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Unit Interval Orders and the Dot Action on the Cohomology of Regular Semisimple Hessenberg Varieties

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Abstract. Motivated by a 1993 conjecture of Stanley and Stembridge, Shareshian and Wachs conjectured that the characteristic map takes the dot action of the symmetric group on the cohomology of a regular semisimple Hessenberg variety to $\omega X_G(t)$, where $X_G(t)$ is the chromatic quasisymmetric function of the incomparability graph G of the corresponding natural unit interval order, and ω is the usual involution on symmetric functions. We prove the Shareshian–Wachs conjecture. Our proof uses the local invariant cycle theorem of Beilinson–Bernstein–Deligne to obtain a surjection from the cohomology of a regular Hessenberg variety of Jordan type λ to a space of local invariant cycles; as λ ranges over all partitions, these spaces collectively contain all the information about the dot action on a regular semisimple Hessenberg variety. Using a palindromicity argument, we show that in our case the surjections are actually isomorphisms, thus reducing the Shareshian–Wachs conjecture to computing the cohomology of a regular Hessenberg variety. But this cohomology has already been described combinatorially by Tymoczko; we give a bijective proof (using a generalization of a combinatorial reciprocity theorem of Chow) that Tymoczko's combinatorial description coincides with the combinatorics of the chromatic quasisymmetric function.

Keywords: chromatic quasisymmetric function, indifference graph, local invariant cycles, palindromic

1 Introduction

Let G be the incomparability graph of a unit interval order (also known as an *indifference graph*), i.e., a finite graph whose vertices are closed unit intervals on the real line, and whose edges join overlapping unit intervals. It is a longstanding conjecture [16] related to various deep conjectures about immanants that if G is such a graph, then

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the so-called *chromatic symmetric function* X_G studied by Stanley [17] is e-positive, i.e., a nonnegative combination of elementary symmetric functions. (In fact, Stanley and Stembridge conjectured something seemingly more general, but Guay-Paquet [8] has reduced their conjecture to the one stated here.) Early on, Haiman [10] proved that the expansion of X_G in terms of Schur functions has nonnegative coefficients, and Gasharov [6] showed that these coefficients enumerate certain combinatorial objects known as *P-tableaux*. It is well known that if χ is a character of the symmetric group S_n, then the image of χ under the so-called characteristic map ch

$$\operatorname{ch} \chi := \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma)\, p_{\text{cycletype}(\sigma)} \tag{1.1}$$

(where p here denotes the power-sum symmetric function) is a nonnegative linear combination of Schur functions, with the coefficients giving the multiplicities of the corresponding irreducible characters of S_n. One may therefore suspect that X_G is the image under ch of the character of some naturally occurring representation of S_n, but until recently, there was no candidate, even conjecturally, for such a representation.

Meanwhile, independently and seemingly unrelatedly, De Mari, Procesi, and Shayman [5] inaugurated the study of *Hessenberg varieties*. Let $\mathbf{m} = (m_1, m_2, \ldots, m_{n-1})$ be a weakly increasing sequence of positive integers satisfying $i \le m_i \le n$ for all i, and let $s : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation. The (type A) Hessenberg variety $\mathscr{H}(\mathbf{m}, s)$ is defined by

$$\mathscr{H}(\mathbf{m}, s) := \{\text{complete flags } F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n : sF_i \subseteq F_{m_i} \text{ for all } i\}. \tag{1.2}$$

The geometry of a Hessenberg variety depends on the Jordan form of s. If the Jordan blocks have distinct eigenvalues then we say that $\mathscr{H}(\mathbf{m}, s)$ is *regular*, and if s is diagonalizable then we say that $\mathscr{H}(\mathbf{m}, s)$ is *semisimple*. Of particular interest to us is that there is a representation, called the *dot action*, of S_n on the cohomology of regular semisimple Hessenberg varieties. To the best of our knowledge, this dot action was first defined by Tymoczko, who asked for a complete description of it [20]; e.g., one can ask if there is a combinatorial formula for the multiplicities of the irreducible representations and/or for the character values. (Note that Tymoczko defines the dot action in terms of something called the *moment graph*; the moment graph, and hence the dot action, depends only on $\mathbf{m}$ and not on the choice of regular semisimple s.)

