ELSEVIER Discrete Mathematics 145 (1995) 73–82

DISCRETE MATHEMATICS

On the Dinitz conjecture and related conjectures

Timothy Y. Chow

143 Albany Street #002C, Cambridge, MA 02139, USA

Received 13 December 1993

Abstract

We present previously unpublished elementary proofs by Dekker and Ottens (1991) and Boyce (private communication) of a special case of the Dinitz conjecture. We prove a special case of a related basis conjecture by Rota, and give a reformulation of Rota's conjecture using the Nullstellensatz. Finally we give an asymptotic result on a related Latin square conjecture.

1. Introduction

For each positive integer n, let $D(n)$ be the following claim: 'For each pair of integers i and j such that $1 \leqslant i,j \leqslant n$, let S_{ij} be any set with n (distinct) elements. Then we can pick one element a_{ij} from each set S_{ij} such that the a_{ij} form a partial Latin square, i.e., $a_{ij} \neq a_{ij'}$ for all i and all $j \neq j'$ and $a_{ij} \neq a_{i'j}$ for all j and all $i \neq i'$'. The following innocent-looking conjecture by Dinitz (see [7]) has eluded solution for 15 years.

Conjecture 1. $D(n)$ is true for all n.

For each positive integer n, let $R(n)$ be the following statement: 'If $B_1, \ldots, B_n$ are n bases of $\mathbb{C}^n$, not necessarily distinct or disjoint, then there exists an $n \times n$ matrix such that the elements in the ith row are precisely the elements of B_i and such that for each j the elements in the jth column form a basis'. Rota [13] made the following conjecture in 1989.

Conjecture 2. $R(n)$ is true for all n.

For each positive integer n, let $A(n)$ be the statement that the number of even $n \times n$ Latin squares is not equal to the number of odd $n \times n$ Latin squares, and let $H(n)$ be the statement that the number of row-even $n \times n$ Latin squares is not equal to the number of row-odd $n \times n$ Latin squares. (A Latin square is even or odd according to whether the product of the signs of all its row and column permutations is $+1$ or -1;

SSDI 0012-365X(94)00055-N

'row-odd' and 'row-even' are defined the same way except that only row permutations are considered.) We have the following conjectures.

Conjecture 3. $A(n)$ is true for all even n.

Conjecture 4. $H(n)$ is true for all even n.

Conjecture 3 was first posed by Alon and Tarsi [1], who also showed that if n is an even integer then $A(n)$ implies $D(n)$. Huang et al. [10] have shown that if n is an even integer then $A(n)$ and $H(n)$ are equivalent, and $A(n)$ implies $R(n)$. The equivalence of $A(n)$ and $H(n)$ is also shown in [11].

All of the above conjectures are still open, although some partial results are known. If $n \leqslant 2$ everything is trivial. As noted in Chetwynd and Häggkvist [3], $D(3)$ has been verified by a case-by-case analysis. Alon and Tarsi have verified that $A(n)$ is true for $n = 4$ and $n = 6$. The conjecture has also been verified by computer for $n = 8$.

Let $R'(n)$ denote the statement $R(n)$ with '$\mathbb{C}^n$' replaced by 'a rank n matroid'. Chan [4] has verified $R'(3)$, and Wild [14] has verified $R'(n)$ for the special case of strongly base-orderable matroids.

In this paper, we shall present the following:

1. Previously unpublished proofs by Dekker and Ottens [5] and Boyce [2] of a special case of Conjecture 1, namely where $S_{ij} = S_{ij'}$ for all i,j,j'. (Boyce's proof also appears in [11].) This special case was first proved by Rota but his proof uses advanced techniques of supersymmetric algebra and is also unpublished. The proofs we present are entirely elementary. Part of the reason for giving the proofs here is to provide an adequate reference, since none exists at present.
2. A proof of $R(3)$.
3. A new conjecture that is equivalent to Conjecture 2.
4. An asymptotic result that suggests that the number of row-even Latin squares and the number of row-odd Latin squares are asymptotically equal.

2. Special case of Conjecture 1

Theorem 1. *Let n be a positive integer and let $S_1, \ldots, S_n$ be n sets (not necessarily distinct or disjoint), each with n (distinct) elements. Then there exists an $n \times n$ matrix such that the elements in the ith row are precisely the elements of S_i and such that for each j the elements in the jth column are all distinct.*

Clearly this is equivalent to the special case described in the introduction.

