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Descents, Quasi-Symmetric Functions, Robinson-Schensted for Posets, and the Chromatic Symmetric Function

TIMOTHY Y. CHOW tchow@alum.mit.edu
Tellabs Research Center, 1 Kendall Square, Cambridge, MA 02139

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Abstract. We investigate an apparent hodgepodge of topics: a Robinson-Schensted algorithm for $(\mathbf{3}+\mathbf{1})$-free posets, Chung and Graham's G-descent expansion of the chromatic polynomial, a quasi-symmetric expansion of the path-cycle symmetric function, and an expansion of Stanley's chromatic symmetric function X_G in terms of a new symmetric function basis. We show how the theory of P-partitions (in particular, Stanley's quasi-symmetric function expansion of the chromatic symmetric function X_G) unifies them all, subsuming two old results and implying two new ones. Perhaps our most interesting result relates to the still-open problem of finding a Robinson-Schensted algorithm for $(\mathbf{3}+\mathbf{1})$-free posets. (Magid has announced a solution but it appears to be incorrect.) We show that such an algorithm ought to "respect descents," and that the best partial algorithm so far—due to Sundquist, Wagner, and West—respects descents if it avoids a certain induced subposet.

Keywords: $(\mathbf{3}+\mathbf{1})$-free poset, chromatic polynomial

1. Introduction

The theory of P-partitions continues to spawn new ideas more than twenty years after its birth. Our main object of interest here is one such outgrowth, namely the expansion of Stanley's chromatic symmetric function in terms of Gessel's fundamental quasi-symmetric functions $Q_{S,d}$ (reproduced as Theorem 1 below). Although innocent-looking, this expansion has numerous ramifications, some of them surprising. The purpose of this paper is to explore some of these offshoots.

In Section 3, we recall the result, stating it in a way that differs slightly from the usual formulation. The justification for this modification of standard terminology is that it shows more clearly the relationship with two other closely related results in the literature: Chung and Graham's G-descent expansion of the chromatic polynomial [3, Theorem 2] and the expansion of the path-cycle symmetric function in terms of the $Q_{S,d}$ [2, Proposition 7]. The original proofs of these latter two results did not appeal directly to Stanley's expansion; here we show that the G-descent result and an important special case of the path-cycle symmetric function result are essentially special cases of Stanley's result. Apart from being a pleasing unification of previously disparate results, this provides some evidence that this new formulation of Stanley's theorem is the "right" one.

In Section 4, we investigate the implications of Theorem 1 for Robinson-Schensted algorithms for $(\mathbf{3}+\mathbf{1})$-free posets, a topic that has attracted some recent attention ([9, Section 3.7] and [15]). This is perhaps the most interesting part of this paper, especially for nonspecialists in symmetric functions. An algebraic argument of Gasharov shows that there ought to exist a Robinson-Schensted algorithm for $(\mathbf{3}+\mathbf{1})$-free posets that generalizes the usual Robinson-Schensted algorithm, but no such algorithm is currently known. The best partial result is due to Sundquist, Wagner, and West [15], who provide an algorithm that is valid only on a proper subclass of the class of $(\mathbf{3}+\mathbf{1})$-free posets—the so-called "beast-free" $(\mathbf{3}+\mathbf{1})$-free posets. In Section 4 we show that a Robinson-Schensted algorithm for $(\mathbf{3}+\mathbf{1})$-free posets should "respect descents." The Sundquist-Wagner-West algorithm respects descents only on a proper subclass of the class of beast-free $(\mathbf{3}+\mathbf{1})$-free posets, so it probably needs to be modified even for beast-free $(\mathbf{3}+\mathbf{1})$-free posets.

Finally, in Section 5, we investigate the connection with a new symmetric function basis that was introduced in [2]. The theorem proved here hopefully provides more evidence that this symmetric function basis is a worthy object of study.

2. Preliminaries

We shall assume that reader is familiar with the basic facts about set partitions, posets, permutations, and so on; a good reference is [13]. Our notation for symmetric functions and partitions for the most part follows that of Macdonald [8]. If λ is an integer partition, we write $r_\lambda!$ for $r_1!r_2!\cdots$, where r_i is the number of parts of λ of size i. We will always take our symmetric functions in countably many variables. In addition to the usual symmetric function bases, we shall need the *augmented monomial symmetric functions* $\tilde{m}_\lambda$ [4], which are defined by

$$\tilde{m}_\lambda \stackrel{\text{def}}{=} r_\lambda!\, m_\lambda,$$

where m_λ of course denotes the usual monomial symmetric function. We will sometimes use *set* partitions instead of *integer* partitions in subscripts; for example, if π is a set partition then the expression p_π is to be understood as an abbreviation for $p_{\text{type}(\pi)}$. We will use ω to denote the involution that sends s_λ to $s_{\lambda'}$.

