THE RAMANUJAN JOURNAL 3, 55–72 (1999)

Additive Partitions and Continued Fractions

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Received April 24, 1997; Accepted September 23, 1997

Abstract. A set S of positive integers is *avoidable* if there exists a partition of the positive integers into two disjoint sets such that no two distinct integers from the same set sum to an element of S. Much previous work has focused on proving the avoidability of very special sets of integers. We vastly broaden the class of avoidable sets by establishing a previously unnoticed connection with the elementary theory of continued fractions.

Key words: combinatorial number theory, additive number theory, Beatty's theorem, intermediate fractions, avoidable sets

1991 Mathematics Subject Classification: Primary–11J70; Secondary–05A17, 11B75, 11P81

1. Introduction

Many well-known problems in number theory involve studying the set S of numbers that can be represented as sums of elements of another set A of positive integers. In some cases, A is sparse and we want to know if S nevertheless contains a lot of integers—e.g., S may contain all integers (as in Waring's problem) or all even integers (as in Goldbach's conjecture). In other cases, A is dense and we want to know if S nevertheless avoids a lot of integers—e.g., S may avoid long arithmetic progressions or have low natural density (see, for instance, [10] and the references therein).

The problem that we study in this paper is a relatively little-known variation on the latter theme. A set $S \subset \mathbb{N}$ is said to be *avoidable* if there exists a partition of $\mathbb{N}$ into two disjoint sets A and B such that no two distinct elements of A sum to an element of S and no two distinct elements of B sum to an element of S. We say that the partition $\{A, B\}$ *avoids* S or that S is *avoided by* $\{A, B\}$. If the pair of sets A and B is unique, then S is said to be *uniquely avoidable.* We are interested in the question of which sets are avoidable (or uniquely avoidable).

The theory of avoidable sets has existed for twenty years, yet surprisingly little is known. Only a few special sets of integers have been shown to be avoidable. For example, the following theorem of Evans [4] is typical.

Theorem 1. *Let* $S = \{s_n\}$ *be a set of positive integers such that* $s_1 < s_2$, $(s_1, s_2) = 1$, *and* $s_n = s_{n-1} + s_{n-2}$ *for* $n > 2$. *Then* S *is uniquely avoidable.*

One might hastily conclude from the scarcity of general theorems in this area that the topic of avoidable sets is not very fruitful. We hope to show in this paper that in fact the subject is deeper than it appears at first glance and that much more remains to be discovered. Our main evidence for this claim lies in the following two theorems.

Theorem 2. *Let α be an irrational number between 1 and 2, and define*

$$A_\alpha \stackrel{\text{def}}{=} \{n \in \mathbb{N} \mid \textit{the integer multiple of } \alpha \textit{ nearest } n \textit{ is greater than } n\},$$
$$B_\alpha \stackrel{\text{def}}{=} \{n \in \mathbb{N} \mid \textit{the integer multiple of } \alpha \textit{ nearest } n \textit{ is less than } n\}.$$

Let S_α be the set of all positive integers avoided by the partition $\{A_\alpha, B_\alpha\}$. Then S_α contains all numerators of continued fraction convergents of α.

Theorem 3. *Let α, A_α, B_α, and S_α be as in Theorem 2. Then every element of S_α is either the numerator of a convergent of α, the numerator of an intermediate fraction, or twice the numerator of a convergent.*

In the next three sections we prove these theorems and show how they significantly generalize many previous results. The reader is expected to be familiar with the elementary theory of continued fractions as given in, for example, [8, chapters I and II] or [9, chapter 7]. However, we will give explicit references for some of the less trivial facts. We also use the notation $\lfloor x \rfloor$ and $\{x\}$ for the integer and fractional parts of x respectively.

In the remaining sections we investigate some other questions connected with avoidable sets and present some open problems.

2. Proof of Theorem 2

Lemma 1. *Let α be a positive real number and let p/q be a continued fraction convergent of α. Then $\lfloor n\alpha \rfloor = \lfloor np/q \rfloor$ for all integers n lying strictly between 0 and q.*

Proof: Since p/q is a convergent, $|\alpha - p/q| \le 1/q^2$ [8, Eq. (30)], so

$$\left| n\alpha - \frac{np}{q} \right| \le \frac{n}{q^2} \le \frac{q-1}{q^2} < \frac{1}{q},$$

or

$$\frac{np}{q} - \frac{1}{q} < n\alpha < \frac{np}{q} + \frac{1}{q} \tag{2.1}$$

for $0 < n < q$. Now $(p, q) = 1$, so for $0 < n < q$ we have

$$\frac{1}{q} \le \frac{np}{q} - \left\lfloor \frac{np}{q} \right\rfloor \le \frac{q-1}{q}$$

or

$$\left\lfloor \frac{np}{q} \right\rfloor + \frac{1}{q} \le \frac{np}{q} \le \left\lfloor \frac{np}{q} \right\rfloor + \frac{q-1}{q}.$$

Combining this with (2.1) yields

$$\left\lfloor \frac{np}{q} \right\rfloor < n\alpha < \left\lfloor \frac{np}{q} \right\rfloor + 1,$$

and so $\lfloor n\alpha \rfloor = \lfloor np/q \rfloor$ for $0 < n < q$. □

Lemma 2. *Let α be a positive real number and let p/q be a continued fraction convergent of α. Then for $0 < n < q$,*

$$\{n\alpha\} < \frac{\alpha}{2} \Longrightarrow \left\{ \frac{np}{q} \right\} \le \frac{p}{2q}$$

and

$$\left\{ \frac{np}{q} \right\} < \frac{p}{2q} \Longrightarrow \{n\alpha\} < \frac{\alpha}{2}.$$

Proof: We prove only the first implication; the proof of the second implication follows a similar line of reasoning in reverse.

The lemma is trivial if $\alpha = p/q$, so assume from now on that $\alpha \neq p/q$.

