BACK COPIES OF RMM

Some back issues of *RECREATIONAL MATHEMATICS MAGAZINE* are still available at the prices listed below:

February 1961 (Reprint)	65¢
*April 1961	65¢
*August 1961	65¢
*October 1961	65¢
December 1961	65¢
April 1962	75¢
August 1962	75¢
December 1962	75¢

* Indicates that *very few* of these issues are left. There are no more June 1961, February 1962, June 1962, or October 1962 issues of RMM in stock.

All orders for back issues must be accompanied by payment and refunds will be made in the event issues are no longer available.

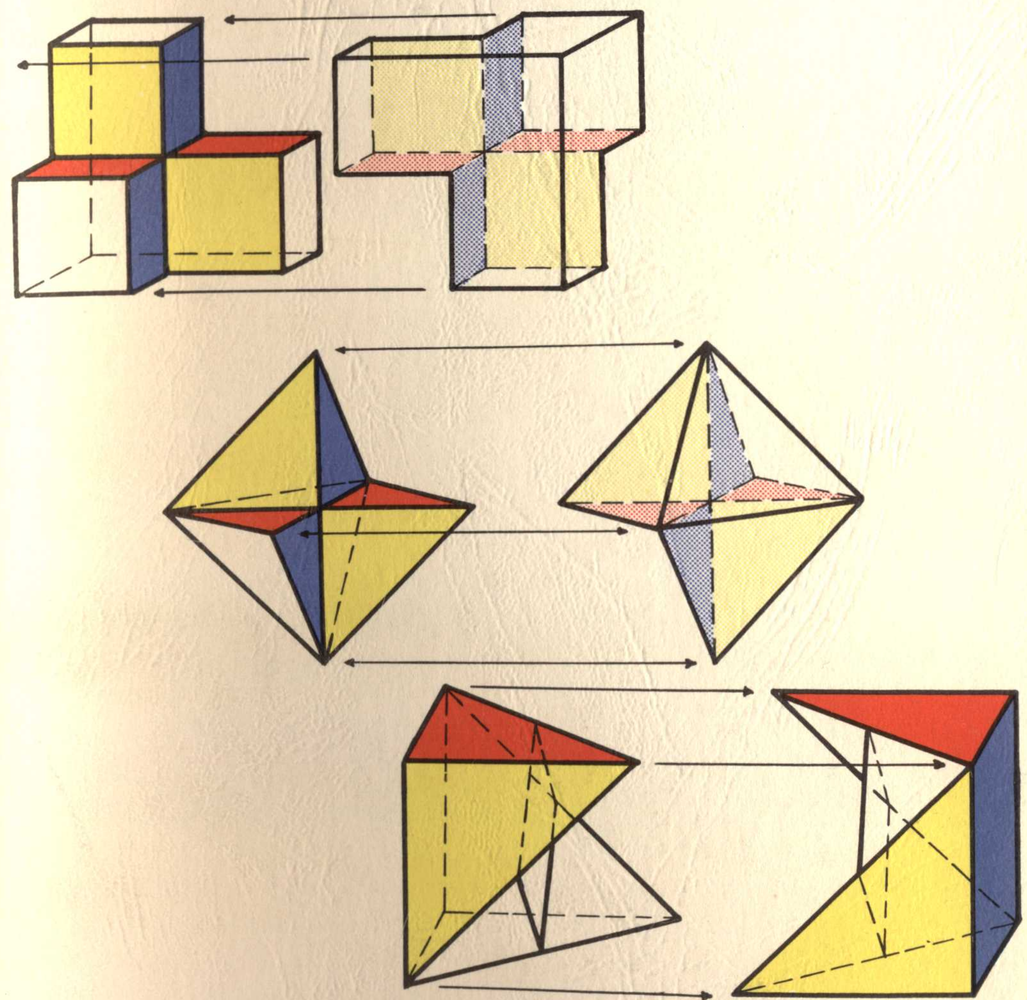
Mail all orders to:

RMM
Box 35
Kent, Ohio

ISSUE NO. 13
FEBRUARY 1963

RECREATIONAL MATHEMATICS magazine

75¢



SOLID
Dissections

All correspondence relating to book reviews, including submission of books for review, should be sent directly to the Book Review Editor:

Dmitri E. Thoro
Mathematics Department
San Jose State College
San Jose 14, California

All correspondence and material relating to the Junior Department should be sent to the Junior Department Editor:

Howard C. Saar
1014 Lindell Avenue
Petoskey, Michigan

All correspondence and manuscripts relating to alphametics, algebraic and number theory and original problems in these categories should be sent to the Associate Editor:

J. A. H. Hunter
88 Bernard Avenue, Apt. 602
Toronto 5, Ontario
Canada

All answers to any puzzles, problems, alphametics, etc., posed in RMM, all other correspondence and manuscripts covering areas not covered by the Book Review, Junior Department, or Associate Editors, correspondence concerning subscriptions, changes of address, reprints, and advertising should be sent to:

RMM - Editor
Box 35
Kent, Ohio

RECREATIONAL MATHEMATICS magazine

Box 35 Kent, Ohio

FEBRUARY 1963 ISSUE NUMBER 13

Published Summer 1963

Published and Edited by Joseph S. Madachy
Associate Editor J. A. H. Hunter
Junior Department Editor Howard C. Saar
Book Review Editor Dmitri E. Thoro

-CONTENTS-

ARTICLES

DEPARTMENTS

MISCELLANEOUS

ILLUSTRATIONS

RECREATIONAL MATHEMATICS MAGAZINE is published bimonthly by Joseph S. Madachy. Second-class postage paid at Kent, Ohio. Subscription rates (worldwide) $3.25 per year, 75¢ per copy. All correspondence concerning subscriptions, changes of address, and advertising should be sent to The Editor - RMM, Box 35, Kent, Ohio. See inside front cover for manuscript and other mailings. See back cover for back issue prices and availability. RECREATIONAL MATHEMATICS MAGAZINE is printed by the Commercial Press, Inc., Kent, Ohio.

FROM THE EDITOR

Again, we are late - but soon we may be able to make some headway. See page 17 for our plans for catching up to a fairly normal publication schedule.

* * * * *

Some readers may notice that the lines of type in this issue are just a bit longer (29 picas - $4\frac{13}{16}$ inches) instead of the former 28 picas ($4\frac{5}{8}$ inches). This slight increase gives us the equivalent of about 1½ extra lines per page without changing the top-to-bottom type-space. Since each issue contains at least 48 pages, this adds about 72 lines per issue - or the equivalent of about 1½ pages of material without adding pages to each issue. You get more for your money - while we do it at no extra cost.

Previously (in the February 1962 issue of RMM) we had introduced several changes in format which, then, increased content. If we can think of any other reasonable methods for getting more material per page we will follow through.

* * * * *

The next issue of RMM will be the *Combined Issue* (April 1963 and June 1963 issues). See page 17 for more details. Here is a partial list of articles to appear in that issue:

A Clerihew ABC of Mathematics *by J. A. Lindon* (England)
Some delightful mathematical poetry.

Something New Behind the 8-Ball *by Ronald Bergman* (New York)
All about the new *elliptical* billiard table.

Magic Knight Tours on Square Boards - Part 2 *by T. H. Willcocks* (England)
A sequel to Mr. Willcocks' article in the December 1962 issue of RMM (pages 9-13).

Dissections Into Unequal Squares *by Ray C. Ellis* (Massachusetts)
Squaring the square and squaring rectangles - the first of a series of articles.

The Eight-Rook Problem *by David Smith* (Florida)
In how many ways can you place 8 rooks on a chessboard so that no rook attacks another?

Infinite Geometry *by Donald L. Vanderpool* (Pennsylvania)
A glimpse into some unusual geometries.

Quadrilles *by Wade E. Philpott* (Ohio)
A beautiful piece of work on a domino recreation - with some completely new results.

That's only a sample! We'll have our usual departments - some of them expanded for the *Combined Issue*.

15 June 1963 J. S. M.

GEOMETRIC MAGIC SQUARES

by Boris Kordemskii
Moscow, U.S.S.R.

A geometric magic square is an $n \times n$ array of n^2 distinct integers with the property that the *product* of the n integers in any row, column, or main diagonal is equal to the same magic constant, P.

An ordinary magic square generally consists of the first n^2 integers arranged in an $n \times n$ array such that the *sum* of the n integers in any row, column, or main diagonal is equal to the same magic constant, S.

Figure 1 shows a 4x4 ordinary magic square with a magic constant of 34.

In this article we shall show several methods by which geometric magic squares may be formed.

15	10	3	6
4	5	16	9
14	11	2	7
1	8	13	12

Figure 1

The first method, the *Exponential Method*, is based on a well-known rule for the multiplication of numbers with exponents. The product of a^p, a^q, and a^r is equal to a^{p+q+r}. If the base (in this case a) is the same the numbers are multiplied together by adding the exponents.