If d is a positive integer, we use the notation $[d]$ for the set $\{1, 2, \ldots, d\}$.

Following Gessel [7] and Stanley [12], we define a power series in the countably many variables $x = \{x_1, x_2, \ldots\}$ to be *quasi-symmetric* if the coefficients of

$$x_{i_1}^{r_1} x_{i_2}^{r_2} \cdots x_{i_k}^{r_k} \quad \text{and} \quad x_{j_1}^{r_1} x_{j_2}^{r_2} \cdots x_{j_k}^{r_k}$$

are equal whenever $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$. For any subset S of $[d-1]$ define the *fundamental* quasi-symmetric function $Q_{S,d}(x)$ by

$$Q_{S,d}(x) = \sum_{\substack{i_1 \le \cdots \le i_d \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \cdots x_{i_d}.$$

Sometimes we will write $Q_{S,d}$ for $Q_{S,d}(x)$ if there is no danger of confusion.

If g is a symmetric or quasi-symmetric function in countably many variables and of bounded degree, then we shall write $g(1^n)$ for the polynomial in the variable n obtained by setting n of the variables equal to one and the rest equal to zero. An important example of this procedure is given in the following proposition, whose (easy) proof we leave as an exercise.

Proposition 1 *For any* $S \subseteq [d-1]$,

$$Q_{S,d}(1^n) = \binom{n+d-|S|-1}{d}.$$

Throughout, the unadorned term *graph* will mean a finite simple labelled undirected graph. If G is a graph we let $V(G)$ and $E(G)$ denote its vertex set and edge set respectively. A *stable partition* of G is a partition of $V(G)$ such that every block is a stable set, i.e., no two vertices in the same block are connected by an edge. Stanley's *chromatic symmetric function* X_G is defined by

$$\mathrm{X}_G \stackrel{\text{def}}{=} \sum_{\pi} \tilde{m}_{\pi},$$

where the sum is over all stable partitions π of G. For motivation for the definition of X_G, see [12]. Here we will just mention that an equivalent definition of X_G is

$$\mathrm{X}_G = \sum_{\kappa : V(G) \to \mathbb{N}} \left(\prod_{v \in V(G)} x_{\kappa(v)} \right),$$

where the sum is over all *proper colorings* κ, i.e., maps $\kappa : V(G) \to \mathbb{N}$ such that $\kappa(u) \neq \kappa(v)$ whenever u is adjacent to v. One can check that $\mathrm{X}_G(1^n)$ is just the chromatic polynomial of G.

3. The fundamental theorem

The fundamental result in this subject is Stanley's expansion of the chromatic symmetric function in terms of Gessel's fundamental quasi-symmetric functions. We shall now present this result; more precisely, as mentioned in the introduction, we shall present a reformulation of the result, and then we will go on to show how this reformulation subsumes Chung and Graham's G-descent expansion of the chromatic polynomial and a special case of the expansion of the path-cycle symmetric function in terms of the $Q_{S,d}$.

A number of details will be omitted from the proofs in this section because the arguments consist mostly of definition-chasing.

We need some definitions. The first of these looks trivial but is actually one of the most important.

Definition A *sequencing* of a graph or a poset with a vertex set V that has cardinality d is a bijection $s : [d] \to V$.

It is helpful to think of a sequencing as the sequence $s(1), s(2), \ldots, s(d)$ of vertices. The reason we claim that this definition is important is that the usual approach to this subject regards a permutation of some kind (either of $[d]$ or of V) as the fundamental object of interest, but as we shall see below, it is often sequencings that are most natural to consider. Even the standard approach often finds it necessary to resort to inverse maps at certain points to convert permutations to sequencings; by focusing on sequencings directly we obviate this.

"Dual" to the notion of a sequencing is a *labelling*, which is a bijection $\alpha : V \to [d]$. A labelling α of a poset is *order-reversing* if $\alpha(x) > \alpha(y)$ whenever $x < y$.

A sequencing s of a graph G induces an acyclic orientation $\mathfrak{o}(s)$ of G: if $i < j$ and $s(i)$ is adjacent to $s(j)$, then direct the edge from $s(j)$ to $s(i)$. The acyclic orientation in turn induces a poset structure $\bar{\mathfrak{o}}(s)$ on the vertex set of G: make $s(i)$ less than $s(j)$ whenever $s(j)$ points to $s(i)$ and then take the transitive closure of this relation.

Let G be a graph with d vertices. If α is a labelling of G and s is a sequencing of G, then we say that *s has an α-descent at i* (for $i \in [d-1]$) if the permutation $\alpha \circ s$ has a descent at i. The *α-descent set $D(\alpha, s)$ of s* is the set

$$\{i \in [d-1] \mid s \text{ has an } \alpha\text{-descent at } i\}.$$

(It is helpful to visualize this by visualizing a numerical label on each element of the sequence $s(1), s(2), \ldots, s(d)$; the sequence of labels is the one-line representation of the permutation $\alpha \circ s$ and the descents occur at the descents of this permutation.)