Assume that $\{n\alpha\} < \alpha/2$. We wish to show that $\{np/q\} \le p/2q$. By Lemma 1, $\lfloor n\alpha \rfloor = \lfloor np/q \rfloor$. Therefore

$$n\alpha - \left\lfloor \frac{np}{q} \right\rfloor < \frac{\alpha}{2}$$

and hence

$$\left\{ \frac{np}{q} \right\} = \frac{np}{q} - \left\lfloor \frac{np}{q} \right\rfloor < \frac{np}{q} - n\alpha + \frac{\alpha}{2} = \frac{p}{2q} + \frac{p(2n-1)}{2q} - \frac{\alpha(2n-1)}{2},$$

i.e.,

$$\left\{ \frac{np}{q} \right\} < \frac{p}{2q} + (2n-1)\left(\frac{p}{2q} - \frac{\alpha}{2} \right). \tag{2.2}$$

If $p/q < \alpha$ then we are done. Otherwise, $p/q > \alpha$ because by assumption $\alpha \neq p/q$. Since p/q is a convergent of α, we have $|p/q - \alpha| \le 1/q^2$, and therefore (2.2) implies

$$\begin{aligned} \left\{ \frac{np}{q} \right\} &< \frac{p}{2q} + \frac{2n-1}{2q^2} \\ &< \frac{p}{2q} + \frac{1}{q}. \end{aligned} \tag{2.3}$$

We now split into two cases.

Case 1. *p is even.* Since p is even, $\{np/q\} \leq p/2q$ if and only if

$$\left\{\frac{np}{q}\right\} < \frac{p}{2q} + \frac{1}{q},$$

and we are done, by (2.3).

Case 2. *p is odd.* Since p is odd, $\{np/q\} \leq p/2q$ if and only if

$$\left\{\frac{np}{q}\right\} < \frac{p+1}{2q}.$$

Hence, in light of (2.3), it suffices to show that $\{np/q\} \neq (p+1)/2q$.

Suppose towards a contradiction that $\{np/q\} = (p+1)/2q$. Since $p/q > \alpha$, p/q must be the ith convergent of α for some odd number i by [8, Theorem 4]. Now

$$p_j q_{j-1} - p_{j-1} q_j = (-1)^{j-1}$$

for all $j \geq 1$ by [8, Theorem 2]. By taking $j = i$ we deduce that

$$pq_{i-1} \equiv 1 \pmod q$$

since i is odd. By assumption $\{np/q\} = (p+1)/2q$, so

$$qq_{i-1}\left(\frac{np}{q} - \left\lfloor \frac{np}{q} \right\rfloor\right) = qq_{i-1}\left(\frac{p+1}{2q}\right).$$

If we multiply this out and reduce modulo q we obtain

$$n \equiv q_{i-1}\left(\frac{p+1}{2}\right) \pmod q.$$

Hence

$$2n \equiv q_{i-1}(p+1) \equiv 1 + q_{i-1} \pmod q$$

or

$$2n - 1 \equiv q_{i-1} \pmod q.$$

Since $0 < n < q$, this congruence implies that $2n-1$ equals either q_{i-1} or $q + q_{i-1}$.

Let a_{i+1} denote the $(i+1)$st partial quotient of α, which exists because $\alpha \neq p/q$. Then $q_{i+1} = a_{i+1}q + q_{i-1}$ by [8, Theorem 1], and

$$\left|\frac{p}{q} - \alpha\right| \leq \frac{1}{qq_{i+1}} = \frac{1}{q(a_{i+1}q + q_{i-1})}$$

by [8, Theorem 9]. Combining these facts with (2.2) yields

$$\begin{aligned}
\left\{\frac{np}{q}\right\} &< \frac{p}{2q} + (2n-1)\left(\frac{p}{2q} - \frac{\alpha}{2}\right) \\
&\le \frac{p}{2q} + (q + q_{i-1})\left(\frac{p}{2q} - \frac{\alpha}{2}\right) \\
&= \frac{p}{2q} + \frac{q + q_{i-1}}{2}\left(\frac{p}{q} - \alpha\right) \\
&\le \frac{p}{2q} + \frac{q + q_{i-1}}{2} \cdot \frac{1}{q(a_{i+1}q + q_{i-1})} \\
&= \frac{p}{2q} + \frac{1}{2q} \cdot \frac{q + q_{i-1}}{a_{i+1}q + q_{i-1}} \\
&\le \frac{p}{2q} + \frac{1}{2q}.
\end{aligned}$$

This contradicts $\{np/q\} = (p+1)/2q$, as desired. □

Lemma 3. *Let α be an irrational number between* 1 *and* 2 *and let p/q be a convergent of α that is greater than α. Then $\{q\alpha\} > \alpha/2$ provided $q \neq 1$.*

Proof: We first prove the lemma for the case in which p/q is the first convergent of α. Let a_1 denote the first partial quotient of α and let p_1/q_1 denote the first convergent. We need to show that $p_1 - q_1\alpha < 1 - \alpha/2$. Since $p_1 = a_1 + 1$ and $q_1 = a_1$, this is equivalent to

$$a_1 + 1 - a_1\alpha < 1 - \alpha/2,$$

or

$$\alpha > 1 + \frac{1}{2a_1 - 1}.$$

But this holds, because $a_1 = q_1 = q \neq 1$ so $2a_1 - 1 > a_1$.

In general, our goal is to show that $p - q\alpha < 1 - \alpha/2$. Now since $p/q > \alpha$, p/q cannot be the zeroth convergent, and hence

$$0 < p - q\alpha \le p_1 - q_1\alpha,$$

so the general case follows from the special case proved above. □

Proof of Theorem 2: Fix a convergent p/q of α. We begin by showing that no two distinct integers in A_α sum to p. First, we may assume that $q \neq 1$, since $q = 1$ and $1 < \alpha < 2$ together imply $p = 1$ or $p = 2$, and no two distinct positive integers can sum to 1 or 2. For the rest of this proof we assume that $q \neq 1$, $p \neq 1$, and $p \neq 2$.

Assume now that x and y are integers in A_α that sum to p; we shall show that $x = y$. Let $m\alpha$ be the positive integral multiple of α nearest x. Then since $x \in A_\alpha$, $m\alpha > x$, and moreover $m\alpha < x+1$ for otherwise $(m-1)\alpha$ would be closer to x than $m\alpha$. Therefore, we may write $x = \lfloor m\alpha \rfloor$, and similarly we may write $y = \lfloor n\alpha \rfloor$ for some positive integer n.

We claim that $m < q$. First of all, $m \le q$, for if m were larger than q then $x = \lfloor m\alpha \rfloor$ would be at least p (since $|q\alpha - p| < 1$), but x is necessarily less than p since $x + y = p$. The remaining possibility is that $m = q$ and $x = p - 1$, but then Lemma 3 tells us that $(m-1)\alpha$ would be closer to x than $m\alpha$ would be. Thus $m < q$, and similarly $n < q$.