It is clear that if we use the integers of an ordinary magic square as the exponents of some base, we can form a geometric magic square. Figure 2 shows an ordinary magic square and Figures 3a and 3b show how conversion to geometric magic squares is accomplished. The magic constant, P, for Figure 3a is 2^{15} or 32,768 and for Figure 3b it is 3^{15} or 14,348,907.

4	9	2
3	5	7
8	1	6

Figure 2

2^4	2^9	2^2
2^3	2^5	2^7
2^8	2^1	2^6

16	512	4
8	32	128
256	2	64

Figure 3a

3^4	3^9	3^2
3^3	3^5	3^7
3^8	3^1	3^6

81	19683	9
27	243	2187
6561	3	729

Figure 3b

The second method of setting up magic squares with a constant product is also based on an elementary idea: *viz.*, that the processes of addition and multiplication have a number of common features as do subtraction and division. As a result all letter formulas of ordinary magic squares with constant sums turn into letter formulas of geometric magic squares, if multiplication is substituted for addition and division is substituted for subtraction.

Thus, the known formula of Kraitchik*, Figure 4, is easily turned into a formula for a geometric magic square as shown in Figure 5. 5*b* is derived from 5*a* if all the elements in 5*a* are multiplied by xy and if $m=1$.

$m+x$	$m-(x+y)$	$m+y$
$m-(x-y)$	m	$m+(x-y)$
$m-y$	$m+(x+y)$	$m-x$

Figure 4

mx	$\frac{m}{xy}$	my
$\frac{my}{x}$	m	$\frac{mx}{y}$
$\frac{m}{y}$	mxy	$\frac{m}{x}$

a

x^2y	1	xy^2
y^2	xy	x^2
x	x^2y^2	y

b

Figure 5

In the Russian literature, the formula of *V. P. Ermakov* (1884) is known for a magic square of the fourth order (Figure 6). Those who wish may likewise convert it into an appropriate formula for a geometric magic square.

A	C	D	B
D	B	A	C
B	D	C	A
C	A	B	D

	$a+b$	$-a-b$	
$c-d$	$-a-c$	$a-c$	$c+d$
$-c+d$	$-a+c$	$a+c$	$-c-d$
	$a-b$	$-a+b$	

Figure 6

In both the Kraitchik and Ermakov formulas, the proper choices of x and y and of A, B, C, D, a, b, c, and d must be made to avoid duplicate integers in the resulting geometric magic square. This may be done by choosing different primes for their values.

* Maurice Kraitchik, *Mathematical Recreations*, Dover Publications, N. Y., 1953, page 148, figure 28.

The third method is a means of forming a geometric magic square of any order n with *the least possible constant*, P, for the square of the given order. For this the following is necessary:

1. Take the smallest number having n^2 divisors and write down all the divisors of that number.

2. Take any ordinary magic square of the nth order and for all of its elements substitute the divisors found according to a definite rule which will be clear from the examples which follow.

An integer, N, greater than 1, can be represented uniquely as the product of prime numbers. That is,

$$N = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k},$$

where $p_1, p_2, \ldots p_k$ are different prime numbers and $a_1, a_2, \ldots a_k$ are positive integers. Then the total number of divisors of N is equal to

$$(a_1+1)(a_2+1) \cdots (a_k+1).$$

For example, if a and b are prime, then $N=a^2b^2$ has $(2+1)(2+1)$ or 9 divisors. They are $1, a, a^2, b, ab, a^2b, b^2, ab^2, a^2b^2$. Nine divisors of of the number a^2b^2 make it possible to set up a geometric magic square of the third order. To do so we take the ordinary magic square shown in Figure 1 and successively substitute for the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 the nine divisors of a^2b^2 *in the sequence in which they are written above* giving Figure 7. If we take as a and b the lowest prime numbers, 2 and 3, we get a square of the third order with the smallest possible product $P=216$ utilizing different integers (Figure 8).

b	a^2b^2	a
a^2	ab	b^2
ab^2	1	a^2b

Figure 7

3	36	2
4	6	9
18	1	12

Figure 8

The number $N_1=a^3b^3$ has $(3+1)(3+1)$ or 16 divisors. $N_2=abcd$ also has 16 divisors $[(1+1)(1+1)(1+1)(1+1)=16]$. If $a=2$, $b=3$, $c=5$, $d=7$, then $N_1=216$ and $N_2=210$. Since N_2 is the smallest of these numbers having 16 divisors, we may construct a geometric magic square of the fourth order with the smallest possible constant, P. We arrange the sequence of the divisors of the number $abcd$: $1, a, b, ab, c, ac, bc, abc, d, ad, bd, abd, cd, acd, bcd, abcd$. Now we take any ordinary magic square of the fourth order, for example, the one which is drawn in the familiar engraving *Melancholy* by Durer (Figure 9), and form a geometric magic square with a constant $P=44100$ by sequential substitution as indicated above. The set-up is shown in Figure 10.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Figure 9

abcd	*b*	*a*	*cd*
c	*ad*	*bd*	*abc*
d	*ac*	*bc*	*abd*
ab	*bcd*	*acd*	1

Figure 10

The reader may wish to try his hand at composing geometric magic squares with constant products from the divisors of the numbers $N_3 = a^4b^4$ and $N_4 = a^5b^5$. Will these constants, P_3 and P_4, be the lowest possible for geometric magic squares of the 5th and 6th orders?