A connection between these two apparently unrelated topics has been conjectured by Shareshian and Wachs [12, 13]. Motivated by the e-positivity conjecture, they have generalized X_G to something they call the *chromatic quasisymmetric function* $X_G(t)$ of a graph, which is a polynomial in t with power series coefficients that reduces to X_G when $t = 1$. They also noted that if we are given a sequence $\mathbf{m}$ as above, and we let $G(\mathbf{m})$ be the undirected graph on the vertex set $\{1, 2, \ldots, n\}$ such that i and j are adjacent if

$i < j \le m_i$, then $G(\mathbf{m})$ is an indifference graph, and moreover that every indifference graph is isomorphic to some $G(\mathbf{m})$. They then made the following conjecture. Let ω denote the usual involution on symmetric functions [15, Section 7.6].

Conjecture 1. *If $\chi_{\mathbf{m},d}$ denotes the dot action on the cohomology group H^{2d} of the regular semisimple Hessenberg variety $\mathscr{H}(\mathbf{m}, s)$, then* $\operatorname{ch}\chi_{\mathbf{m},d}$ *equals the coefficient of t^d in $\omega X_{G(\mathbf{m})}(t)$.*

This conjecture is intriguing not only because it would answer Tymoczko's question, but it would open up the possibility of proving the e-positivity conjecture by geometric techniques.

The main result of the present paper is a proof of Conjecture 1. The linchpin of our proof is the following result.

Theorem 1. *Let λ be a partition of n. Let s be a regular element with Jordan type λ, and let $S_\lambda := S_{\lambda_1} \times \cdots \times S_{\lambda_\ell}$ be a Young subgroup of the symmetric group S_n. Consider the restriction of $\chi_{\mathbf{m},d}$ to S_λ. Then the dimension of the subspace fixed by S_λ equals the Betti number β_{2d} of $\mathscr{H}(\mathbf{m}, s)$.*

It is a standard fact (Proposition 1 below) from the representation theory of S_n that the dimension of the subspace fixed by S_λ in a representation χ is the coefficient of m_λ in the monomial symmetric function expansion of $\operatorname{ch}\chi$, and therefore these dimensions completely determine χ. So Theorem 1 reduces the computation of the dot action on the cohomology of a regular semisimple Hessenberg variety to the computation of the cohomology of regular (but not necessarily semisimple) Hessenberg varieties. However, this latter task has already been largely carried out by Tymoczko [19], who has given a combinatorial description of the Betti numbers β_{2d} for all Hessenberg varieties in type A. So with Theorem 1 in hand, all that remains to prove Conjecture 1 is to give a bijection between Tymoczko's combinatorial description and the combinatorics of $\omega X_{G(\mathbf{m})}(t)$. To do this, we first compute the coefficients $c_{d,\lambda}(\mathbf{m})$ of $t^d m_\lambda$ in the monomial symmetric function expansion of $\omega X_{G(\mathbf{m})}(t)$. We do this with a generalization of a combinatorial reciprocity theorem of Chow (Theorem 2). This yields a description of $c_{d,\lambda}(\mathbf{m})$ that is almost, but not quite, identical to Tymoczko's description of β_{2d}; we show that the descriptions are equivalent by describing a explicit bijection between the two (Theorem 4). As a corollary (Corollary 2), we derive the fact that the Betti numbers of regular Hessenberg varieties form a palindromic sequence (even though the varieties are not smooth), because Shareshian and Wachs have proved that $\omega X_{G(\mathbf{m})}(t)$ is palindromic.