Proof 1 (Dekker and Ottens [5]). The basic idea is to fill in the matrix row by row, rearranging previously placed elements if conflicts arise.

The elements of S_1 can be placed in the first row in an arbitrary order. Let us now assume that the first $i-1$ rows $(2 \leqslant i \leqslant n)$ have been filled in accordance with the conditions of the theorem. We now fill in the ith row with the elements of S_i. Let $a_1, \ldots, a_n$ be the elements of S_i. We place a_1 in the first column, a_2 in the second column, and so on, until a conflict occurs, i.e., for some j and some $r < i$, the element a_j appears in column j and row r, so that there are two occurrences of a_j in column j (one in row i and one in row r), violating the conditions of the theorem. We now proceed to rearrange some of the elements in the first $i-1$ rows in such a way that no two elements in the same column are alike. If we can do this we will be done, for then we can continue placing elements, rearranging elements in previous rows if necessary, until all the elements are placed.

Simple counting shows that there must exist a column, say column j', such that a_j does not appear in column j'. Note that $j \neq j'$. Switch the elements in columns j and j' in row r. This relieves the conflict between the two a_j's but may create new conflicts in columns j and j'.

We now carry out the following switching procedure. Let a *pair* consist of an element in column j of one of the first $i-1$ rows together with the element in column j' of the same row. Define *switched pairs* and *unswitched pairs* in the obvious way. (The pair in row r is the unique switched pair at this point.) A step of the switching procedure consists of switching every unswitched pair that conflicts with some switched pair. (Note that this changes unswitched pairs to switched pairs, and this may create new conflicts.) The procedure terminates when no unswitched pair conflicts with a switched pair. The procedure must eventually terminate since there are only finitely many rows.

We claim that when the procedure terminates, the conditions of the theorem are satisfied. To check this, we need only consider columns j and j'. The procedure guarantees that there is no conflict between an unswitched pair and a switched pair. There can also be no conflict between two unswitched pairs or between two switched pairs, because if there were, there would have been a conflict before the switching procedure began, contrary to the induction hypothesis. There can be no conflict between the element a_j in column j with any other element in column j because the only other occurrence of a_j in columns j and j' was moved to column j' at the first step. So it remains only to show that the element in row i and column j' (call it b; if $j < j'$ so that b does not exist we are done, so assume b exists) does not conflict with any other element in column j'.

We prove the stronger statement that at no point during the switching procedure is there a conflict in column j'. This is clearly true before any switching and also after the first switch, in row r, because initially a_j did not appear in column j'. Suppose as an induction hypothesis that at some stage of the switching procedure there are no conflicts in column j'. Some unswitched pairs may conflict with switched pairs (let P be the set of these switched pairs that conflict); by assumption the conflicts must occur in column j. The switching procedure flips these unswitched pairs. These pairs cannot create conflicts in column j' after flipping, for they cannot conflict with other switched pairs (otherwise they would have conflicted before the switching procedure began) and they cannot conflict with elements in column j' that have not been switched, including the element b (otherwise these unswitched elements would have conflicted with pairs in the set P before the switching procedure began). Thus there are no conflicts in column j' after the switches are made. Hence by induction there is never a conflict in column j' and the proof is complete. □

Proof 2 (Boyce [2]). Generalize the theorem: suppose we have n sets $S_1, \dots, S_n$ with k elements each and no element appears in more than k sets. Then we claim that there is an $n \times k$ matrix with the following properties.

1. The elements in the ith row of the matrix are precisely the elements of S_i.
2. The elements in each column of the matrix are all distinct.

We proceed by induction on k. Clearly $k = 1$ is trivial. To complete the induction we need to show that if we have n sets with k elements each, and each element occurs in no more than k sets, then there is a system of distinct representatives that includes the elements that appear exactly k times, i.e., then there is a set $S = \{a_1, \dots, a_n\}$ such that $a_i \in S_i$, the a_i are all distinct, and every element that appears in exactly k of the sets is in S.

Suppose there are r elements $a_1, \dots, a_r$ that occur exactly k times each. Given any $s \leqslant r$ of these elements, note that at least s of the sets S_i contain at least one of these s elements, for each set contains only k elements and there are sk occurrences of the s elements. Thus by the well-known Hall marriage theorem (see, e.g., [9, Theorem 5.1.1]), we can find r distinct sets $S_{i_1}, \dots, S_{i_r}$ such that $a_j \in S_{i_j}$ for $1 \leqslant j \leqslant r$. This gives us a partial system S' of distinct representatives.