* * * * *

Later this year RMM will publish another set of unusual magic squares (see RMM No. 5, October 1961, pages 24-29; RMM No. 7, February 1962, pages 14-15). Perhaps some unusual examples of geometric magic squares will be found by our readers. If so, we would certainly like to see them.

We introduced the idea of *anti*-magic squares in the February 1962 issue of RMM (*Anti-Magic Squares* by J. A. Lindon, pages 16-19). An anti-magic square is an $n \times n$ array of the integers from 1 to n^2 such that sums of the rows, columns, and two main diagonals are all *different*. Further, we restrict the $2(n+1)$ different sums to either of the series of integers from $\frac{1}{2}n(n^2-1)$ to $\frac{1}{2}n(n^2+3)+1$ (A to Z) or from $(A-1)$ to $(Z-1)$. For example, the 4th order sequences of different sums are 30 to 39 or 29 to 38; 5th order sequences of different sums are 59 to 70 or 60 to 71; 6th order - from 105 to 118 or 104 to 117; etc.

There are still no known systematic methods of forming anti-magic squares, nor do we have any catalog or listing of the general properties of anti-magic squares.

* * * * *

ROYAL V. HEATH FANS -
See page 33.

MAILING LABEL CODES? SUBSCRIPTION RENEWALS?
See page 17.

MULTIGRADES

by D. C. Cross
Birmingham, England

Many mathematicians, both amateur and professional, are interested in multigrades, of which the following are examples:

$$1^n + 6^n + 8^n = 2^n + 4^n + 9^n \qquad \text{for } n = 1, 2.$$
$$1^n + 5^n + 8^n + 12^n = 2^n + 3^n + 10^n + 11^n \qquad \text{for } n = 1, 2, 3.$$

One property of a multigrade is that you can add the same quantity to each term without affecting the relationship. This can be proved quite simply, the proof for a 3rd order multigrade being as follows:

Say we have $A^n + B^n + C^n + D^n = E^n + F^n + G^n + H^n$, for $n = 1, 2, 3$.
Obviously, $(A+k)+(B+k)+(C+k)+(D+k) = (E+k)+(F+k)+(G+k)+(H+k)$.
Also, $(A+k)^2+(B+k)^2+(C+k)^2+(D+k)^2 = A^2+B^2+C^2+D^2+2k(A+B+C+D)+4k^2$
$= E^2+F^2+G^2+H^2+2k(E+F+G+H)+4k^2$
$= (E+k)^2+(F+k)^2+(G+k)^2+(H+k)^2$.
And, $(A+k)^3+(B+k)^3+(C+k)^3+(D+k)^3$
$= A^3+B^3+C^3+D^3+3k(A^2+B^2+C^2+D^2)+3k^2(A+B+C+D)+4k^3$
$= E^3+F^3+G^3+H^3+3k(E^2+F^2+G^2+H^2)+3k^2(E+F+G+H)+4k^3$
$= (E+k)^3+(F+k)^3+(G+k)^3+(H+k)^3$.

This means, taking a simple example, that we can alter the second of our original two multigrades to read:

$$2^n + 6^n + 9^n + 13^n = 3^n + 4^n + 11^n + 12^n \qquad \text{for } n = 1, 2, 3.$$

Forming a multigrade is easy. Start with a simple equality such as:

$$1 + 4 = 2 + 3$$

Now add 4 to each term: $5 + 8 = 6 + 7$.

Then, we can obtain a 2nd order multigrade by "switching sides" and combining. The formal proof for this is given later.

$$1^n + 4^n + 6^n + 7^n = 2^n + 3^n + 5^n + 8^n \qquad \text{for } n = 1, 2.$$

When adding 4 to each term, we chose that increase as the smallest that would result in the 2nd order multigrade with all terms different.