We can now state Stanley's theorem.

Theorem 1 *Let G be a graph with d vertices. Suppose that to each sequencing s of G there is associated an order-reversing labelling α_s of $\bar{\mathfrak{o}}(s)$. Suppose further that $\alpha_s = \alpha_{s'}$ whenever s and s' are two sequencings of G that induce the same acyclic orientation of G. Then*

$$\mathrm{X}_G = \sum_{\text{all sequencings } s} Q_{D(\alpha_s, s), d}.$$

Sketch of proof: The basic idea here can be traced back to [11]. For each acyclic orientation $\mathfrak{o}$ of G, let s be some sequencing that induces $\mathfrak{o}$ and define $\omega_{\mathfrak{o}} : \bar{\mathfrak{o}} \to [d]$ to be the order-reversing bijection α_s. Let $\mathscr{L}(\bar{\mathfrak{o}}, \omega_{\mathfrak{o}})$ be the set of all linear extensions of $\bar{\mathfrak{o}}$, regarded as permutations of $[d]$ via $\omega_{\mathfrak{o}}$, and if e is a permutation let $D(e)$ denote the descent set of e. Then [12, Theorem 3.1], combined with [12, Eq. (8)], states that

$$\mathrm{X}_G = \sum_{\mathfrak{o}} \sum_{e \in \mathscr{L}(\bar{\mathfrak{o}}, \omega_{\mathfrak{o}})} Q_{D(e), d}, \tag{3.1}$$

where the first sum is over all acyclic orientations $\mathfrak{o}$ of G.

Now there is a bijection between the set of all sequencings of G and the set of ordered pairs $\{(\mathfrak{o}, e) \mid e \in \mathscr{L}(\bar{\mathfrak{o}})\}$—given a sequencing s, let $\mathfrak{o}$ be the acyclic orientation induced by s and let $e = \omega_{\mathfrak{o}} \circ s = \alpha_s \circ s$. Theorem 1 then follows from (3.1) once we verify that $D(e)$ corresponds to $D(\alpha_s, s)$ under this bijection. □

Chung and Graham [3, Theorem 2] have shown that if the chromatic polynomial of a graph is expanded in terms of the polynomial basis

$$\binom{x+k}{d}_{k=0,\ldots,d} \tag{3.2}$$

then the coefficients can be interpreted in terms of what they call *G-descents.* In their paper, Chung and Graham give a sketch of a somewhat complicated proof of this result, and remark that while in principle it follows from Brenti's expansion [1, Theorem 4.4] (which in turn is essentially what one obtains by specializing Theorem 1 via the map $g \mapsto g(1^n)$), the implication is not particularly direct. However, Chung and Graham's result follows directly from Theorem 1 by choosing the α_s appropriately and then specializing from symmetric functions to one-variable polynomials, as we shall now see.

Again, we need some definitions. To *peel* a poset P is to remove its minimal elements, then to remove the minimal elements of what is left, and so on. The *rank* $\rho(x)$ of an element $x \in P$ is the stage at which it is removed in the peeling process.

Next we give the definition of Chung and Graham's concept of a G-descent, translated into our terminology. Let G be a graph with d vertices. Let β be a labelling of G and let s be a sequencing of G. If v is a vertex of G then we define $\rho(v)$ by using the poset structure $\bar{\mathbf{o}}(s)$. We then say that *s has a CG β-ascent at i* (for $i \in [d-1]$) if either

1. $\rho(s(i)) < \rho(s(i+1))$ or
2. $\rho(s(i)) = \rho(s(i+1))$ and $\beta(s(i)) < \beta(s(i+1))$.

The *CG β-ascent set of s* is defined in the obvious way. We then have the following result.

Corollary 1 *If G is a graph with d vertices and a labelling β, then*

$$\mathrm{X}_G = \sum_S N_S\, Q_{S,d}, \tag{3.3}$$

where the sum is over all subsets $S \subseteq [d-1]$ and N_S is the number of sequencings of G with CG β-ascent set S.

Sketch of proof: The appropriate choices of α_s in Theorem 1 are as follows. Given a sequencing s, arrange the vertices of G in the following "peeling order": first take the elements of highest rank in $\bar{\mathbf{o}}(s)$, then the elements of next highest rank, and so on; arrange elements with the same rank in decreasing order of their β-labels. Now define the labelling α_s by setting $\alpha_s(v) = j$ where j is the position of v in the peeling order. It is now straightforward to check that the CG β-ascent set of s coincides with the α_s-descent set of s. □

Chung and Graham's result [3, Theorem 2] now follows as a special case of Corollary 1. For if we apply the map $g \mapsto g(1^n)$ to (3.3), the left-hand side specializes to the chromatic polynomial of G ([12, Proposition 2.2]) and by Proposition 1 the right-hand side specializes to the binomial coefficient sum

$$\sum_S N_S \binom{n+d-|S|-1}{d} = \sum_k N_k \binom{n+k}{d},$$

where N_k is the number of sequencings with $d-1-k$ CG β-ascents, i.e., with k CG β-descents (where CG β-descents are defined in the natural way). This is exactly the result of Chung and Graham [3, Theorem 2].