By Lemma 1, it follows that $x = \lfloor mp/q \rfloor$ and $y = \lfloor np/q \rfloor$. Now since $0 < m < q$ and $0 < n < q$, we have

$$\left\{\frac{mp}{q}\right\} = \frac{r}{q} \quad \text{and} \quad \left\{\frac{np}{q}\right\} = \frac{s}{q}$$

for some integers r and s with $0 < r < q$ and $0 < s < q$. From $x + y = p$ it follows that

$$\frac{mp}{q} + \frac{np}{q} = p + \frac{r+s}{q}.$$

Multiplying both sides by q, we see that $r + s = kp$ for some positive integer k. Now $1 < \alpha < 2$, so $1 \le p/q \le 2$, or $q \le p \le 2q$. Since $0 < r < q$ and $0 < s < q$, it follows that $0 < r + s < 2q \le 2p$, so $k = 1$, i.e., $r + s = p$.

Now x and y are in A_α, so $\{m\alpha\} < \alpha/2$ and $\{n\alpha\} < \alpha/2$. By the first part of Lemma 2, this implies

$$0 < \frac{r}{q} \le \frac{p}{2q} \quad \text{and} \quad 0 < \frac{s}{q} \le \frac{p}{2q}.$$

Since $r + s = p$, these inequalities force $r = s = p/2$. Thus

$$\frac{mp}{q} - \left\lfloor \frac{mp}{q} \right\rfloor = \frac{np}{q} - \left\lfloor \frac{np}{q} \right\rfloor,$$

so $mp \equiv np \pmod{q}$. Since $(p, q) = 1$, this implies $m \equiv n \pmod{q}$, and since $0 < m < q$ and $0 < n < q$, we must have $m = n$. This proves that $x = y$, as required.

Now there are exactly $\lfloor (p-1)/2 \rfloor$ pairs of distinct positive integers whose sum is p. To finish the proof, it suffices to show that A_α contains exactly one integer from each of these pairs (for then B_α cannot contain two distinct integers that sum to p). To show this, it suffices to show that A_α contains $\lfloor (p-1)/2 \rfloor$ integers *other than* $p/2$ that are less than p, since we have already shown that A_α cannot contain *both* integers from a "bad" pair.

The arguments we gave for showing that $x = \lfloor m\alpha \rfloor$ for some m with $0 < m < q$ show that the elements of A_α less than p are in one-to-one correspondence with integers n in the range $0 < n < q$ such that $\{n\alpha\} < \alpha/2$. Thus, from the second part of Lemma 2, it suffices to find $\lfloor (p-1)/2 \rfloor$ integers n in the range $0 < n < q$ such that

$$\left\{\frac{np}{q}\right\} < \frac{p}{2q}$$

and $\lfloor n\alpha \rfloor \ne p/2$.

Let $np \bmod q$ denote the remainder when np is divided by q. We are seeking integers n such that $0 < n < q$ and

$$np - q\left\lfloor \frac{np}{q} \right\rfloor < \frac{p}{2}.$$

The left-hand side lies between 0 and q and it is congruent to np modulo q, so it equals $np \bmod q$. Now p and q are relatively prime, so as n ranges from 1 to $q-1$, $np \bmod q$ also takes on each value from 1 to $q-1$ exactly once. Therefore we can indeed find $\lfloor (p-1)/2 \rfloor$ integers n with the desired property: simply take the $\lfloor (p-1)/2 \rfloor$ integers n such that $np \bmod q < p/2$. It remains only to show that for no such n can $\lfloor n\alpha \rfloor$ equal $p/2$.

Suppose to the contrary that some such n satisfies $\lfloor n\alpha \rfloor = p/2$. By Lemma 1, $\lfloor n\alpha \rfloor = \lfloor np/q \rfloor$. Therefore

$$np - \frac{qp}{2} < \frac{p}{2},$$

i.e., $n < (q+1)/2$ or $n \le (q-1)/2$. We therefore have

$$\frac{p}{2} \le n\alpha \le \frac{(q-1)\alpha}{2},$$

or $p \le (q-1)\alpha$, which is not possible. □

Remark. Michael Bennett (personal communication) has suggested the possibility that a positive integer p is avoided by (A_α, B_α) if and only if p is the numerator of a fraction p/q satisfying $1 \le p/q \le 2$, $(p, q) = 1$, and the properties listed in Lemmas 1, 2, and 3. He has made considerable progress towards a proof.

3. Proof of Theorem 3

We begin with a careful statement of a well-known fact that is sometimes sloppily stated.

Lemma 4. *Let α be a positive real number and let p_{n-1}/q_{n-1} and p_n/q_n be two consecutive convergents of the continued fraction representation of α, with $n > 0$. Then*

$$|q_n\alpha - p_n| \le |q_{n-1}\alpha - p_{n-1}|,$$

with equality if and only if $n = 1$ and $\alpha = a_0 + 1/2$ for some integer a_0. On the other hand, if c/d is a fraction with $0 < d < q_n$, then

$$|d\alpha - c| \ge |q_{n-1}\alpha - p_{n-1}|.$$

Proof: This is essentially [9, Theorems 7.12 and 7.13], except that we have stated the theorem for arbitrary real α instead of only for irrational α. The proofs are easily modified to cover the general case. (We remark that the lemma also follows from [8, Theorem 17], but the statement of this latter theorem is slightly incorrect; p_0/q_0 fails to be a best approximation of the second kind whenever $\alpha \geq a_0 + 1/2$.) □

Lemma 5. *Let α be a positive real number and let p_{n-1}/q_{n-1} and p_n/q_n be two consecutive convergents of the continued fraction representation of α, with $n > 0$. Suppose p is an integer such that*

$$|q_n\alpha - p| < |q_{n-1}\alpha - p_{n-1}|.$$

Then $p = p_n$.

Proof:
Case 1. $|q_{n-1}\alpha - p_{n-1}| \leq 1/2$. Then by Lemma 4,

$$|q_n\alpha - p_n| \leq |q_{n-1}\alpha - p_{n-1}| \leq 1/2,$$

and therefore $q_n\alpha$ is at least 1/2 away from any integer other than p_n. Thus if p satisfies

$$|q_n\alpha - p| < |q_{n-1}\alpha - p_{n-1}| \leq 1/2,$$

we must have $p = p_n$.