To build up a 3rd order multigrade, from the above, we now add 8 to each term. This gives:

$$9^n + 12^n + 14^n + 15^n = 10^n + 11^n + 13^n + 16^n \qquad \text{for } n = 1, 2.$$

"Switching sides" and combining, as before, these give us:

$$1^n+4^n+6^n+7^n+10^n+11^n+13^n+16^n = 2^n+3^n+5^n+8^n+9^n+12^n+14^n+15^n,$$

for $n = 1, 2, 3$.

If, in this last transformation, we had added 6 to each side instead of 8, the further "switching" operation would have given a 3rd order multigrade in which the terms 7 and 8 would have appeared on both sides of the identity. The reader may care to check this. Omitting those identical terms, we would have been left with:

$$1^n + 4^n + 6^n + 9^n + 11^n + 14^n = 2^n + 3^n + 5^n + 10^n + 12^n + 13^n \qquad \text{for } n = 1, 2, 3.$$

We can continue in this way, *ad infinitum*, building up multigrades of successively higher orders.

The proof for the "switching" procedure follows. It is given only for derivation of a 2nd order from a 1st order multigrade, but for successively higher order transformations the procedure can be proved similarly.

We start with $x + y = (x - z) + (y + z)$

Then, $(x + k) + (y + k) = (x - z + k) + (y + z + k)$

Switching: $x + y + (x - z + k) + (y + z + k) = (x - z) + (y + z) + (x + k) + (y + k)$

Now take each side separately, summing the squares of the terms:

$$x^2 + y^2 + (x - z + k)^2 + (y + z + k)^2 = 2(x^2 + y^2 + z^2 + k^2 - xz + xk + yz + yk)$$
$$(x - z)^2 + (y + z)^2 + (x + k)^2 + (y + k)^2 = 2(x^2 + y^2 + z^2 + k^2 - xz + xk + yz + yk)$$

This proves that the "switching procedure" yields a valid 2nd order multigrade for all values of x, y, z, and k.

Two mathematicians, *Prouhet* and *Tarry*, some years ago proposed a problem about multigrades which still interests mathematicians. In effect, one is asked to find a multigrade of the nth order having $(n + 1)$ terms on each side, for various values of n. For $n = 3$, we have already derived multigrades (i.e., 3rd order) having eight terms, and six terms, each side: we now require one with four terms, each side. It may be interesting to derive this here.

We start with: $1 + 5 = 2 + 4$

Adding 4 to each term, and carrying out the "switching" procedure, we get: $1^n + 5^n + 6^n + 8^n = 2^n + 4^n + 5^n + 9^n$ for $n = 1, 2$.

Cancelling the 5, which appears in both sides, we have:

$$1^n + 6^n + 8^n = 2^n + 4^n + 9^n \quad \text{for } n = 1, 2.$$

Adding 7 to each term, "switching" and cancelling identical terms, this leads to: $1^n + 6^n + 11^n + 16^n = 2^n + 4^n + 13^n + 15^n$ for $n = 1, 2, 3$.

This is one solution of the *Prouhet* and *Tarry* problem for 3rd order multigrades, there being four terms each side. Of course there are an infinite number of other 3rd order solutions. It will be noticed that the L.H.S. terms are evenly spaced in this case: another solution with evenly spaced terms is:

$$1^n + 14^n + 27^n + 40^n = 5^n + 7^n + 34^n + 36^n \quad \text{for } n = 1, 2, 3.$$

J. A. H. Hunter, whose valued help throughout this paper is gratefully acknowledged, suggests a general identity which provides all 3rd order solutions of the *Prouhet* and *Tarry* problem subject to the L.H.S. terms being evenly spaced:

$$0^n + (x^2 + y^2)^n + (2x^2 + 2y^2)^n + (3x^2 + 3y^2)^n$$
$$= (2x^2 - 3xy + y^2)^n + (3y^2 - xy)^n + (3x^2 + xy)^n + (x^2 + 3xy + 2y^2)^n \quad \text{for } n = 1, 2, 3.$$

In this identity, in order to derive terms in *ascending* order, we select any x and y subject to $13x \geqq 19y$, and $13x \leqq 21y$. Having derived the the terms, we can then add any desired number to each term and so obtain the required four terms in each side: a zero term would not be acceptable in the final solution.

It may be mentioned here, although perhaps obvious, that all terms in a multigrade may be divided by any common factor, or multiplied by any number, without affecting its validity.

The following identity gives a solution of the 3rd order *Prouhet* and *Tarry* problem. With suitable choice of x, y, z (i.e., no term can equal zero), this covers all possible cases in which the terms on one side are not evenly spaced.

$$x^n + (y - z)^n + (3y + 2z - 2x)^n + (2y + 3z - x)^n$$
$$= (2y - x)^n + (x - z)^n + (y + 2z)^n + (3y + 3z - 2x)^n \quad \text{for } n = 1, 2, 3.$$

Readers who are interested in finding general solutions for higher orders may feel tempted to face the somewhat laborious algebraical work that is involved. Inherently, the process entails no principles other than those already outlined.