Our second corollary involves the expansion of the path-cycle symmetric function Ξ_D in terms of the $Q_{S,d}$ [2, Proposition 7]. (We shall not give the formal definition of the path-cycle symmetric function here because we will not need it; suffice it to say that it is a certain symmetric function invariant Ξ_D that can be associated to any digraph D.) For certain digraphs D, Ξ_D coincides with the chromatic symmetric function X_G of some graph G, and therefore, in these cases, [2, Proposition 7] gives an interpretation of the coefficients of the $Q_{S,d}$-expansion of X_G. This interpretation is ostensibly different from the one given by Theorem 1, but as we shall show presently, it again follows directly from Theorem 1 via suitable choices of α_s.

We shall now make these somewhat vague remarks precise. Let P be a poset with d vertices. If s is a sequencing of P, we say that s has a *descent at* i (for $i \in [d-1]$) if $s(i) \not< s(i+1)$. The *descent set* $D(s)$ *of* s is again defined in the obvious way. The *incomparability graph* $\mathrm{inc}(P)$ of P is the graph with the same vertex set as P and in which two vertices are adjacent if and only if they are incomparable elements of P.

An acyclic, transitively closed digraph is equivalent to a poset. According to [2, Proposition 2], the path-cycle symmetric function of such a digraph coincides with the chromatic symmetric function of the incomparability graph of the equivalent poset. Therefore, what [2, Proposition 7] says in this case is the following.

Corollary 2 *Let P be a poset with d vertices. Then*

$$X_{\mathrm{inc}}(P) = \sum_{\text{all sequencings } s} Q_{D(s),d}.$$

We now claim that this result can also be derived from Theorem 1.

Sketch of proof: We define the α_s as follows. Let s be any sequencing of $\mathrm{inc}(P)$. The maximal elements of $\bar{\mathbf{o}}(s)$ form a stable set in $\mathrm{inc}(P)$ and therefore a chain in P; call the minimal (with respect to the ordering of P, not of $\bar{\mathbf{o}}(s)$) element of this chain v_1, and set $\alpha_s(v_1) = 1$. Now delete v_1 and repeat the procedure, i.e., let v_2 be the P-minimal element among the $\bar{\mathbf{o}}(s)$-maximal elements of the deleted graph, and set $\alpha_s(v_2) = 2$. Continue in this way until $\alpha_s(v)$ is defined for all v. We leave to the reader the (straightforward although not entirely trivial) task of verifying that the α_s-descents of s (considered as a sequencing of $\mathrm{inc}(P)$) coincide with the descents of s (considered as a sequencing of P). $\square$

4. Robinson-Schensted and (3 + 1)-free Posets

A poset is said to be $(\mathbf{3}+\mathbf{1})$-*free* if it contains no induced subposet isomorphic to the disjoint union of a three-element chain with a singleton. Unless otherwise noted, all posets in this section are assumed to be $(\mathbf{3}+\mathbf{1})$-free.

Gasharov [6] has proved a remarkable result about the expansion of $\mathrm{X}_{\mathrm{inc}(P)}$ in terms of Schur functions. To state it we must first recall the notion of a P-tableau. If P is any poset, a (*standard*) P-*tableau* is an arrangement of the elements of P into a Ferrers shape such that the rows are strictly increasing (i.e., each row is a chain) and the columns are weakly increasing (by which we mean that if u appears *immediately above* v [when the tableau is drawn English style] then $u \not> v$). Each element of P appears exactly once in the tableau. Then Gasharov's result is the following.

Theorem 2 *Let P be a $(\mathbf{3}+\mathbf{1})$-free poset. Then*

$$\mathrm{X}_{\mathrm{inc}(P)} = \sum_{\lambda} f_P^{\lambda} s_{\lambda},$$

where f_P^{λ} is the number of P-tableaux of shape λ.