In this extended abstract, we give complete definitions and theorem statements (without proofs, which of course are in the full version) of the combinatorial ingredients described above. There is no space to provide details of the geometric ingredients, or even a complete definition of the dot action, so we limit ourselves to the following brief comments about the proof of Theorem 1. The idea is to show that Tymoczko's dot action coincides with the monodromy action for the family $\mathscr{H}^{\mathrm{rs}}(\mathbf{m}) \to \mathfrak{g}^{\mathrm{rs}}$ of Hessenberg varieties over the space of regular semisimple $n \times n$ matrices. This allows us to apply results from the theory of local systems and perverse sheaves to questions involving the dot action. In particular, the local invariant cycle theorem of Beilinson–Bernstein–Deligne [3, Corollaire 6.2.9] implies that there is a surjective map from the cohomology of a regular Hessenberg variety to the space of local invariants of the monodromy action near a regular element s in the space $\mathfrak{g}$ of all $n \times n$-matrices. We prove a general theorem showing that the local invariant cycle map is an isomorphism if and only if the Betti numbers of the special fiber are palindromic in a suitable sense; but the Betti numbers *are* palindromic in our case (by Corollary 2), so the local invariant cycle map is indeed an isomorphism in our case. Then we show that the local invariant cycles near a regular element s with Jordan type λ coincide with S_λ invariants of the dot action on the regular semisimple Hessenberg variety. The latter fact is proved by a monodromy argument that uses the Kostant section. (This Kostant section argument and some other ingredients of the proof were inspired by Ngô's paper on the Hitchin fibration [11].)

1.1 Previous work and acknowledgments

Prior to our work, Conjecture 1 was already known for some graphs G: a complete graph (trivial), a complete graph minus an edge [18], a complete graph minus a path of length three (Tymoczko, unpublished), and a path (by piecing together known results as explained in [13]). Teff also showed that it would suffice to prove the conjecture for all *connected* graphs G. In a different direction, Abe, Harada, Horiguchi and Masuda [2] proved that the multiplicity of the trivial representation is indeed as predicted by Conjecture 1. (Hearing about this development and reading the last paragraph of [1], which explains how to compute the multiplicity of the trivial representation in terms of the regular nilpotent Hessenberg variety, partially inspired our own proof.) They also computed the ring structure on regular semisimple Hessenberg varieties of type $(m_1, n, \ldots, n)$, and deduced Conjecture 1 in that case from the computation.

Shortly after this work was completed, Guay–Paquet [9] announced an independent proof of Conjecture 1 using completely different methods.

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2 Preliminaries

We fix some notation that will be used throughout the paper.

2.1 General notation

We let $\mathbb{P}$ denote the positive integers. If $n \in \mathbb{P}$, we let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

The vector $\mathbf{m} = (m_1, \ldots, m_{n-1})$ will always denote a *Hessenberg function*, by which we mean a sequence of positive integers satisfying

1. $m_1 \leq m_2 \leq \cdots \leq m_{n-1} \leq n$, and
2. $m_i \geq i$ for all i.

We also define

$$|\mathbf{m}| := \sum_{i=1}^{n-1} (m_i - i). \tag{2.1}$$

Given $\mathbf{m}$, let $P(\mathbf{m})$ be the poset on the vertex set $[n]$ whose order relation $\prec$ is given by

$$i \prec j \iff j \in \{m_i + 1, m_i + 2, \ldots, n\}.$$

Such a poset is called a *natural unit interval order*. (It is a theorem that every unit interval order in the sense defined in the introduction is isomorphic to some natural unit interval order.) The *incomparability graph* $G(\mathbf{m})$ is the undirected graph on the vertex set $[n]$ in which i and j are adjacent if and only if i and j are incomparable in $P(\mathbf{m})$. In other words, if $i < j$ then i and j are adjacent in $G(\mathbf{m})$ if and only if $j \leq m_i$.