Higher order solutions can also be found that are not fully general, although giving infinite numbers of particular numerical solutions. For example, I derived the following very simple semi-general solution for 5th order multigrades: so far as I know, it has not appeared in print before:

$$0^n + (2a^2 + 3a + 3)^n + (4a^2 - 3a - 2)^n + (6a^2 + 9a + 4)^n + (8a^2 + 3a - 1)^n + (10a^2 + 6a + 2)^n$$
$$= (a + 2)^n + (2a^2 - 2a - 2)^n + (4a^2 + 7a + 3)^n + (6a^2 - a - 1)^n + (8a^2 + 8a + 4)^n + (10a^2 + 5a)^n$$
$$\text{for } n = 1, 2, 3, 4, 5.$$

This, for example, leads to:

$$1^n + 9^n + 18^n + 38^n + 47^n + 55^n = 3^n + 5^n + 22^n + 34^n + 51^n + 53^n$$
$$\text{for } n = 1, 2, 3, 4, 5,$$

if we take $a = 2$, and then add 1 to each term throughout.

This has been intended only as a brief introduction to a subject that can provide endless fun for number enthusiasts. Quite apart from general solutions, they may enjoy building up numerical multigrades of 5th, 6th, and higher order by following the "switching" procedure that has been outlined.

Mr. Cross was born in Nova Scotia, educated at Bridgewater High School and Dalhousie University, Nova Scotia. Since 1949 he has lived in England, teaching in a Birmingham school. Surprisingly, until recently mathematics was only a hobby – he now teaches the subject – but he has contributed papers to the British mathematical journals *Mathematical Gazette* and *Eureka*.

****

SEA DISTANCE

Arising from a point raised by Mr. C. E. Branscome, of Lacoochee, Florida, it may be interesting to note that the distance of the sea horizon may be calculated quickly but approximately by the formula:

$$D = \frac{4\sqrt{h}}{3},$$

h being height of eye in feet, and D the distance in miles.

****

See page 17.

ALPHAMETICS

With this issue RMM has published a total of 66 different alphametics and cryptarithms. The score by country: Canada - 39½; USA - 15½; England - 9; Denmark - 1; Unknown - 1. In the June 1962 issue of RMM we tried to prod US puzzlists to catch up to Canada. The score, then, was 75% from Canada and only 16% from the US. *Now*, it's 60% and 24%, respectively. Can we hope for more representation from *other* countries?

* * * * *

```
 ANTS
 CANT
-----
 SCAN
```

Our ANTS, of course, are obviously very odd.
(*Jonathan Khuner*; Berkeley, California)

* * * * *

In this unusual alphametic each separate column, including its digit in the final total, adds up to a sum that is divisible by 8. For example, the sum of E, A, A, and T is so divisible.
(*George Propper*; Bronx, New York)

```
 OBEY
  SAM
  SAM
 ----
 VOTE
```

* * * * *

Here we have the detailed calculation of the cube of a 4-digit number, in two separate operations. In the first operation we derive the square of that number: in the second we multiply the 4-digit number by its square. The little x's indicate the positions of the digits. What is the original 4-digit number?

```
        xxxx                    xxxx
        xxxx                 xxxxxxx
       -----                 -------
       xxxxx                    xxxx
       xxxx                   xxxx
      xxxx                   xxxx
     xxxx                   xxxx
     -------              xxxxx
     xxxxxxx             xxxxx
                         -----------
                         xxxxxxxxxxx
```

For this original and most intriguing example of the rare "*no digits*" form of puzzle, we are indebted to *Willy Enggren* of Copenhagen, Denmark.

* * * * *

There can be no doubts as to what we have here.

So what is this PRIME?
(*A. G. Bradbury*; North Bay, Ontario)

```
  THIS
  SURE
    IS
 -----
 PRIME
```

* * * * *

```
  FLAT
    AS
  ----
  xxxx
 xOLD
 -----
 ACTOR
```

The wise ones "give up" in time, but you won't have to here!
(*A. G. Bradbury*; North Bay, Ontario)

SOUP, FISH, AND FINITE GEOMETRIES

by Ronald C. Read
Kingston, Jamaica

The other evening I invited my two mathematician friends, Professor Lovelace and Professor Pennywell, to dine out with me. I have dined with these two worthy gentlemen many times before, and I know their ways; so it was no surprise to me when, with the arrival of the soup, they produced pencils and paper, and began an earnest discussion in the peculiar jargon that is the stock in trade of the professional mathematician. Usually when this happens I let them get on with it; but this time I was determined to get into the conversation somehow. After all, I was paying for the meal!