It would be nice to have a direct bijective proof of Theorem 2 (Gasharov's proof is not). In [14] Stanley remarks that when P is a chain, f_P^{λ} is just the number of standard Young tableaux of shape λ, so a bijective proof of Theorem 2 is provided by the Robinson-Schensted correspondence. (For background on Robinson-Schensted and tableaux, see [10].) Stanley further remarks that Magid [9, Section 3.7] has produced a generalization of the Robinson-Schensted correspondence that provides the desired bijective proof of Theorem 2. However, the exposition in [9, Section 3.7] is difficult to follow, and to the best of my understanding there is an error in the construction. Let P be the four-element poset whose Hasse diagram looks like an uppercase "N." (We shall refer to this poset as *Poset N.*) Label the vertices a, b, c, and d from left to right and from top to bottom, as in reading English. Then the two sequences *dacb* and *dbca* appear to generate the same pair of tableaux under Magid's insertion algorithm, which should not happen since the insertion algorithm is supposed to give a bijection between sequencings of the poset and pairs of tableaux. It is possible that I am misinterpreting Magid's algorithm, but if I am correct then the problem of finding a bijective proof of Theorem 2 is still open. The best partial result is due to Sundquist et al. [15], who provide an algorithm that gives the desired bijection for a certain proper subclass of $(\mathbf{3}+\mathbf{1})$-free posets.

We shall say more about the algorithm in [15] in a moment, but our main purpose here is to observe that combining Corollary 2 with Theorem 2 gives us some insight into the kind of Robinson-Schensted algorithm we want. If $\lambda \vdash d$ then from [12, Eq. (15)] we have

$$s_{\lambda} = \sum_{S} f_S^{\lambda} Q_{S,d},$$

where the sum is over all $S \subseteq [d-1]$ and f_S^{λ} is the number of standard Young tableaux with shape λ and descent set S. Combining this with Theorem 2 yields

$$X_{\mathrm{inc}(P)} = \sum_S \sum_\lambda f_P^\lambda f_S^\lambda Q_{S,d}.$$

In other words, the coefficient of $Q_{S,d}$ in $X_{\mathrm{inc}(P)}$ is the number of ordered pairs (T, T') where T is a P-tableau and T' is a standard Young tableau with the same shape and with descent set S.

Comparing this with Corollary 2, we see that not only does there exist a bijection between sequencings of P and ordered pairs (T, T') with T a P-tableau and T' a standard Young tableau, but there exists such a bijection with the further property that it respects descents. (It is well known that this is true in the case of the usual Robinson-Schensted algorithm.) It is therefore natural to hope for an algorithm that also respects descents. For one thing, this would provide an alternative proof of Gasharov's theorem.

We might ask if the Sundquist-Wagner-West algorithm respects descents, at least for the class of $(\mathbf{3}+\mathbf{1})$-free posets to which it is applicable. The answer is no, and the sequencings of Poset N mentioned above in connection with Magid's algorithm are also counterexamples for this question. However, we do have the following result.

Theorem 3 *The Sundquist-Wagner-West algorithm respects descents when restricted to the class of* $(\mathbf{3}+\mathbf{1})$*-free posets that do not contain Poset N as an induced subposet.*

Proof: The Sundquist-Wagner-West algorithm applies to a more general class of objects than we have been discussing here, but in our present context, it reduces to the following. Let P be a $(\mathbf{3}+\mathbf{1})$-free poset. Given a sequencing s of P, construct an ordered pair (T, T') where T is a P-tableau and T' is a standard Young tableau by *inserting* $s(1)$, $s(2)$, and so on in turn. The P-tableau T will be the insertion tableau, and T' will be the recording tableau. The recording is done in the normal way and requires no comment. To insert an element $s(i)$ into T, observe that each row R of T is a chain. (This property is trivial to begin with and it will be easy to see that it is preserved at each stage of the insertion process.) Therefore, since P is $(\mathbf{3}+\mathbf{1})$-free, $s(i)$ is incomparable to at most two elements of R. If $s(i)$ is incomparable to zero elements of R, then the situation is indistinguishable from standard Robinson-Schensted, so proceed in the expected way: append $s(i)$ to the end of R if $s(i)$ is greater than every element of R; otherwise, let $s(i)$ bump the smallest element of R greater than $s(i)$ and proceed inductively by inserting the bumped element into the next row of the insertion tableau. If $s(i)$ is incomparable to exactly one element of R, make $s(i)$ bump that one element. Finally, if $s(i)$ is incomparable to two elements of R, make $s(i)$ "skip over" R and inductively insert $s(i)$ into the next row. We remark that it is easy to show that if there are any elements in R incomparable to $s(i)$, then these elements must be in a single consecutive block and that $s(i)$ must be greater than everything to the left of this block and less than everything to the right of this block. Keeping this fact in mind will make it easier to follow the arguments below.

Sundquist, Wagner and West prove that the above algorithm produces a bijection if P is what they call "beast-free" in addition to being $(\mathbf{3}+\mathbf{1})$-free. Since the beast contains Poset N as an induced subposet, the bijection is valid for the posets that we are concerned with here.