Next note that given any s sets, their union must contain at least k distinct elements. Applying the Hall marriage theorem again, we can find a set S as desired. (Here we need the part of the Hall marriage theorem that says that if S' is a partial system of distinct representatives and a complete system of distinct representatives exists, then there is a complete system of distinct representatives that includes the elements of S', though not necessarily associating them with the same sets. See for example [9, Theorem 5.1.3].) This proves the generalized version of the theorem, and setting $k = n$ we are done. □

3. Special case of Conjecture 2

Theorem 2. *Let V be a vector space of dimension 3, and let B_1, B_2, and B_3 be three bases of V. Then there exists a 3×3 matrix such that the elements in the ith row are precisely the elements of B_i and such that for each j the elements in the jth column form a basis.*

Proof. The proof makes heavy use of coordinates. All coordinates will be taken with respect to the basis B_1. Let (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be the coordinates of the three elements of B_2. We proceed by filling in the 3×3 matrix row by row.

Without loss of generality we fill in the first row from left to right with the elements $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ of B_1 in that order. There are $3! = 6$ ways to fill in the second row with the elements of B_2. Let us denote the six different partially filled-in matrices that result by $F^1, F^2, \ldots, F^6$. For each k, denote the (i,j)th entry of F^k by f^k_{ij}.

For each k, let $n^k_j = f^k_{1j} \times f^k_{2j}$, for $1 \leqslant j \leqslant 3$, where $\times$ here denotes vector cross product. We claim that if for some k the vectors n^k_1, n^k_2 and n^k_3 are linearly independent, then we are done. To see this, let $B_3 = \{v_1, v_2, v_3\}$ and for each i let S_i be the set of columns j such that $\{f^k_{1j}, f^k_{2j}, v_i\}$ is linearly independent. (Note that this is equivalent to the set of columns j such that the corresponding n^k_j is not perpendicular to v_i.) Now each S_i contains at least one member, for if v is one of the elements of B_3, then at least one of the n^k_j is not perpendicular to v, since v is nonzero (being a member of the basis B_3) and the n^k_j are linearly independent by hypothesis. Next, given any pair $\{S_i, S_{i'}\}$ of the S's, note that their union contains at least two members. For otherwise there must be a pair of the n^k_j such that v_i and $v_{i'}$ are both perpendicular to both n^k_j in this pair, implying that v_i and $v_{i'}$ are parallel, contradicting the fact that B_3 is a basis. Finally, the union of all the S_i's must contain all three columns, for otherwise one of the n^k_j must be perpendicular to all the elements of the basis B_3—i.e., one of the n^k_j must be zero, contradicting the assumption that the n^k_j are linearly independent. Thus by the Hall marriage theorem, we can place the elements of B_3 in the last row in such a way as to fulfill the conditions of the theorem, and we are done, as claimed.

So let us assume that for all k, the set $\{n^k_1, n^k_2, n^k_3\}$ is linearly dependent, i.e., the determinant of the matrix whose jth column is the coordinates of n^k_j is zero. This gives us six equations. Computing explicitly, we find that these six equations are equivalent to

$$x_1y_2z_3 = x_2y_3z_1 = x_3y_1z_2 \quad \text{and} \quad x_1y_3z_2 = x_2y_1z_3 = x_3y_2z_1.$$

Let $a = x_1y_2z_3$, i.e., the common value of the first three expressions above, and let $b = x_1y_3z_2$, the common value of the second three expressions above. Note that

$$a^3 = x_1x_2x_3y_1y_2y_3z_1z_2z_3 = b^3.$$

Now B_2 is a basis, so the determinant of the matrix whose columns are the coordinates of the elements of B_2 is nonzero. This gives us the equation

$$x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_1y_3z_2 - x_2y_1z_3 - x_3y_2z_1 \neq 0,$$

from which it follows that $a \neq b$. (Since $a^3 = b^3$, this gives us a contradiction in fields with unique cube roots, e.g., the reals, and completes the proof in this case, but in general we are not yet done.) In particular, none of the nine coordinates of the elements of B_2 is zero, because that would imply $a^3 = b^3 = 0$ and hence a $a = b = 0$.