Case 2. $|q_{n-1}\alpha - p_{n-1}| > 1/2$. Then repeated application of Lemma 4 implies that $|q_0\alpha - p_0| > 1/2$, where p_0/q_0 is the zeroth convergent. Now $q_0 = 1$ and $p_0 = \lfloor\alpha\rfloor$; therefore the fractional part of α exceeds 1/2, forcing the first partial quotient a_1 to equal one. Then $p_1 = \lfloor\alpha\rfloor + 1$ and $q_1 = 1$, so

$$|q_1\alpha - p_1| = 1 - |q_0\alpha - p_0| < 1/2.$$

This shows us that Case 2 arises only if $n = 1$ and $\alpha - \lfloor\alpha\rfloor > 1/2$. When this happens, the only integer $p \neq p_1$ that has a chance of satisfying

$$|q_1\alpha - p| < |q_0\alpha - p_0|$$

is the integer on the other side of $q_1\alpha$ from p_1. But for this p,

$$|q_1\alpha - p| = 1 - |q_1\alpha - p_1| = |q_0\alpha - p_0|,$$

so the desired inequality cannot in fact be satisfied even for this p. This proves the lemma. □

Lemma 6. *Let α be a positive real number and let p_{n-1}/q_{n-1}, p_n/q_n, and p_{n+1}/q_{n+1} be three consecutive convergents of the continued fraction representation of α, with $n > 0$. Let c/d be a fraction whose denominator satisfies $q_{n-1} < d < q_{n+1}$. If*

$$|q_{n-1}\alpha - p_{n-1}| > |d\alpha - c|, \tag{3.1}$$

then either c/d is in lowest terms and is an intermediate fraction having a denominator between q_n and q_{n+1}, or $c/d = p_n/q_n$. (*In the latter case, c/d is not necessarily in lowest terms*).

Proof: Assume that there exist c and d satisfying $q_{n-1} < d < q_{n+1}$ and (3.1) but such that c and d are not the numerator and denominator of an intermediate fraction (in lowest terms) having a denominator between q_n and q_{n+1}. We will show that $c/d = p_n/q_n$ in several steps. First we construct a fraction c'/d' satisfying (3.1) and $q_{n-1} \le d' < q_n + q_{n-1}$. (When we say that c'/d' satisfies (3.1) we mean of course (3.1) with c and d replaced by c' and d' respectively.) We then show that $c'/d' = p_n/q_n$, and finally we show that this in turn implies that $c/d = p_n/q_n$.

Let a_{n+1} be the $(n+1)$st partial quotient of α. By definition [8, equation (20)], the intermediate fractions with denominators between q_n and q_{n+1} are (in lowest terms) the fractions of the form

$$\frac{p_{n-1} + kp_n}{q_{n-1} + kq_n} \quad \text{where } 0 < k < a_{n+1}.$$

Let $p = c - p_{n-1}$ and let $q = d - q_{n-1}$. Then $0 < q < a_{n+1}q_n$, so there exist unique integers k_0 and j with $0 \le j < q_n$ and $0 \le k_0 < a_{n+1}$ such that $q = k_0 q_n + j$. Let $i = p - k_0 p_n$. By assumption, c and d are not simultaneously equal to the numerator and denominator of any one of the above intermediate fractions, and $q \ne 0$, so i and j cannot both be zero. Also by assumption, c/d satisfies (3.1), so

$$\begin{aligned} |q_{n-1}\alpha - p_{n-1}| &> |(q_{n-1} + q)\alpha - (p_{n-1} + p)| \\ &= |q_{n-1}\alpha - p_{n-1} + k_0(q_n\alpha - p_n) + j\alpha - i|. \end{aligned} \tag{3.2}$$

The quantities $q_{n-1}\alpha - p_{n-1}$ and $q_{n+1}\alpha - p_{n+1}$ lie on the same side of zero with $q_{n+1}\alpha - p_{n+1}$ being closer. (If α is rational and p_{n+1}/q_{n+1} is the last convergent of α, then $q_{n+1}\alpha - p_{n+1}$ is actually *equal* to zero, but this does not invalidate the argument in the next sentence, which is all that we use the observation in the previous sentence for.) Note that since $0 \le k_0 < a_{n+1}$, the quantity

$$X \stackrel{\text{def}}{=} q_{n-1}\alpha - p_{n-1} + k_0(q_n\alpha - p_n)$$

lies between $q_{n-1}\alpha - p_{n-1}$ and $q_{n+1}\alpha - p_{n+1}$; in particular, X lies between $q_{n-1}\alpha - p_{n-1}$ and zero. (It could equal the former; this will not be a problem.)

We now claim that

$$|j\alpha - i| \ge |q_{n-1}\alpha - p_{n-1}|. \tag{3.3}$$

If $j \ne 0$, then this follows from Lemma 4 because $j < q_n$. If $j = 0$, then $i \ne 0$, and the left-hand side is a positive integer whereas the right-hand side is at most one. This establishes (3.3).

Now (3.3) forces $j\alpha - i$ and $q_{n-1}\alpha - p_{n-1}$ to have opposite sign. For if they had the same sign, then (3.3) together with the fact that X lies between $q_{n-1}\alpha - p_{n-1}$ and zero would imply that adding $j\alpha - i$ to X would result in a number with absolute value greater than that of $q_{n-1}\alpha - p_{n-1}$, contradicting inequality (3.2).

Next we claim that

$$|q_{n-1}\alpha - p_{n-1} + j\alpha - i| < |q_{n-1}\alpha - p_{n-1}|.$$

To see this, consider first the case where $q_{n-1}\alpha - p_{n-1} < 0$. Then $j\alpha - i > 0$ and $q_n\alpha - p_n > 0$, and in light of (3.3) we have

$$\begin{aligned} 0 \le q_{n-1}\alpha - p_{n-1} + j\alpha - i &\le q_{n-1}\alpha - p_{n-1} + k_0(q_n\alpha - p_n) + j\alpha - i \\ &= |q_{n-1}\alpha - p_{n-1} + k_0(q_n\alpha - p_n) + j\alpha - i| \\ &< |q_{n-1}\alpha - p_{n-1}|, \end{aligned}$$

where the last inequality follows from (3.2). This establishes the claim in this case, and the same argument *mutatis mutandis* covers the case $q_{n-1}\alpha - p_{n-1} > 0$.

This last claim, however, just says that if we set $c' = p_{n-1} + i$ and $d' = q_{n-1} + j$, then c'/d' satisfies (3.1), and since $0 \le j < q_n$, we have $q_{n-1} \le d' < q_n + q_{n-1}$. This completes the first step of our argument. We now wish to show that $c'/d' = p_n/q_n$.

We may assume that $d' \ge q_n$, by (3.1) and Lemma 4. Let us now assume towards a contradiction that $d' > q_n$. For simplicity we shall assume that $q_{n-1}\alpha - p_{n-1} < 0$; the reader can check that the argument is easily modified to handle the case $q_{n-1}\alpha - p_{n-1} > 0$.