To show that descents are respected in this algorithm, we proceed by a straightforward case-by-case analysis. Suppose first that $s(i) \not< s(i+1)$. We wish to show that $i+1$ appears in a lower row than i in the recording tableau. We claim first that when $s(i+1)$ is inserted, it cannot be appended at the end of row 1. To see this, back up and think about what could have happened when $s(i)$ was inserted. If $s(i)$ did not skip over row 1, then $s(i+1)$ could not then be appended to row 1 since $s(i) \not< s(i+1)$. If on the other hand $s(i)$ did skip over row 1, then $s(i)$ must be incomparable to two elements in row 1, and because P is $(\mathbf{3}+\mathbf{1})$-free, we must have $s(i) > s(i+1)$, and $s(i+1)$ cannot be appended to row 1 because this would force $s(i)$ to be greater than everything in row 1, contradiction.

Now if $s(i)$ is appended to the end of row 1 then we are done. Otherwise, each of $s(i)$ and $s(i+1)$ gives rise to an element to be inserted into row 2; call these two elements u and v respectively. (They need not be distinct from $s(i)$ and $s(i+1)$ but they must be distinct from each other.) By induction it suffices to show that $u \not< v$. We have several cases.

1. Suppose $u = s(i)$, i.e., suppose $s(i)$ skips over row 1. If $v = s(i+1)$ then we are done. Otherwise, suppose towards a contradiction that $s(i) < v$. Consider the situation before the insertion of $s(i)$. Since $v \neq s(i+1)$, v must be in row 1, and since $s(i) < v$, $s(i)$ is less than everything to the right of v. But $s(i)$ is incomparable to two elements in row 1, so there must exist at least two elements in row 1 to the left of v. Let q and r be the two elements in row 1 immediately preceding v. Now $s(i+1)$ bumped v so $q < r < s(i+1)$. Since $s(i) \not< s(i+1)$, we have $s(i) \not< q$ and $s(i) \not< r$. But since $s(i)$ is less than v and everything to the right of v, we must have $s(i) \not> q$ and $s(i) \not> r$ for otherwise there could not be two elements in row 1 incomparable to $s(i)$. Therefore $s(i) \not> s(i+1)$ and $s(i)$ together with $q < r < s(i+1)$ is a $(\mathbf{3}+\mathbf{1})$, contradiction.
2. Suppose $u \neq s(i)$ and $s(i) < u$. If $s(i+1) = v$ then since $s(i) \not< s(i+1) = v$ and $s(i) < u$ we must have $u \not< v$ and we are done. So we may assume that $s(i+1) \neq v$. Suppose towards a contradiction that $u < v$. Then when $s(i+1)$ is inserted into row 1 it bumps something (namely v) that is greater than u and thus greater than $s(i)$. Since $s(i)$ is sitting in row 1 when $s(i+1)$ is inserted, this forces $s(i+1) > s(i)$, contradiction.
3. Suppose that $u \neq s(i)$ and that $s(i)$ and u are incomparable. We have two subcases: either $s(i+1) \neq v$ or $s(i+1) = v$. In the former case, suppose towards a contradiction that $u < v$. Since $v > u$, v must be sitting in row 1 to the right of $s(i)$ just before $s(i+1)$ bumps it. Therefore $v > s(i)$ and hence $s(i+1) > s(i)$ (since $s(i+1)$ bumps v and not $s(i)$), contradiction. In the latter case, again suppose towards a contradiction that $u < v$. Since $s(i)$ and u are incomparable, this implies that $s(i+1) = v \not< s(i)$, i.e., that $s(i)$ and $s(i+1)$ are incomparable. Then $u < s(i+1)$ together with $s(i)$ form a $(\mathbf{2}+\mathbf{1})$, so that $s(i)$ is one of the two elements in row 1 incomparable to $s(i+1)$ that cause $s(i+1)$ to skip over row 1. Let w be the other element in row 1 incomparable to $s(i+1)$; then w is either the immediate successor or the immediate predecessor of $s(i)$—and therefore of u before u was bumped by $s(i)$. Actually, though, w cannot be a predecessor of u since this would make $s(i+1) > w$. Combining all this information, we see that $s(i) < w > u < s(i+1)$ together form an induced subposet isomorphic to Poset N, contradiction.

To complete the proof of the theorem we just need to show that if $s(i) < s(i+1)$ then we do not obtain a descent in the recording tableau. If $s(i+1)$ is appended to the end of row 1 then we are done. If $s(i)$ is appended to the end of row 1 then so is $s(i+1)$ and again we are done. Therefore, as before, it is enough by induction to show that "$u < v$".

Suppose first that $s(i)$ bumps some element u from row 1. If $s(i+1)$ also bumps some element v from row 1 then since $s(i) < s(i+1)$ we must have $u < v$, so by induction we are done; therefore we may assume that $s(i+1)$ skips over row 1. Suppose towards a contradiction that $u \not< s(i+1)$. Then $s(i+1)$ is not greater than the element in row 1 immediately to the right of u, but $s(i+1)$ *is* greater than $s(i)$, which displaces u. Therefore, after the insertion of $s(i)$, the two elements in row 1 incomparable to $s(i+1)$ must be the two elements q and r in row 1 immediately to the right of $s(i)$. Hence $y \not< u$, but then $u < q < r$ and y form a $(\mathbf{3}+\mathbf{1})$, contradiction.