Next we claim that n^k_j and $n^k_{j'}$ are nonparallel whenever $j \neq j'$. For example, choose the k such that

$$f^k_1 = (x_1, y_1, z_1), \qquad f^k_2 = (x_2, y_2, z_2), \qquad f^k_3 = (x_3, y_3, z_3).$$

Then for this k,

$$n_1^k = (0, -z_1, y_1), \qquad n_2^k = (z_2, 0, -x_2), \qquad n_3^k = (-y_3, x_3, 0).$$

Clearly the only way for two of these n_j^k to be parallel is for one of the x_i, y_i or z_i to be zero, which we have already shown is impossible. Similar arguments apply to all the F^k.

For each k, let $d^k = n_1^k \times n_2^k$. We have just shown that n_1^k and n_2^k are nonparallel, so $d^k \neq 0$, and d^k is a normal to the plane spanned by the n_j^k. Direct computation shows that the coordinates of the six d^k are

$$(x_2z_1, y_1z_2, z_1z_2), \quad (x_2z_3, y_3z_2, z_3z_2), \quad (x_1z_2, y_2z_1, z_2z_1),$$

$$(x_1z_3, y_3z_1, z_3z_1), \quad (x_3z_1, y_1z_3, z_1z_3), \quad (x_3z_2, y_2z_3, z_2z_3).$$

We claim that no two of these six vectors are parallel. For example, suppose (x_2z_1, y_1z_2, z_1z_2) and (x_1z_2, y_2z_1, z_2z_1) are parallel. Since their third coordinates are equal, this implies that they are equal. In particular, $x_2z_1 = x_1z_2$. Multiplying both sides by y_3, we obtain $a = b$, a contradiction. Or suppose that (x_2z_1, y_1z_2, z_1z_2) and (x_2z_3, y_3z_2, z_3z_2) are parallel. Multiply the former vector by z_3 and the latter vector by z_1 and compare coordinates to deduce $y_1z_3 = y_3z_1$ (recall that we have shown that the z_i cannot be zero). Multiplying both sides by x_2 again yields $a = b$, a contradiction. The remaining cases are handled similarly.

Thus, for at least one value of k, d^k is not parallel to any of the vectors in B_3. It follows from this that for this k, each vector in B_3 is perpendicular to at most one of the n_j^k (recall that no two of the n_j^k are parallel). A Hall marriage theorem argument similar to that given for the case where the n_j^k are linearly independent now shows that for this k, the third row of F^k can be filled in with the elements of B_3 in such a way that the conditions of the theorem are satisfied, thus completing the proof. □

4. Reformulation of Conjecture 2

We begin with some general definitions. Fix a positive integer n. Let S be the set of all n-tuples $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ where each σ_i is a permutation of n. If $M = (M_{i,j})$ is a matrix and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is an element of S, then define M^σ to be the matrix whose (i,j) entry $M_{i,j}^\sigma$ is $M_{i,\sigma_i(j)}$. (Intuitively, M^σ is obtained by letting σ_i permute the ith row of M.)

Now let $\{x_{ijk}\}_{1 \leq i,j,k \leq n}$ be a set of n^3 independent indeterminates, and let R be the polynomial ring $\mathbb{C}[\{x_{ijk}\}]$. Let A be the matrix whose (i,j) entry $A_{i,j}$ is the vector $(x_{ij1}, x_{ij2}, \ldots, x_{ijn})$. For each $\sigma \in S$, let

$$f_\sigma = \prod_{j=1}^{n} \det_{ik}(A_{ijk}^\sigma),$$

where $\det_{ik}(A^{\sigma}_{ijk})$ denotes the element of R obtained by taking the determinant of the matrix whose (i,k) entry is the kth coordinate of $A^{\sigma}_{i,j}$. (Intuitively, f_σ is the 'product of the determinants of the columns of A^{σ}'.) Next let

$$d = \prod_{i=1}^{n} \det_{jk}(A_{ijk}),$$

where $\det_{jk}(A^{\sigma}_{ijk})$ is the determinant of the matrix whose (j,k) entry is the kth coordinate of $A_{i,j}$. (Intuitively, d is the 'product of the determinants of the rows of A'.) Finally, let $C(n)$ be the statement that d^r lies in the ideal generated by $\{f_\sigma\}_{\sigma\in S}$ for some positive integer r. We then have the following theorem, which immediately yields a reformulation of Conjecture 2.