Using this assumption, we have $q_n\alpha - p_n > 0$ and

$$d'\alpha - c' = q_{n-1}\alpha - p_{n-1} + j\alpha - i \ge 0.$$

Since $0 < d' < q_n + q_{n-1} \le q_{n+1}$, by Lemma 4 $d'\alpha - c'$ must be further away from zero than $q_n\alpha - p_n$, i.e.,

$$d'\alpha - c' > q_n\alpha - p_n > 0.$$

Therefore $d'\alpha - c' - (q_n\alpha - p_n)$ is closer to zero than $d'\alpha - c'$ is, and $d'\alpha - c'$ is in turn closer to zero than $q_{n-1}\alpha - p_{n-1}$ is, since c'/d' satisfies (3.1). This means that the fraction

$$\frac{c' - p_n}{d' - q_n}$$

is a better approximation of the second kind to α than p_{n-1}/q_{n-1} is. But $d' - q_n < q_{n-1}$, so this contradicts Lemma 4.

Therefore, $d' = q_n$. By Lemma 5, the inequality (3.1) implies that $c' = p_n$. It remains to show that $c/d = p_n/q_n$. This is straightforward:

$$\frac{c}{d} = \frac{p_{n-1} + i + k_0 p_n}{q_{n-1} + j + k_0 q_n} = \frac{c' + k_0 p_n}{d' + k_0 q_n} = \frac{p_n + k_0 p_n}{q_n + k_0 q_n} = \frac{p_n}{q_n}.$$

This completes the proof. □

Remark. In fact, the converse of Lemma 6 holds, but we do not need this fact.

Proposition 1. *Let α be an irrational number such that $1 < \alpha < 2$. Let c be a positive integer that is neither the numerator of an intermediate fraction nor the numerator of a convergent. Then there exists a convergent p/q such that p and $c - p$ are either both in A_α or both in B_α.*

Proof: Choose d to minimize the quantity $|d\alpha - c|$. We now wish to let n be the largest positive integer such that $q_n < d$, but we must first check that such an n exists. Since α is between 1 and 2, its zeroth convergent equals one and its first convergent equals $(a_1+1)/a_1$ for some integer $a_1 > 0$. From the definition of intermediate fraction we see that every fraction of the form $(i+1)/i$ with $1 \le i \le a_1$ is an intermediate fraction or a convergent. Since c is not the numerator of any of these, $c \ge a_1 + 2$. In particular, $c \ge 3$ so $d \ge 2$. Therefore there exist integers $N \ge 0$ such that $q_N < d$. Let n be the largest such integer. We claim that $n \ge 1$. For if $n = 0$, then $d \le q_1 = a_1$, and since $\alpha < (a_1+1)/a_1$, we have $d\alpha < a_1 + 1$, contradicting $c \ge a_1 + 2$. It therefore makes sense to speak of the convergent p_{n-1}/q_{n-1}, and we shall do so.

By our choice of n, $q_n < d \le q_{n+1}$. Now $q_n\alpha - p_n$ and $q_{n-1}\alpha - p_{n-1}$ have opposite signs. Let m be the element of the set $\{n, n-1\}$ such that $q_m\alpha - p_m$ has the same sign as $d\alpha - c$. We shall argue that p_m and $c - p_m$ are in the same set (i.e., they are either both in A_α or both in B_α). We claim that to prove this it suffices to show that

$$|q_m\alpha - p_m| \le |d\alpha - c|. \tag{3.4}$$

To see that this does indeed suffice, begin by noting that

$$(d - q_m)\alpha - (c - p_m) = (d\alpha - c) - (q_m\alpha - p_m).$$

Now in view of (3.4) and the fact that $d\alpha - c$ and $q_m\alpha - p_m$ have the same sign, this implies that $(d - q_m)\alpha - (c - p_m)$ has the same sign as $d\alpha - c$; moreover, $(d - q_m)\alpha$ is closer to $c - p_m$ than $d\alpha$ is to c, so $(d - q_m)\alpha$ is the multiple of α closest to $c - p_m$. It follows that $c - p_m$ is in the same set as c. From (3.4) we see that $q_m\alpha$ is the multiple of α closest to p_m, so the fact that $d\alpha - c$ and $q_m\alpha - p_m$ have the same sign implies that c is in the same set as p_m. Hence $c - p_m$ and p_m are in the same set, as required.

We are reduced to proving (3.4). We handle first the special case where $d = q_{n+1}$. By assumption, $c \ne p_{n+1}$, so c must be the integer on the opposite side of $d\alpha$ from p_{n+1}. Therefore

$$|d\alpha - c| = 1 - |q_{n+1}\alpha - p_{n+1}|$$

and $m = n$. We need to show that

$$1 - |q_{n+1}\alpha - p_{n+1}| \ge |q_n\alpha - p_n|.$$

But this follows from the inequalities

$$|q_{n+1}\alpha - p_{n+1}| \le |q_n\alpha - p_n| < 1/2,$$

which hold because $n \ge 1$.

If $d \neq q_{n+1}$, so that $q_n < d < q_{n+1}$, then Lemma 6 provides the key to proving (3.4). For if $c/d = p_n/q_n$, then $d\alpha - c$ is just a multiple of $q_n\alpha - p_n$ and hence $m = n$, and (3.4) is obvious. Otherwise, Lemma 6 (together with Lemma 4) tells us that the only way for (3.4) to be violated is for m to equal $n-1$ and for c/d to equal an intermediate fraction in lowest terms. But this contradicts the fact that c is not the numerator of an intermediate fraction. This completes the proof. □

Theorem 3 follows immediately from Proposition 1.

4. Relationship of Theorems 2 and 3 with prior work

Theorem 1 subsumes several earlier results: the case $s_1 = 1$ and $s_2 = 2$ was first posed by Silverman [12] and solved independently by numerous people, and the case $s_1 = 1$ and s_2 arbitrary was proved by Alladi, Erdős, and Hoggatt [1]. Evans [4] also showed that if $S = \{s_n\}$ satisfies $2 \mid s_1 s_2$, $(s_1, s_2) = 1$, and $s_n = s_{n-1} + s_{n-2}$ for $n > 2$, then S is uniquely avoidable, but that if $s_1 > s_2$ and $2 \nmid s_1 s_2$, then S is not avoidable.