It remains to consider the case when $s(i)$ skips over row 1. If $s(i+1)$ also skips over then we are done. We have two remaining subcases: either the element v that $s(i+1)$ bumps is larger than $s(i+1)$ or else v and $s(i+1)$ are incomparable.

In the former case, let w_1 and w_2 be the two elements in row 1 incomparable to $s(i)$ (just prior to the insertion of $s(i)$). Since $s(i) < s(i+1)$ we must have $s(i+1) \not< w_1$ and $s(i+1) \not< w_2$. Since by assumption $s(i+1)$ bumps something larger than itself, we must have $y > w_1$ and $y > w_2$. Therefore v must lie to the right of w_1 and w_2, so $v > s(i)$, which is want we want to show.

In the latter case, suppose towards a contradiction that $s(i) \not< v$. We cannot have $v < s(i)$ because then $v < s(i) < s(i+1)$, contradicting the incomparability of v and $s(i+1)$. So v is incomparable to $s(i)$. Consider row 1 just before the insertion of $s(i+1)$; $s(i)$ is incomparable to two elements in row 1, and one of these is v. The other one, which we may call w, must be either the immediate predecessor or the immediate successor of v. If w is the immediate successor of v then this forces $s(i+1) < w$ and since $s(i) < s(i+1)$ this implies $s(i) < w$, contradiction. Therefore w is the immediate predecessor of v. Combining this information we see that $s(i) < s(i+1) > w < v$ is an induced subposet isomorphic to Poset N, contradiction. □

Possibly, then, the Sundquist-Wagner-West algorithm needs to be modified not only in the case of posets containing the "beast" but also beast-free posets that contain Poset N. However, so far I have not been able to find a modification of the Sundquist-Wagner-West algorithm with all the properties we would like it to have.

5. The symmetric function basis $\{\xi_\lambda\}$

In [2] a new symmetric function basis, which we shall denote by $\{\xi_\lambda\}$ (in place of the original but more cumbersome notation $\{\tilde{\Xi}_\lambda\}$), is introduced. For completeness we repeat the definition here. For each integer partition λ, let D_λ denote the digraph consisting of a disjoint union of directed paths such that the ith directed path has λ_i vertices. If F is a subset of the set $E(D_\lambda)$ of edges of D_λ, then the spanning subgraph of D_λ with edge set F is a disjoint union of directed paths. The multiset of sizes of these directed paths forms an integer partition which we denote by $\pi(F)$. The number of parts of $\pi(F)$ is denoted by $\ell(\pi(F))$. Then the symmetric function ξ_λ is defined by

$$\xi_\lambda = \sum_{F \subseteq E(D_\lambda)} \frac{\tilde{m}_{\pi(F)}}{\ell(\pi(F))!},$$

where the sum is over all subsets F of $E(D_\lambda)$. Two things that make this basis interesting are that it provides a symmetric function generalization of the polynomial basis (3.2), and that there is a linear involution that exchanges the ξ's with the monomial symmetric functions. See [2] for details.

In [2, Theorem 3] it is stated that X_G is ξ-positive (i.e., that its expansion in terms of the ξ_λ has nonnegative coefficients). The proof, however, is not given there. My original proof of this claim was a direct argument giving a combinatorial interpretation of the coefficients in this expansion in terms of Chung and Graham's G-descents. However, a different proof will be presented here that is perhaps more illuminating, since it shows how the result follows from Theorem 1.

We need a technical lemma. If π and σ are set partitions, write $\pi \le \sigma$ for "π refines σ". If $\pi \le \sigma$, let k_i denote the number of blocks of σ that are composed of i blocks of π, and following Doubilet [5] define

$$\lambda(\pi,\sigma)! \stackrel{\text{def}}{=} \prod_i i!^{k_i}.$$

Also, given any integer partitions μ and ν, let π be any set partition of type μ and define

$$c_{\mu,\nu} \stackrel{\text{def}}{=} \sum_{\{\sigma \ge \pi \mid \text{type}(\sigma)=\nu\}} \lambda(\pi,\sigma)!.$$

Lemma 1 *The number of subsets F of $E(D_\lambda)$ such that $\pi(F) = \nu$ equals $c_{\nu,\lambda} r_\lambda!/r_\nu!$.*

Proof: See the proof of [2, Proposition 13]. □

If S is a subset of $[d-1]$ then we define the *type* of S to be the integer partition whose parts are the lengths of the subwords obtained by breaking the word $123\ldots d$ after each element of S.