Theorem 3. *For any positive integer n, $C(n)$ is equivalent to $R(n)$.*

Proof. We can restate $R(n)$ as follows: 'For each $i \in \{1,2,\ldots,n\}$, let B_i be a set of n vectors in $\mathbb{C}^n$. Let B be the matrix whose (i,j) entry is the jth vector of B_i. If for every $\sigma \in S$ at least one column of B^{σ} is not a basis, then at least one B_i is not a basis'. Now the property of not being a basis is equivalent to the vanishing of an associated determinant. From this fact, it follows directly from our definitions that $R(n)$ is equivalent to the following statement: 'If P is a point in $\mathbb{C}^{n^3}$ such that f_σ vanishes at P for all $\sigma \in S$, then d also vanishes at P'. In the language of (classical) algebraic geometry, this is equivalent to the statement that d is in the ideal of variety of the ideal generated by the f_σ. By the Nullstellensatz, this is equivalent to $C(n)$, since $\mathbb{C}$ is algebraically closed. (See, for example, [8] for an explanation of this terminology and a proof of the Nullstellensatz.) □

Notice that Theorem 3 holds even for odd n. In principle one can now resolve $R(n)$ for any particular n by a finite computation using Gröbner basis techniques. However, even for small n the computations are prohibitively large.

5. Asymptotic enumeration of row-odd and row-even Latin rectangles

A $k \times n$ Latin rectangle is an array of k rows and n columns with the integers $\{1,2,\ldots,n\}$ in each row and all distinct integers in each column. Row-odd and row-even Latin rectangles are defined in the obvious way. Let $r_o(k,n)$ and $r_e(k,n)$ be the number of row-odd and row-even $k \times n$ Latin rectangles, respectively.

Theorem 4. *Fix any $\varepsilon > 0$. Then for all sufficiently large n, the inequality*

$$\left|\frac{r_o(k,n)}{r_e(k,n)} - 1\right| < \varepsilon$$

holds for all $k <$ $(\log n)^{3/2-\varepsilon}$.

We shall need a result of Erdős and Kaplansky [6].

Proposition 1. *Fix $\varepsilon > 0$. Then there exists a positive constant c such that for all sufficiently large n the following holds: if $k < (\log n)^{3/2-\varepsilon}$ and L is any $k \times n$ Latin rectangle, then the number N of ways of adding a new row to L (while preserving the Latin rectangle property) satisfies*

$$\left|\frac{Ne^k}{n!} - 1\right| < \frac{1}{n^c}.$$

Proof of Theorem 4. The proof is a slight adaptation of the method used by Erdős and Kaplansky [6] to enumerate *all* Latin rectangles. Fix ε and choose c as in Proposition 1. Suppose L is a $k \times n$ Latin rectangle. Let N be the number of ways of adding a new row to L while preserving the Latin rectangle property and let N_e (N_o) be the number of these ways where the new row is an even (odd) permutation. Let A_r be the number of ways of choosing r distinct integers in L with no two in the same column. To compute N, we begin with the total number $n!$ of choices for the new row, and then we subtract those permutations having a clash with L in a given column — summed over all choices of that column, and then we must reinstate those having clashes in two given columns, etc. By the inclusion–exclusion principle, this yields

$$N = \sum_{r=0}^{n} (-1)^r A_r (n-r)!,$$

since $(n-r)!$ is the number of permutations σ of n such that $\sigma(i)$ has been prespecified for r values of i.

We can similarly use the method of inclusion and exclusion to evaluate N_e and N_o. The point here is that if we prespecify r values of a permutation of n, and $n - r \geqslant 2$, then the number of even permutations equals the number of odd permutations — just take any two values that have not been prespecified and exchange them to obtain a bijection between the even and odd permutations. However, if $n - r < 2$, then such a bijection is not available. We can thus write

$$N_e = \left(\sum_{r=0}^{n} \frac{(-1)^r A_r (n-r)!}{2}\right) + \delta_e = \frac{N}{2} + \delta_e,$$

where δ_e is a term that corrects the error arising from the last two terms in the summation. For our present purposes it is enough for us to note that

$$|\delta_e| \leqslant \frac{A_{n-1} + A_n}{2} \leqslant \frac{nk^{n-1} + k^n}{2}.$$

Similarly, $N_o = N/2 + \delta_o$ where $|\delta_o| \leqslant (nk^{n-1} + k^n)/2$.