Our results do not completely subsume Evans's results, because our theorems say nothing about uniqueness. However, our results do generalize Evans's in the following sense: given any set S that Evans has shown to be (uniquely) avoidable—i.e., a set of the form specified in Theorem 1 or in the previous paragraph—we can find an irrational number α between 1 and 2 such that $\{A_\alpha, B_\alpha\}$ avoids S. To see this, consider first the case $s_1 < s_2$. If $s_1 \neq 1$, set $s_0 = s_2 \bmod s_1$. If $s_0 \neq 1$, set $s_{-1} = s_1 \bmod s_0$. Continue in this way until $s_{-k} = 1$ for some $k \geq -1$; this must happen at some point since $(s_1, s_2) = 1$. Now let α be the number whose partial quotients are

$$1,\, s_{-k+1} - 1,\, \left\lfloor \frac{s_{-k+2}}{s_{-k+1}} \right\rfloor,\, \left\lfloor \frac{s_{-k+3}}{s_{-k+2}} \right\rfloor, \ldots, \left\lfloor \frac{s_2}{s_1} \right\rfloor, 1, 1, 1, 1, 1, \ldots$$

It is easy to see that the numerators of the convergents of α are $s_{-k}, s_{-k+1}, s_{-k+2}, \ldots$, so by Theorem 2, $\{A_\alpha, B_\alpha\}$ does indeed avoid S.

If $s_1 > s_2$ and $2 \mid s_1 s_2$, then applying the above argument with s_2 and s_3 in place of s_1 and s_2 shows that for a suitable choice of α, $\{A_\alpha, B_\alpha\}$ avoids every element of S except possibly for s_1. In fact, $\{A_\alpha, B_\alpha\}$ must avoid s_1 as well. For Theorem 1 asserts that $\{A_\alpha, B_\alpha\}$ is the *only* partition avoiding $\{s_2, s_3, \ldots\}$. But the partition avoiding all of S is also unique and it *a fortiori* avoids $\{s_2, s_3, \ldots\}$. Hence $\{A_\alpha, B_\alpha\}$ must coincide with the partition that avoids all of S.

We remark that if $s_1 = 1$ and $s_2 = 2$, so that S is the set of Fibonacci numbers, then it turns out that $\alpha = \tau = (1+\sqrt{5})/2$, and it is well known that the numerators of the convergents of τ are the Fibonacci numbers. In general, for any of Evans's sets, the associated irrational α is some element of $\mathbb{Q}(\sqrt{5})$.

The relevance of continued fractions to the theory of avoided sets has not been noticed explicitly before, but it is implicit in [3]. To explain the main result of [3], we must first recall *Beatty's theorem* [2]. Beatty's theorem states that if α and β are positive irrational numbers such that $1/\alpha + 1/\beta = 1$, then the sets

$$\{\lfloor n\alpha \rfloor \mid n \in \mathbb{N}\} \quad \text{and} \quad \{\lfloor n\beta \rfloor \mid n \in \mathbb{N}\}$$

partition $\mathbb{N}$ into two disjoint sets. As Beatty partitions arise naturally in many contexts, it is natural to ask for the connection between Beatty's theorem and the theory of avoidable sets. In [1] it is proved that the partitions in Theorem 1 cannot be Beatty partitions. (They actually only showed this for the case $s_1 = 1$ but their argument extends easily to the general case.) However, Hoggatt and Bicknell-Johnson [7] and the second author have independently noticed that there is actually a close relationship between Beatty's theorem and avoidable sets. Let $\{A_\tau, B_\tau\}$ denote the partition avoiding the Fibonacci numbers. Observe that $1/\tau + 1/\tau^2 = 1$, so that if we set $\alpha = \tau$ in Beatty's theorem then $\beta = \tau^2$. The observation of Hoggatt-Bicknell-Johnson and the second author is that

$$A_\tau \subseteq \{\lfloor n\tau \rfloor \mid n \in \mathbb{N}\} \quad \text{and} \quad B_\tau \supseteq \{\lfloor n\tau^2 \rfloor \mid n \in \mathbb{N}\}.$$

Thus, we can obtain $\{A_\tau, B_\tau\}$ from a Beatty partition by transferring some elements from one half of the partition to the other.

The main result of [3] is that the elements that need to be transferred have a simple description:

$$\begin{aligned} A_\tau &= \{\lfloor n\tau \rfloor \mid n \in \mathbb{N}\} \setminus \{\lfloor n\tau \rfloor \mid n \in \mathbb{N} \text{ and } \{n\tau\} > \tau/2\} \\ \text{and} \quad & \\ B_\tau &= \{\lfloor n\tau^2 \rfloor \mid n \in \mathbb{N}\} \cup \{\lfloor n\tau \rfloor \mid n \in \mathbb{N} \text{ and } \{n\tau\} > \tau/2\}. \end{aligned} \tag{4.1}$$

Now, it is not hard to see that the right-hand sides of these equations are equivalent to the definitions of A_τ and B_τ given in Theorem 2. Thus, Theorem 2 may be regarded as generalizing the main result of [3] from the case $\alpha = \tau$ to the case of arbitrary irrational α.

Incidentally, the reason we say that continued fractions are "implicit" in [3] is that the key lemma in that paper is really a well-known fact about continued fractions, but at the time, the author was unaware of the theory of continued fractions, and hence did not state the lemma in that language. In fact, only after we proved the main theorems of the present paper did we notice the implicit continued fractions in [3].

One paper whose results we feel should be closely related to Theorems 2 and 3 is Zhu and Shan [13], but so far we have had only partial success in establishing a connection. ([13] is in Chinese, but an English translation is available: [11].) Zhu and Shan consider sets $S = \{s_n\}$ that are defined by a recurrence of the form

$$s_n = s_{n-1} + s_{n-2} + k$$

for some fixed nonnegative integer k. Note that if we set $t_n = s_n + k$, then

$$t_n = t_{n-1} + t_{n-2},$$

so the Zhu-Shan sets may be regarded as "shifted Evans sets." Now if k is even, with $k = 2k'$ for some k', and if $\{A, B\}$ is a partition avoiding the set $\{t_n\}$, then we may subtract

k' from each element of A and from each element of B, discarding any nonpositive integers that result. This produces a partition that avoids $S = \{s_n\}$. This establishes a connection in the case of even k, but we are not sure about odd k.

