The Erasing Marks Conjecture (Timothy Y. Chow, December 2019)

The Stanley–Stembridge e-positivity conjecture for $(\mathbf{3}+\mathbf{1})$-free posets is still open as of this writing but has been proved in some special cases. In particular, in his 1995 paper, Stanley already proved it for a graph that is a path. However, even in this case, I know of no explicit description of a basis for equivariant cohomology with the property that the dot action simply permutes the basis elements. An inductive construction can be extracted from C. Procesi's paper, "The toric variety associated to Weyl chambers" (*Mots*, Lang. Raison. Calc., Hermès, Paris, 1990, pp. 153–161), but it is not very explicit.

In 2015, I formulated the conjecture below (although it was originally not stated as cleanly as it is here). It remains open as far as I know. It is not too hard to show, using for example the work of J. R. Stembridge, "Eulerian numbers, tableaux, and the Betti numbers of a toric variety" (*Discrete Math.* **99** (1992), 307–320), that the conjectured basis has the correct cardinality, so it "only" remains to show that it is linearly independent.

Let S_n denote the symmetric group.

Definition. A *mark set* is a subset $m := \{m_1, m_2, \ldots, m_k\} \subset \{1, 2, \ldots, n-1\}$. By convention we assume that $m_1 < m_2 < \cdots < m_k$.

Definition. Given a mark set m, let $Y(m)$ denote the Young subgroup

$$S_{m_1} \times S_{m_2-m_1} \times S_{m_3-m_2} \times \cdots \times S_{n-m_k}.$$

Let $Y(m)$ act on S_n on the right (i.e., on positions). Call the orbits of this action *m-orbits*.

Example. Let $n = 9$ and let $m = \{1, 2, 4, 7, 8\}$. If we write elements of S_n in one-line notation then we can visualize each m_i as "marking" the space between the m_ith and the $(m_i + 1)$st letter (with a vertical bar, say), and we can visualize $Y(m)$ as permuting elements between marks. So the following elements of S_n are in the same m-orbit:

$$3|6|85|947|1|2 \sim 3|6|58|497|1|2$$

Definition. If $\pi \in S_n$ and m is a mark set, let $p(\pi, m)$ be the polynomial in t defined by the formula

$$p(\pi, m) := \prod_{i=1}^{k} \big(t_{\pi(m_i)} - t_{\pi(m_i+1)}\big).$$

Definition. If m is a mark set then its *erasure* $e(m)$ is the mark set defined by

$$e(m) := \{i \in m : i \neq 1 \text{ and } i-1 \notin m\}.$$

Erasing Marks Conjecture. For each mark set m and each $e(m)$-orbit C, define an equivariant cohomology class of a regular semisimple permutahedral Hessenberg variety (i.e., $h_i = i + 1$ for all i) by putting (on the moment graph) $p(\pi, m)$ at every $\pi \in C$, and putting 0 at every other vertex. Then these classes form a basis for equivariant cohomology.