An *integer partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of a positive integer n is a weakly decreasing sequence of positive integers that sum to n, while a *composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ of a positive integer n is any sequence of positive integers that sum to n. We visualize a composition of n by drawing vertical bars in some subset of the $n - 1$ spaces between consecutive objects in a horizontal line of n objects; the parts α_i are then the numbers of objects between successive bars. Motivated by the equivalence between compositions and sets of bars, we define the following (possibly not quite standard) notation: $|\alpha|$ for the number of bars of α; $\overline{\alpha}$ for the composition that has bars in precisely the positions where α does *not* have bars; and $\alpha \leq \beta$ if the bars of α are a subset of the bars of β.

We write S_n for the symmetric group. If S_n acts in the usual way on a set of size n, and α is a composition of n, then the *Young subgroup* S_α is the subgroup

$$S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_\ell} \subseteq S_n \tag{2.2}$$

comprising all the permutations that permute the first α_1 elements among themselves, the next α_2 elements among themselves, and so on.

An *ordered (set) partition* $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_\ell)$ of a finite set S is a sequence of pairwise disjoint subsets of S whose union is S.

A *sequencing* q of a finite set S of cardinality n is a bijective map $q : [n] \to S$. It is helpful to think of q as the sequence $q(1), \ldots, q(n)$ of elements of S.

By a *digraph* we mean a finite directed graph with no loops or multiple edges but that may have bidirected edges, i.e., it may contain both $u \to v$ and $v \to u$ simultaneously. If D is a digraph, we write $\overline{D}$ for the *complement* of D, i.e., the digraph with the same vertex set as D but with a directed edge $u \to v$ if and only if there does *not* exist a directed edge $u \to v$ in D.

2.2 Symmetric and quasisymmetric functions

We mostly follow the notation of Stanley [15] for symmetric functions. For convenience, we recall some of that notation here. Let $\mathbf{x} = \{x_1, x_2, x_3, \ldots\}$ be a countable set of independent indeterminates. If $\kappa : [n] \to \mathbb{P}$ is a map then we write $\mathbf{x}_\kappa$ for the monomial $x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}$. A formal power series in $\mathbf{x}$ is a *symmetric function* if it is invariant under any permutation of the variables $\mathbf{x}$. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is an integer partition, then the *monomial symmetric function* m_λ is the symmetric function of minimal support that contains the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_\ell^{\lambda_\ell}$.

The *characteristic map* ch is a function that sends characters χ of the symmetric group to symmetric functions via the formula

$$\operatorname{ch} \chi := \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma)\, p_{\mathrm{cycletype}(\sigma)} \tag{2.3}$$

where cycletype(σ) is the integer partition consisting of the cycle sizes of σ, listed with multiplicity in weakly decreasing order, and p denotes the power-sum symmetric function. As we explained in the introduction, the following standard fact is an important ingredient in our proof.

Proposition 1. *Let ρ be a finite-dimensional representation of S_n, and let χ be its character. Let* $\operatorname{ch} \chi = \sum_\lambda c_\lambda m_\lambda$ *be the monomial symmetric function expansion of* $\operatorname{ch} \chi$. *Then c_λ equals the dimension of the subspace fixed by any Young subgroup $S_\lambda \subseteq S_n$. In particular, knowing c_λ for all λ uniquely determines χ.*

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition of n. The *monomial quasisymmetric function* M_α is the formal power series defined by

$$M_\alpha := \sum_{i_1 < \cdots < i_\ell} x_{i_1}^{\alpha_1} \cdots x_{i_\ell}^{\alpha_\ell}, \tag{2.4}$$

where the sum is over all strictly increasing sequences $(i_1, \ldots, i_\ell)$ of positive integers. A formal power series is a *quasisymmetric function* if it is a scalar linear combination of monomial quasisymmetric functions. Note that symmetric functions are always quasisymmetric, but not vice versa.