5. Saturated sets

Following [1] we say that a set $S \subseteq \mathbb{N}$ is *saturated* if it is avoidable and it is maximal (with respect to set inclusion) among all avoidable sets. In [1] it is asked if a saturated set is necessarily uniquely avoidable. Although it might seem plausible to conjecture that the answer is yes, Evans [4] exhibited a saturated set that is avoided in two different ways. In this section we strengthen this result by showing that there exist saturated sets that are avoided by arbitrarily large numbers of partitions.

To state our results more precisely, we first recall another definition from [1]. If $S \subseteq \mathbb{N}$, then the *graph* $G(S)$ of S is the graph whose vertex set is $\mathbb{N}$ and whose edges are the sets $\{x, y\}$ such that $x \neq y$ and $x + y \in S$. It is easily seen that S is avoidable if and only if $G(S)$ is bipartite and that S is uniquely avoidable if and only if $G(S)$ is bipartite and connected. Moreover, if $G(S)$ has k connected components, then the number of partitions avoiding S is 2^{k-1}.

We also say that a set $S \subseteq \mathbb{N}$ is *doubly uniquely avoidable* if it is uniquely avoidable and there exists a unique partition of the odd positive integers into two disjoint sets A and B such that no two distinct elements of A sum to twice an element of S and no two distinct elements of B sum to twice an element of S.

The main result of this section is

Theorem 4. *Let S be a doubly uniquely avoidable set that is maximal* (*with respect to set inclusion*) *among all doubly uniquely avoidable sets. For $k \geq 3$, let*

$$S_k = \{1, 2, 3, 2^2, 2^3, 2^4, \ldots, 2^{k-1}\} \cup \{2^k s \mid s \in S\}.$$

Then S_k is saturated and $G(S_k)$ has k connected components.

That Theorem 4 is not vacuous is guaranteed by the following result.

Proposition 2. *There are uncountably many distinct doubly uniquely avoidable sets that are maximal among all doubly uniquely avoidable sets.*

Proof: Let $S = \{s_n\} \cup \{1, 2\}$ be any set that satisfies the following conditions: $s_1 = 3$, $s_2 = 5$, and s_{n+1} equals either $2s_n - 1$ or $2s_n - 2$ for all $n > 1$. We claim that S is doubly uniquely avoidable.

One of the simplest general methods for demonstrating unique avoidability is induction on n (cf. [6]). For example, to show that S is uniquely avoidable, use the inductive hypothesis that there is a unique partition (into two sets) of the positive integers less than s_n that avoids S. If this is true for all n, then S must be uniquely avoidable. Here the inductive hypothesis is easily checked for small n. To pass from n to $n + 1$, we just need to consider the integers m in the range $s_n \leq m < s_{n+1}$ in succession. Provided s_{n+1} is less than $2s_n$, uniqueness is guaranteed, because then m and $s_{n+1} - m$ are distinct and must be placed in opposite sets, but $s_{n+1} - m$ is less than s_n and it is therefore already determined which half of the partition $s_{n+1} - m$ must be in. In the case at hand $s_{n+1} = 2s_n - 1$ or $s_{n+1} = 2s_n - 2$ so uniqueness follows. To show existence, observe that we just need to ensure that m can be placed in such a way as to avoid all elements of S that are less than $2m$. In the case at hand, there is only one such element of S, namely s_{n+1}, and hence it suffices to put m into the set opposite $s_{n+1} - m$.

The same idea works to show that there is a unique way to partition the odd integers so as to avoid $\{2s \mid s \in S\}$. We begin by placing 1 and 3 in opposite sets so as to avoid $4 = 2 \cdot 2$. The induction argument proceeds as before, with only one additional subtlety: when we are showing existence, there are two elements of S for which we need to check avoidability: $2s_{n+1}$ and $2s_{n+2}$. Conceivably, we might not be able to avoid both of these simultaneously when placing the odd integers m in the range $2s_n \leq m < 2s_{n+1}$. Actually, this potential problem arises only for the numbers $2s_{n+1} - 1$ and $2s_{n+1} - 3$, which may sum to $2s_{n+2}$ if s_{n+2} happens to equal $2s_{n+1} - 2$. However, this is not a problem, because 1 and 3 are in opposite sets, and hence the process of avoiding $2s_{n+1}$ will force $2s_{n+1} - 1$ and $2s_{n+1} - 3$ into opposite sets (provided $s_n > 3$, so that $2s_n - 3$ is distinct from 3; but one can check that no problems occur for $s_n = 3$ either), thus automatically avoiding $2s_{n+2}$ as well. This proves the doubly unique avoidability of S.

We now "saturate" each S by taking a doubly uniquely avoidable set $\bar{S}$ that is maximal among all doubly uniquely avoidable sets and that contains S. The set $\bar{S}$ exists and is unique: the integers that we must add to S are precisely those integers m that are avoided by the unique partition avoiding S *and* whose doubles are avoided by the unique partition of the odd numbers avoiding $\{2s \mid s \in S\}$. There are clearly uncountably many distinct S's, and distinct S's are avoided by distinct partitions $\{A, B\}$, so distinct S's have distinct saturations. This completes the proof. □

We remark that the set of Fibonacci numbers is doubly uniquely avoidable, but we do not need this fact so we omit the proof.

Proof of Theorem 4: We claim that the connected components of $G(S_k)$ are as follows.

$$\begin{aligned}
W &= \{m \in \mathbb{N} \mid m \equiv 0 \pmod{2^k}\}\\
X &= \{m \in \mathbb{N} \mid m \equiv \pm 2^{k-1} \pmod{2^{k+1}}\}\\
Y &= \{m \in \mathbb{N} \mid m \equiv 1, 5, 6 \pmod 8\} \cup \{m \in \mathbb{N} \mid m \equiv -1, -5, -6 \pmod 8\}\\
Z_1 &= \{m \in \mathbb{N} \mid m \equiv 4 \pmod{16}\} \cup \{m \in \mathbb{N} \mid m \equiv -4 \pmod{16}\}\\
Z_2 &= \{m \in \mathbb{N} \mid m \equiv 8 \pmod{32}\} \cup \{m \in \mathbb{N} \mid m \equiv -8 \pmod{32}\}\\
&\vdots\\
Z_{k-3} &= \{m \in \mathbb{N} \mid m \equiv 2^{k-2} \pmod{2^k}\} \cup \{m \in \mathbb{N} \mid m \equiv -2^{k-2} \pmod{2^k}\}
\end{aligned}$$

(If $k = 3$ then there are no Z's.) Moreover, we claim that all these connected components are bipartite. The bipartitions for Y and for the Z's are the ones suggested by their definitions above, and the bipartitions for W and X are the ones forced on them by the doubly unique avoidability of S: take the unique partition of the positive integers avoiding S and multiply each number by 2^k to obtain the correct partition for W, and take the unique partition of the odd integers avoiding $\{2s \mid s \in S\}$ and multiply each number by 2^{k-1} to obtain the correct partition for X.

