Addendum to "Polynomials for Directed Graphs"

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We regret that an incomplete manuscript of the article *Polynomials for directed graphs* appeared in the last volume of this journal [Congressus Numerantium **94** (1993), pp. 187-201]. The following page should have appeared between those numbered 193 and 194 in the published paper.

from a root. (See Proposition 2.5.)

Chung and Graham [4] have introduced an interesting polynomial invariant of unrooted digraphs, the *cover polynomial.* Unlike the polynomials already mentioned in this section, the cover polynomial satisfies a deletion/contraction property somewhat similar to (T3), though it does have the property that the appropriate notion of contraction is not symmetric: the initial and terminal vertices of the edge being contracted are treated differently.

4. Two order-dependent polynomials.

In the discussion subsequent to Proposition 2.6, it is noted that complete recursive descriptions of f_2 and f_3 are not possible. Nevertheless, the notions of 2-isthmus and 2-loop are similar to the notions of isthmus and loop in a graph or matroid. Recall that an edge e in a rooted digraph D is a 2-isthmus iff e is in every maximal $*$ rooted arborescence (i.e., e is in every 2-basis), and is a 2-loop iff it is in no maximal $*$ rooted arborescence. We will define a polynomial on rooted digraphs recursively, using these notions of isthmus and loop. We will also distinguish between 2-loops which are *ordinary loops* (i.e., those in which the initial and terminal vertices coincide), and 2-loops which are not, which we will call *reversed loops.* (The term "loop" means "ordinary loop".)

We now define a polynomial $f_7(D) = f_7(D, O; x, y, z)$ associated to a rooted digraph D whose underlying undirected graph is connected, with respect to an ordering O on $E(D)$. This definition is based on the recursive definition (T3).

Definition 4.1.
(a) If $D = \{*\}$, then $f_7(D) = 1$.
(b) Let e be the first edge (in the ordering O) which emanates from $*$.
1. *$f_7(D) = xf_7(D/e)$ if e is a 2-isthmus.*
2. *$f_7(D) = yf_7(D - e)$ if e is a loop.*
3. *$f_7(D) = f_7(D - e) + f_7(D/e)$ otherwise.*

(c) If no edge emanates from $*$, then let e be the first edge directed into $*$, i.e., e is a reversed loop. Then $f_7(D) = zf_7(D/e)$.

The next proposition can be proven using the definition and induction.

Proposition 4.2. *Suppose D has a spanning arborescence rooted at $*$. Then for any ordering O, $f_7(D)$ is a polynomial in x and y (i.e., no z term appears).*

If D has no spanning $*$ rooted arborescence, then it is still easy to describe the behavior of the variable z. Let $R(D,*)$ denote the set of vertices in D which are reachable from $*$. Let H be the induced rooted subdigraph on $R(D,*)$ and G/H the rooted subdigraph obtained from D by contracting all of H to the single vertex $*$.

Lemma 4.3. $f_7(D) = f_7(H)f_7(G/H)$.