"What are we discussing this evening?" I asked.

"Finite geometries," replied Professor Pennywell.

"What are they?"

"You mean to say you don't know what finite geometries are?" said Professor Lovelace. A modest man is Prof. Lovelace; he is always surprised to discover that a piece of information which *he* has managed to acquire should have eluded others. Prof. Pennywell was more helpful.

"It would be difficult to give a snap definition of them," he said, "but if you really want to know (I nodded to indicate that I did) I believe I can manage it. Let me see - I think I'll start by asking you a question. What is a line?"

"A straight line?"

"If you like."

"A straight line is the shortest distance between two points." I recited. (I pride myself that I remember most of what I learned in High School.)

"Bah! High School nonsense!" exclaimed Prof. Lovelace, in between two mouthfuls of soup. I was deflated.

"You're thinking of Euclid's definition," Prof. Pennywell went on, "Not a very satisfactory one even for his sort of geometry. But I think I put the question badly; what I was getting at was this: a line is a *collection*, or *set* of points."

I thought this over for a moment; something didn't seem quite right. Then I put my finger on the trouble.

"That can't be the whole story," I said, "a line can't be just *any* set of points. Otherwise how would you distinguish a line from, say, a circle, which presumably is also a set of points?"

"Quite right," replied Prof. Pennywell, "lines can't be just any old sets of points. They have to satisfy certain conditions or *axioms*, as they are called. But your soup is getting cold," he added, "finish it up, and we'll go a stage further."

I finished my soup, and we ordered the fish. Prof. Pennywell continued his explanation.

"The conditions that the lines have to satisfy are very simple. The first is that if you pick on any two points then there is exactly one line (a set of points, remember) which contains both of them."

"Or 'joins them' as Euclid put it in his quaint way," interjected Prof. Lovelace.

"That seems straightforward enough," I said, "Like when you join two points on a piece of paper by a ruled line. The two points are then just two of the infinite number of points that make up the line. I can picture that."

"Careful," warned Prof. Lovelace, "pictures are often misleading."

"The other condition," went on Prof. Pennywell, "is that if you take any two lines, they will 'intersect', or 'meet', in exactly one point. That is to say there will be exactly one point which belongs to both the lines."

"I can picture that, too - I think. But just a moment," I said, "aren't you making the assumption that the lines aren't parallel?" (Prof. Lovelace is always saying how important it is to watch out for hidden assumptions.)

"Ah, you're still thinking about Euclid's geometry," came the reply, "But the geometry *we* are talking about is a rather different kind, called 'projective geometry'."

I sighed. If there is one thing about mathematics that my friendship with the Professors has taught me it is that you make up the rules as you go along. And alter them, too, if you feel like it!

"Well, what is projective geometry?" I asked.

"I've been telling you," said Prof. Pennywell, "in fact, you have the whole picture now, more or less. You have a collection of points, and certain sets of these points. These sets you call lines. Now if these lines satisfy the two axioms - let's write them down:

> *Axiom* 1: Given any two points, there is exactly one line containing both of them, and
>
> *Axiom* 2: Given any two lines, there is exactly one point belonging to both of them.

if, as I was saying, the points and lines satisfy these two axioms, then the whole set-up of points and lines is called a projective geometry. To be strictly accurate I should say a *plane* projective geometry."

"I notice that you say '*a* projective geometry'. Does that mean that there are a lot of them?"

"Yes, indeed. There are many ways in which the axioms can be satisfied, and these correspond to different projective geometries. Ah! Here comes the fish."

"I think I get the idea." I said, after some thought. "There is just one question I'd like to ask, though I suspect that I know the answer already. You've defined lines as sets of points satisfying certain conditions or axioms, but what are points?"

"If you suspect that you know the answer already, why not air your suspicions first?" mumbled Prof. Lovelace through a mouthful of fish.

"Well," I said, "it seems to me that it doesn't matter much what a point is. I tend to think of a point as being like a pencil dot on a piece of paper, but it might just as well be - well, anything at all."

"That's exactly it." exclaimed Prof. Pennywell, with encouraging enthusiasm. "You can think of points in whatever way you like. Any collection of things of any kind will serve as the collection of points, provided it can be formed into sets (lines) satisfying the axioms. In mathematical jargon, points are 'undefined'; they are not given any properties because they don't need any. None of this 'a-point-has-position-but-no-magnitude' sort of business. All that matters is the structure that we get when the collection is sorted out into sets which satisfy the axioms."