The idea now is to show that the error δ_e is negligible. Now if $k < (\log n)^{3/2-\varepsilon}$ (or even if $k < n^{1-\varepsilon}$), then

$$\begin{aligned}\left|\frac{\delta_e e^k}{n!}\right| &\leqslant \frac{(nk^{n-1} + k^n)e^k}{2n!} \\ &< \frac{(n^{n(1-\varepsilon)+\varepsilon} + n^{n(1-\varepsilon)}) \exp(n^{1-\varepsilon})}{2n!}\end{aligned}$$

$$\leqslant \frac{n^{\varepsilon} n^{n(1-\varepsilon)} \exp(n^{1-\varepsilon}) \exp(n)}{n^n}$$

$$= \frac{n^{\varepsilon} \exp(n + n^{1-\varepsilon})}{n^{n\varepsilon}}$$

$$\leqslant \exp(-\varepsilon(n-1)\log n + 2n).$$

(Note that we used Stirling's approximation in the third line.) Hence for all sufficiently large n,

$$\begin{aligned}\left|\frac{2N_e e^k}{n!} - 1\right| &= \left|\frac{(N + 2\delta_e)e^k}{n!} - 1\right| \\ &\leqslant \left|\frac{Ne^k}{n!} - 1\right| + 2\left|\frac{\delta_e e^k}{n!}\right| \\ &< \frac{1}{n^c} + 2\exp(-\varepsilon(n-1)\log n + 2n),\end{aligned}$$

by Proposition 1. Now $1/n^c = \exp(-c\log n)$ for any positive constant c, and this tends to zero much more slowly than $\exp(-\varepsilon n \log n)$. More precisely, there exists a positive constant $c' > c$ such that

$$\frac{1}{n^c} + 2\exp(-\varepsilon(n-1)\log n + 2n) < \frac{1}{n^{c'}}$$

for all sufficiently large n. From this we see that the analogue of Proposition 1 with N replaced by N_e and $e^k/n!$ replaced by $2e^k/n!$ is true. We can argue exactly the same way for N_o in place of N_e. It is not hard to see that Theorem 4 now follows by taking the product of the N_e over the k rows of the Latin rectangle and noting that the number of even and odd Latin rectangles remains asymptotically equal as each row is added. □

Note that the bound $k < (\log n)^{3/2-\varepsilon}$ is not sharp. In fact, McKay [12] has improved the bound to $o(n)$. Theorem 4 suggests that Conjecture 4 cannot be resolved by coarse enumeration of row-even and row-odd Latin squares, even though the information demanded by Conjecture 4 is very coarse. More delicate arguments are necessary.

Acknowledgements

I am grateful to Professor Gian-Carlo Rota for bringing this problem to my attention and for numerous helpful discussions.

Note added in proof. Conjecture 1 was recently proved by Fred Galvin, and Arthur Drisko has proved Conjecture 3 for $n = p + 1$ where p is an odd prime.

References

[1] N. Alon and M. Tarsi, Colorings and orientations of graphs, Combinatorica 12 (1992) 125–134.
[2] J. Boyce, private communication.
[3] A. Chetwynd and R. Häggkvist, A note on list colorings, J. Graph Theory 13 (1989) 87–95.
[4] W. Chan, An exchange property of matroids, Discrete Math., to appear.
[5] R. Dekker and A. Ottens, Re: Trivial or non-trivial?, message-ID ⟨6301@dutrum2.tudelft.nl⟩, USENET newsgroup rec.puzzles, 27 Nov 91 12:19:06 GMT.
[6] P. Erdős, and I. Kaplansky, The asymptotic number of Latin rectangles, Amer. J. Math. 68 (1946) 230–236.
[7] P. Erdős, A. Rubin and H. Taylor, Choosability in graphs, Congr. Numer. 23 (1979) 125–157.
[8] W. Fulton, Algebraic Curves (Benjamin, New York, 1969).
[9] M. Hall Jr, Combinatorial Theory (Blaisdell, New York, 1967).
[10] R. Huang, J. Kahn and G.C. Rota, On the relations of various conjectures on Latin squares and straightening coefficients, unpublished manuscript.
[11] J. Janssen, Even and odd Latin squares, Ph.D. Thesis, Lehigh University, PA, 1993.
[12] B. McKay, private communication.
[13] G.C. Rota, private communication.
[14] M. Wild, On Rota's problem about n bases in a rank n matroid, Advances in Math. 108 (1994) 336–345.