To prove these claims, let us begin by observing that two numbers m and m' from distinct components cannot sum to an element of S_k. (Here "component" just means one of W, X, Y, or Z_i as defined above; we use the term "component" for convenience and its use should not be construed as presupposing that these *are* the components of $G(S_k)$ since we have not shown that yet.) To see this, write m and m' in binary notation and note that their rightmost 1's cannot be in the same position. Therefore they cannot sum to a power of 2. Moreover, at least one of them has its rightmost 1 in one of the k least significant bits, and hence $m + m'$ cannot be divisible by 2^k. Finally, $m + m' \neq 3$.

Next, let us show that if the components are partitioned in the way we described, then two distinct numbers m and m' from the same half of a *single* component cannot sum to an element of S_k. In the case of Z_i, $m + m' \equiv 2^{r-1} \pmod{2^r}$ for some $r \leq k$ and hence cannot be divisible by 2^k; moreover it is clear that $m + m'$ cannot equal a power of 2 since $m \neq m'$. A similar argument covers the component Y. As for W and X, the elements are too large to sum to a power of 2 less than 2^k, and they avoid $\{2^k s \mid s \in S\}$ by the doubly unique avoidability of S. Finally, $m + m' \neq 3$ again.

Now observe that W and X are connected because of the doubly unique avoidability of S. Proving that each of Y and Z_i is actually connected can be done using an inductive procedure similar to the one described in the proof of Proposition 2. The details are straightforward and are left to the reader.

It remains to prove saturation. No integer of the form $2^k s$ with $s \notin S$ can be added to S_k, by the maximality of S. Integers m that are not multiples of eight cannot be added to S_k, because they can be represented as sums of distinct elements from the same half of Y, as follows. Observe that modulo 8, we have

$$-3 \equiv -5 - 6, \quad -2 \equiv 1 + 5, \quad -1 \equiv 1 + 6, \quad 1 \equiv -1 - 6,$$
$$2 \equiv -1 - 5, \quad 3 \equiv 5 + 6, \quad 4 \equiv -6 - 6.$$

Except when $m = 1, 2, 3, 4$, these congruences can be converted into pairs of distinct integers summing to m, e.g., if $m = 17$ then $m \equiv 1 \pmod 8$, and $1 \equiv -1 - 6$ so we can write m as the sum of two integers, one congruent to -1 modulo 8 and the other congruent to -6 modulo 8, i.e., $17 = 7 + 10$ or $17 = 15 + 2$.

It remains to show that integers m that are congruent to 8 modulo 16 or 16 modulo 32 and so on cannot be added to S_k. But this is easily proved using the same kind of argument as in the previous paragraph, e.g., $8 \equiv 4 + 4 \pmod{16}$, and this congruence can be converted into a representation of m as a sum of two distinct integers from the same half of Z_1—except when $m = 8$, but 8 is already in S_k. □

6. Open problems

In an earlier version of this paper, we posed as an open question the problem of characterizing S_α precisely, since Theorems 2 and 3 provide only upper and lower bounds. We also conjectured that S_α is always uniquely avoidable. Both of these problems have very recently been solved by Grabiner [5]. In particular, Grabiner has proved the following.

Theorem 5. *If p_n is the numerator of a convergent, then $2p_n \in S_\alpha$ if and only if p_n is odd, and either* (i) *p_{n+1} is odd and $a_{n+1} \geq 3$, or* (ii) *p_{n+1} is even and $a_{n+1} \geq 2$, or* (iii) *$p_n = 1$. If $p_n + kp_{n+1}$ is the numerator of an intermediate fraction, then it is in S_α if and only if either* (i) *p_{n+1} is even, or* (ii) *$k = 1$ and p_n is odd, or* (iii) *$k = a_{n+2} - 1$ and p_{n+2} is odd.*

To convince the reader that Grabiner's results are definitely nontrivial, we remark that it is natural to conjecture that the set S_α consists precisely of numerators of "best approximations" in some sense, but for the most natural notions of "best approximation"—e.g., best approximations of the first kind, or all fractions p/q such that $|\alpha - p/q| \leq 1/q^2$—this conjecture is false. Also, the simple inductive method that we used in the proof of Proposition 2 does not suffice in general to prove the unique avoidability of S_α; large "gaps" can occur in S_α.

One can ask more generally for a characterization of *all* avoidable sets, or all uniquely avoidable sets, or all saturated sets, or all saturated sets whose graph has a given number of connected components. The sets S_α do not exhaust the class of all saturated uniquely avoidable sets. For example, the set

$$\{3, 4, 8, 12, 17, 22, 43, 85, \dots\}$$

(where each subsequent element is twice the previous element, minus one) is uniquely avoidable but it can be shown that its saturation is not equal to any of our sets S_α.

There are several results in the existing literature that can probably be generalized. For example, in [3] it is proved that the set A'_τ defined by

$$A'_\tau = \{\lfloor n\tau \rfloor \mid n \in \mathbb{N}\} \setminus A_\tau$$

satisfies

$$A'_\tau = \{\lfloor n\tau^3 \rfloor \mid n \in A_\tau\}.$$

We hope this result can be generalized, but we are not sure how. As another example, [6] considers Tribonacci numbers, sequences in which each term is the sum of the previous *three* terms in the sequence. It is not even obvious whether this kind of avoidable set has any connection with continued fractions.

Many variations on existing ideas are possible. What if we drop the word "distinct" from the definition of avoidable set? What happens if we consider algebraic integers rather than rational integers? Clearly much more remains to be done.

Acknowledgments

Before Theorem 3 was formulated and proved, Michael Bennett (personal communication) suggested the possibility that every element of S_α might be either the numerator of an intermediate fraction or a multiple of a numerator of a convergent. Without this suggestion we might never have been led to formulate Theorem 3. The first author is also grateful to Michael Bennett for many stimulating discussions.

In a preliminary version of this paper, we formulated Theorem 2 using definitions of A_α and B_α analogous to (4.1). Then Glen Whitney asked us about the $\tau/2$ appearing in (4.1), wondering about its significance. His question helped us discover the elegant description of A_α and B_α in Theorem 2.

The first author was supported by a National Science Foundation postdoctoral fellowship and did part of the work for this paper while a general member of the Mathematical Sciences Research Institute.

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