The *fundamental quasisymmetric function* F_α of Gessel [7] is defined by

$$F_\alpha := \sum_{\beta \geq \alpha} M_\beta. \tag{2.5}$$

By inclusion-exclusion,

$$M_\alpha = \sum_{\beta \geq \alpha} (-1)^{|\beta|-|\alpha|} F_\beta. \tag{2.6}$$

2.3 Hessenberg varieties

As mentioned in the introduction, if $\mathbf{m}$ is a Hessenberg function and $s : \mathbb{C}^n \to \mathbb{C}^n$ is a linear transformation, then we define the *Hessenberg variety* (of type A, which is the only type that we consider in this paper) by

$$\mathscr{H}(\mathbf{m}, s) := \{\text{complete flags } F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n : sF_i \subseteq F_{m_i} \text{ for all } i\}.$$

If the Jordan blocks of s have distinct eigenvalues then we say that $\mathscr{H}(\mathbf{m}, s)$ is *regular*, if s is diagonalizable then we say that $\mathscr{H}(\mathbf{m}, s)$ is *semisimple*, and if s is nilpotent then we say that $\mathscr{H}(\mathbf{m}, s)$ is *nilpotent*. Since $\mathscr{H}(\mathbf{m}, s)$ can equal $\mathscr{H}(\mathbf{m}, s')$ for $s \neq s'$ (e.g., if $s' - s$ is a constant), this is a very minor abuse of terminology.

Hessenberg varieties are projective as they are closed subschemes of the projective variety of complete flags. If s is regular semisimple then they are smooth, but in general they may be singular, and sometimes not reduced. This explains why we cannot simply cite a duality theorem to prove palindromicity.

3 The chromatic quasisymmetric function

Given a graph G whose vertex set is a subset of $\mathbb{P}$, Shareshian and Wachs [13] define the *chromatic quasisymmetric function* $X_G(\mathbf{x}, t)$ of G.

Definition 1. *Let G be a graph whose vertex set V is a finite subset of $\mathbb{P}$. Let $C(G)$ denote the set of all proper colorings of G, i.e., the set of all maps $\kappa : V \to \mathbb{P}$ such that adjacent vertices are always mapped to distinct positive integers. Then*

$$X_G(\mathbf{x}, t) := \sum_{\kappa \in C(G)} t^{\operatorname{asc} \kappa} \mathbf{x}_\kappa, \tag{3.1}$$

where

$$\operatorname{asc} \kappa := |\{\{u, v\} : \{u, v\} \textit{ is an edge of } G \textit{ and } u < v \textit{ and } \kappa(u) < \kappa(v)\}|.$$

For brevity, we sometimes write $X_G(t)$ for $X_G(\mathbf{x}, t)$. It will be convenient for us to restate the definition of $X_G(t)$ in terms of monomial quasisymmetric functions.

Proposition 2. *Let G be a graph whose vertex set V is a finite subset of $\mathbb{P}$. Then*

$$X_G(\mathbf{x}, t) = \sum_{\sigma=(\sigma_1,\ldots,\sigma_\ell)} t^{\operatorname{asc}\sigma} M_{|\sigma_1|,\ldots,|\sigma_\ell|}, \tag{3.2}$$

where the sum is over all ordered partitions σ of V such that every σ_i is a stable set of G (i.e., there is no edge between any two vertices of σ_i), and $\operatorname{asc}\sigma$ is the number of edges $\{u, v\}$ of G such that $u < v$ and v appears in a later part of σ than u does.

We remark that if we set $t = 1$ then the chromatic quasisymmetric function specializes to the *chromatic symmetric function* X_G of Stanley [17].

3.1 Reciprocity

If f is a symmetric function, then a "reciprocity theorem," loosely speaking, is a result that gives a combinatorial interpretation of ωf, where ω is a well-known involution on symmetric functions [15, Section 7.6]. Since Conjecture 1 concerns $\omega X_G(t)$ rather than $X_G(t)$ itself, one might expect a reciprocity theorem to be relevant. This is indeed the case. Specifically, the coefficients of the monomial symmetric function expansion of $\omega X_G(t)$ play an important role in our arguments, so we now introduce some notation for them.