Theorem 4 *Let g be any symmetric function. If a_λ and $b_{S,d}$ are constants such that*

$$g = \sum_\lambda a_\lambda \xi_\lambda \quad \text{and} \quad g = \sum_{S,d} b_{S,d} Q_{S,d},$$

then

$$a_\lambda = \sum_{\{S \mid \text{type}(S)=\lambda\}} b_{S,d}.$$

Proof: It is not difficult to see that it suffices to prove the theorem for the case $g = \xi_\mu$. For d a positive integer and S a subset of $[d-1]$, define

$$M_{S,d} \stackrel{\text{def}}{=} \sum_{\substack{i_1 \le i_2 \le \cdots \le i_d \\ i_j < i_{j+1} \text{ iff } j \in S}} x_{i_1} x_{i_2} \cdots x_{i_d}.$$

Then

$$m_\lambda = \sum_{\{S \mid \mathrm{type}(S)=\lambda\}} M_{S,d} \quad \text{and} \quad Q_{S,d} = \sum_{T\supseteq S} M_{T,d},$$

where in the first summation d is the size of λ. By an inclusion-exclusion argument,

$$m_\lambda = \sum_{\{S \mid \mathrm{type}(S)=\lambda\}} \sum_{T\supseteq S} (-1)^{|T|-|S|} Q_{T,d}.$$

Let $q_{\lambda,T,d}$ be the coefficient of $Q_{T,d}$ in m_λ. We compute

$$\sum_{\{T \mid \mathrm{type}(T)=\nu\}} q_{\lambda,T,d}.$$

Observe that there is a bijection between subsets of type λ and orderings of the parts of λ: given a subset $S \subseteq [d-1]$ of type λ, take the sequence of the lengths of the subwords of the word $123\ldots d$ obtained by breaking after each element of S. Thinking of such subwords as directed paths, we see that for any fixed S of type λ, the number of subsets $T \supseteq S$ such that $\mathrm{type}(T) = \nu$ is just the number of subsets F of $E(D_\lambda)$ satisfying $\pi(F) = \nu$, which from Lemma 1 is

$$\frac{r_\lambda!}{r_\nu!} c_{\nu,\lambda}.$$

Now there are $\ell(\lambda)!/r_\lambda!$ subsets S of type λ, and if $\mathrm{type}(S) = \lambda$ and $\mathrm{type}(T) = \nu$ then

$$(-1)^{|T|-|S|} = (\operatorname{sgn}\nu)(\operatorname{sgn}\lambda).$$

Putting all this together, we see that

$$\sum_{\{T \mid \mathrm{type}(T)=\nu\}} q_{\lambda,T,d} = \frac{\ell(\lambda)!}{r_\nu!} c_{\nu,\lambda}(\operatorname{sgn}\nu)(\operatorname{sgn}\lambda).$$

But, again from Lemma 1,

$$\xi_\mu = \sum_\lambda \frac{r_\mu!}{r_\lambda!} c_{\lambda,\mu} \frac{r_\lambda!}{\ell(\lambda)!} m_\lambda.$$

Hence if $g = \xi_\mu$, then

$$\begin{aligned}\sum_{\{S \mid \mathrm{type}(S)=\nu\}} b_{S,d} &= \sum_\lambda \frac{r_\mu!}{r_\lambda!} c_{\lambda,\mu} \frac{r_\lambda!}{\ell(\lambda)!} \cdot \frac{\ell(\lambda)!}{r_\nu!} c_{\nu,\lambda}(\operatorname{sgn}\nu)(\operatorname{sgn}\lambda)\\ &= \frac{r_\mu!}{r_\nu!} \sum_\lambda (\operatorname{sgn}\nu) c_{\nu,\lambda} (\operatorname{sgn}\lambda) c_{\lambda,\mu}\\ &= \delta_{\mu\nu},\end{aligned}$$

because $((\operatorname{sgn}\lambda)c_{\lambda,\mu})$ is the matrix of ω with respect to the augmented monomial symmetric function basis (by [5, Appendix 1, #9]) and ω is an involution. This completes the proof. $\square$

It follows as an immediate corollary that any symmetric function (such as X_G or s_λ) that is Q-positive is also ξ-positive, and moreover if there is a combinatorial interpretation of the $Q_{S,d}$-coefficients then it carries over into a combinatorial interpretation of the ξ_λ coefficients.

We should caution the reader, however, that ξ_λ is *not* Q-positive. Nor is it true that the only $Q_{S,d}$'s in the $Q_{S,d}$-expansion of ξ_λ with nonzero coefficients are those with $\text{type}(S) = \lambda$. Thus, while Theorem 4 allows one to translate combinatorial *interpretations of the coefficients* of the Q-expansion of a symmetric function g into combinatorial interpretations of the coefficients of the ξ-expansion of g, there is no guarantee that combinatorial *proofs* can be so translated. Some tricky reshuffling of combinatorial information occurs in the transition from the $Q_{S,d}$'s to the ξ_λ's. In fact, I do not know of a direct combinatorial proof that the ξ_λ-expansion of the Schur functions enumerates Young tableaux according to descents.

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