On Galvin's Proof of the Dinitz Conjecture

G. W. Peck*, Math Dept., MIT

Abstract: We present a shortened version of Galvin's elegent short proof of the Dinitz conjecture on list colorability of a product of two n vertex complete graphs, that is suitable for presentation to undergraduates or high school students.

*research supported in part by NSA ....

An n by n Latin square can be described as an assignment of integers from 1 to n to the squares of an n by n chessboard, so that the Latin condition holds: no number appears twice in any row or column. Latin squares of any size are easy to produce; for example we can assign the number (i+j)(modn)+1 to the square in the ith row and jth column

Dinitz raised the following question: suppose each square of an n by n chessboard has a list of n numbers, which lists may bear arbitrary relation to one another. Can one always assign a number to each square, so as to preserve the Latin condition while choosing the number in any square from the list associated with that square?

Since it is so easy to construct Latin squares, which correspond to having the list (1,....,n) at each square, and since allowing lists to vary does not seem to make the construction problem more difficult, it is intuitively sensible that the answer should be yes. It was surprising that nobody was able to resolve this conjecture for a number of years. The proof below is however so easy that one wonders how it could have been passed over for so long.

An affirmative answer is equivalent to the statement which we will not define here, that the direct product of two copies of the complete graph on n vertices is n list-colorable.

We prove the claim as a corollary of a version of Gale's Stable Marriage Theorem, which we now describe and, for completeness, prove. It is usually described in human terms which makes it very easy to describe the proot. For our purposes read row for boy and column for girl in the discussion below.

Suppose there are n boys and n girls, each boy is acquainted with a subset of the girls and vice versa. Suppose further that each person ranks those members of the opposite sex that he or she knows. A legal set of marriages is a collection of boy-girl pairs that avoids bigamy while pairing each paired boy to a girl he knows and vice versa. Such a set is called unstable if any boy-girl pair who are not married to one another, say Alfred and Beth prefer one another to their respective mates, Anne and Bryan. The stable marriage theorem is the statement that there is always a stable set of marriages, that is one without any instability.

The traditional proof of this theorem is by describing a constructive procedure that produces a stable set of marriages. Each boy in the first round proposes to the girl at the top of his list. Each girl with more than one proposal rejects all but the one of these she likes best. Each rejected boy crosses the rejector off his list and all the boys again propose to the perhaps new favorite on the next round. Since some list gets shorter with each rejection, eventually rejections must stop, at which point each girl has at most one suitor. who she then marries. The resulting set of marriages is stable.

(Notice that some girls may get no suitors, and some men may exhaust their lists with only rejections, and they will then remain single in this stable set of marriages.)

It is obvious that this set of marriages avoids bigamy. That it is stable is also easy. If Alfred, wed to Anne (or single), prefers Beth to Anne (or singleness), he must have proposed to Beth and been rejected. But then Beth must have married someone who she prefers to Alfred; otherwise she would not have rejected him. This argument proves the theorem.

We now prove Dinitz' conjecture. We first choose a Latin square of entries in our n by n places. We then use it to define a system of preferences among the rows (boys) and columns (girls). If the (i,j)th entry in the Latin square is bigger than the (i,k)th we say that row i prefers column k to column j. For reference imagine that we draw an arrow from the (i,j) th box to the (i,k)th if this happens, that is if row i preferes column k to j.

Similarly if the (j,i)th entry in our Latin square is smaller than the (k,i)th we say that column i prefers row k to row j, and draw an arrow from (j,i) to (j,k).

We can verify that there are exactly n-1 arrows emanating from each square; it the square (i,j) has value t in our Latin square, there are t-1 arrows to squares with smaller value in the ith row, and n-t arrows to squares with larger values in the jth column.

We now give a constructive proof of Dinitz conjecture by describing a procedure, in this framework, that will always produce an assignment of numbers with the desired property.

We order the numbers that are on any of the lists in numerical order as a1,a2, ...

1. Find a stable set of marriages given that only rows and columns whose intersection square's list includes a1 know each other so that only such squares can marry; among these use the preferences described by the arrows defined above restricted to such squares.

2. Assign a1 to the squares in this stable set of marriages.

3. Remove all other entries from the lists at the squares in this stable set.

Repeat these steps in turn with a1 replaced by a2, a3, ...

This procedure obviously leads to an assignment of aj's to squares of our n by n board. The Latin condition holds because the stable set of marriages obtained for each aj is by definition non-bigamous. Each pair receives an assignment on its list, since all marriages for any aj were restricted to pairs that had aj on their list. Our proof is therefore complete if we show that every pair (i,j) is married for some ak.

For each entry ak on the list at (i,j) the condition of stability tells us that i must prefer its mate in the stable set for ak to j, or j must prefer its mate in this stable set to i. In either case there must be an arrow out of (i,j) pointing to a square in that stable marriage set which receives the assignment ak.

But there are only n-1 arrows emanating from (i,j) in our arrow assignment, and there are n ak's on (i,j)'s list. By a version of the pigeonhole principal, (i,j) must be married for some ak, and must therefore receive an assignment, which completes our proof.

This proof is essentially the same as that of Galvin, but it avoids definitions and inductive arguments, which, while simple enough (and useful for generalizations), are unnecessary here.

References

Alon and x Tarsi , ~~Galvin~~ Galvin

Dinitz