Definition 2. *Given a Hessenberg function $\mathbf{m}$, we let $c_{d,\lambda}(\mathbf{m})$ be the coefficients defined by the following expansion of $\omega X_{G(\mathbf{m})}(\mathbf{x}, t)$ in terms of monomial symmetric functions:*

$$\omega X_{G(\mathbf{m})}(\mathbf{x}, t) = \sum_d t^d \sum_\lambda c_{d,\lambda}(\mathbf{m}) m_\lambda. \tag{3.3}$$

Our starting point is the observation that Chow [4, Theorem 1] has proved a reciprocity theorem for a symmetric function invariant of a digraph called the *path-cycle symmetric function* Ξ_D. There is a certain precise sense in which Ξ_D is equivalent to Stanley's X_G in the case of posets, but the nice thing about reciprocity for Ξ_D is that it naturally yields a combinatorial interpretation for the coefficients of the monomial symmetric function expansion of $\omega\Xi_D$, which is not immediately evident from Stanley's reciprocity theorem [17, Theorem 4.2] for X_G. This fact suggests the following plan: Generalize Ξ_D to $\Xi_D(t)$ (just as Shareshian and Wachs have generalized X_G to $X_G(t)$), prove reciprocity for $\Xi_D(t)$, and read off the desired combinatorial interpretation of $c_{d,\lambda}(\mathbf{m})$. This plan works, and we now show how to carry it out.

We define the *path quasisymmetric function* $\Xi_D(\mathbf{x}, t)$ of a digraph D; as its name suggests, it enumerates paths only and not cycles (since for our present purposes we do not care about enumerating cycles), and it has a definition analogous to that of the chromatic quasisymmetric function.

Definition 3. *Let D be a digraph whose vertex set V is a subset of $\mathbb{P}$. An* ordered path cover *of D is an ordered pair (q, β) such that q is a sequencing of V, $\beta = (\beta_1, \ldots, \beta_\ell)$ is a composition of $n := |V|$, and*

$$q(\beta_{i-1}+1) \to q(\beta_{i-1}+2) \to \cdots \to q(\beta_i)$$

is a directed path in D for all $i \in [\ell]$ (adopting the convention that $\beta_0 = 0$). Define

$$\Xi_D(\mathbf{x}, t) := \sum_{(q,\beta)} t^{\operatorname{asc} q} M_\beta \tag{3.4}$$

where the sum is over all ordered path covers (q, β) of D and $\operatorname{asc} q$ *is the number of pairs $\{u, v\}$ of vertices of D such that*

1. *either $u \to v$ and $v \to u$ are both edges of D or neither one is,*
2. *$u < v$, and*
3. *v appears later in the sequencing q than u does.*

For brevity, we sometimes write $\Xi_D(t)$ for $\Xi_D(\mathbf{x}, t)$.

Although we are ultimately interested in expansions in terms of monomial *symmetric* functions, it turns out that the proofs are more naturally stated in terms of monomial *quasisymmetric* functions. So we need to describe the action of ω on monomial quasisymmetric functions.

Definition 4. *The linear map ω on quasisymmetric functions is defined by the following action on monomial quasisymmetric functions.*

$$\omega M_\beta := (-1)^{|\beta|} \sum_{\alpha \le \beta} M_\alpha. \tag{3.5}$$

It is known (e.g., see the proof of [17, Theorem 4.2]) that the usual map ω is characterized by the equation $\omega F_\alpha = F_{\overline{\alpha}}$, so the following proposition confirms that our definition of ω coincides with the standard one.

Proposition 3. $\omega F_\alpha = F_{\overline{\alpha}}$.

We are ready for the reciprocity theorem for $\Xi_D(t)$.

Theorem 2. *Let D be a digraph whose vertex set V is a subset of $\mathbb{P}$. Then $\omega \Xi_D(\mathbf{x}, t) = \Xi_{\overline{D}}(\mathbf{x}, t)$.*

Theorem 2 gives us a nice combinatorial interpretation of $c_{d,\lambda}(\mathbf{m})$.

