More generally, as pointed out by an anonymous referee, if  $k$  is odd, then a wheel with  $k - 1$  copies of each of its  $k$  spokes yields a counterexample to  $C(k)$  if the  $I_i$  are not required to be independent.

In closing, we speculate that Conjecture 2 might be provable using the following strategy. First, develop a modified version of  $C(2)$  that says that the conclusion holds provided certain “obstructions” (such as  $M(K_4)$  and  $J$ ) are absent. Then use Rado’s theorem (12.2.2 of [11]), or a suitable strengthening of it, to construct a first column of  $G$  in such a way that the remaining  $2n$  elements are obstruction-free. Applying the modified version of  $C(2)$  would then yield the desired result. The analysis of obstructions should hopefully be tractable since there are only three columns to consider.

**Acknowledgments.** I wish to thank Jonathan Farley, Patrick Brosnan, and James Oxley for useful discussions, and a referee for correcting an error in my counterexample based on  $J$ .

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