Corollary 1. *Let $\mathbf{m}$ be a Hessenberg function, and let $D(\mathbf{m})$ denote the digraph on $[n]$ that has an edge $u \to v$ if and only if $v \prec u$ in P. Then for any composition α whose parts are a permutation of the parts of λ, $c_{d,\lambda}(\mathbf{m})$ equals the number of ordered path covers (q, α) of $\overline{D(\mathbf{m})}$ with* $\operatorname{asc} q = d$.

4 Betti numbers of regular Hessenberg varieties

The main result of this section is that if $\mathscr{H}(\mathbf{m}, s)$ is a regular Hessenberg variety and s has Jordan type λ, then its Betti number β_{2d} equals $c_{d,\lambda}(\mathbf{m})$.

Tymoczko [19, Theorem 7.1] has already done a lot of the work needed to prove this result, by showing that Hessenberg varieties admit a paving (or cellular decomposition) by affine spaces, and obtaining a combinatorial interpretation of the dimensions of the cells. For regular Hessenberg varieties, Tymoczko's theorem simplifies as follows. If λ is an integer partition of n then by a *tableau of shape* λ we mean any filling of the boxes of the Young diagram of λ with one copy each of the numbers $1, 2, \ldots, n$.

Theorem 3 (Tymoczko)**.** *Let $\mathscr{H}(\mathbf{m}, s)$ be a regular Hessenberg variety and let the partition λ encode the sizes of the Jordan blocks of s. Then $\mathscr{H}(\mathbf{m}, s)$ is paved by affines. The nonempty cells are in bijection with tableaux T of shape λ with the property that k appears in the box immediately to the left of j only if $k \le m_j$. The dimension of a nonempty cell is the sum of:*

1. *the number of pairs i, k in T such that*
   - *(a) i and k are in the same row,*
   - *(b) i appears somewhere to the left of k,*
   - *(c) $k < i$, and*
   - *(d) if j is in the box immediately to the right of k then $i \le m_j$;*
2. *the number of pairs i, k in T such that*
   - *(a) i appears in a lower row than k, and*
   - *(b) $k < i \le m_k$.*

It remains for us to establish a correspondence between the combinatorics of Theorem 3 and the combinatorics of $\omega X_{G(\mathbf{m})}(t)$, or equivalently (by the results of the previous section) the combinatorics of ordered path covers.

Definition 5. *If X is a topological space and i is an integer, we write β_i or $\beta_i(X)$ for the i-th Betti number* $\dim H^i(X, \mathbb{C})$ *of X.*

Theorem 4. *Let $\mathscr{H}(\mathbf{m}, s)$ be a regular Hessenberg variety and let the Jordan type of s be λ. Then the Betti number β_{2d} of $\mathscr{H}(\mathbf{m}, s)$ equals $c_{d,\lambda}(\mathbf{m})$, and $\beta_i = 0$ for i odd.*

Theorem 4 is proved bijectively, by matching up the combinatorics of ordered path covers with the above combinatorial description by Tymoczko.

Shareshian and Wachs proved palindromicity for the chromatic quasisymmetric function, so we can now deduce the following palindromicity result for Hessenberg varieties.

Corollary 2. *Let $\mathscr{H}(\mathbf{m}, s)$ be a regular Hessenberg variety with s of type λ as in Theorem 4. Set*

$$q = q_{\mathscr{H}(\mathbf{m},s)} := \sum_{i \in \mathbb{Z}} \beta_i t^{i-|\mathbf{m}|}.$$

Then $q(t) = q(t^{-1})$.

This concludes the combinatorial portion of our proof. At this point in the full version of the paper, we continue with the geometric portion of the proof, but in this extended abstract, we do not have the space to say anything about the geometric portion beyond what we already sketched in the introduction.

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