"I'm beginning to get the hang of it," I said, "But we started off to define 'finite geometries'. Where does the finite part come in?"

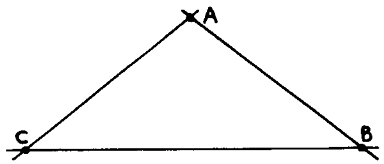
"Well now," Prof. Pennywell went on, "if you picture points as dots on a piece of paper, and lines as the sort of things that you draw with a pencil and a ruler (as you seem to do), then you are liable to think that there must necessarily be an infinite number of points on each line, and therefore an infinite number of points altogether. But this need not be so. It is quite possible to have a *finite* collection of points, sorted out into sets (lines) satisfying the required conditions. Then we say we have a finite geometry."

"Can you give me an example?"

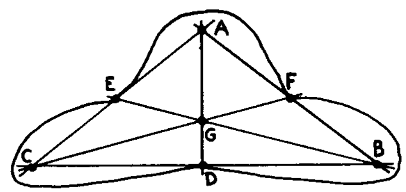
"Certainly. The simplest example is that in which there are three points. Let's call them A, B, and C."

"Very original," gibed Prof. Lovelace.

"There are three lines," went on Prof. Pennywell, ignoring the interruption, "one containing the two points A and B, another the points B and C, and the other the points C and A. This geometry can be represented diagrammatically - and I *do* mean diagrammatically - by this figure which I hope is more or less self-explanatory."



"I can see that the two axioms are satisfied," I said, after a pause during which the fish plates were cleared, and we ordered the next course, "but it doesn't look as though that geometry would be very - well, very interesting."



"It isn't," said Prof. Pennywell, "in fact it is not usually considered worthy of being called a geometry at all. So let's take the next example. In this one there are 7 points and 7 lines, with 3 points on each line and 3 lines through each point. Here is its diagram."

"What's the wiggly curve going round the whole thing for?" I asked.

"That represents one of the lines of the geometry," said Prof. Pennywell. "You see, in this geometry each line contains exactly 3 points; A, F, and B, for example, make up one line. These points are therefore 'collinear'. In the diagram I have tried to make these sets of points *look* as though they were collinear, and have drawn a line (that is to say, a pencil line) through them to emphasize their collinearity. However, there is one set of points which make up a line but which cannot be made to look collinear - in our case the points D, E, and F. The wiggly curve is meant to indicate that these 3 points do, nevertheless, make up a line of the geometry."

"I think I get it - but it does rather spoil the diagram."

"Can't be helped. But perhaps it would be easier if we dispensed with the diagram altogether. Suppose we let the letters A, B, C, D, E, F, and G represent the points of this geometry."

"Since the points are undefined," interrupted Prof. Lovelace, pausing for a moment from his battle with a steak that was not as tender as it might have been, "why not say that they *are* the points of the geometry."

"All right then," agreed Prof. Pennywell, "Let A, B, C, D, E, F, and G *be* the points of the geometry. There will be 7 lines, each a set of 3 points, and they are as follows:

$$AFB,\ BDC,\ CEA,\ AGD,\ BGE,\ CGF,\ DEF.$$

It is easy to check that the axioms are satisfied - there is exactly one set containing two given letters (points), and exactly one letter common to any two given sets. So we have a geometry."

"That's most interesting. Can you have a geometry with any number of points?"

"Oh no, not at all. The possible numbers are fairly limited. You can classify these geometries by the number of points belonging to each line. Suppose that there are $n+1$ points to each line . . . "

"Why $n+1$? Why not n?" I asked.

"No reason really. It's a little more convenient for the mathematical analysis, that's all. Well, if there are $n+1$ points to every line, then there will be $n+1$ lines through each point; that is to say, each point will belong to $n+1$ of the sets that are the lines of the geometry."

"Is that obvious?"

"Not quite, but it is very easy to prove. Try it when you get home. Now a geometry with $n+1$ points to each line will contain $n(n+1)+1$ points, and the same number of lines. That isn't difficult to prove either - you can try that as well."

"At least it checks with your two examples," I said. "When $n=1$ we have $1{\cdot}2+1=3$ points; with $n=2$ there are $2{\cdot}3+1=7$ points, as in your examples. So I suppose that if $n=3$ there would be . . . - by the way, can n take *any* value?"

"No, it can't," replied Prof. Pennywell, "and that is where the subject begins to become interesting. Not a great deal is known about the possible values that n can take. One general result, proved some time ago, is that you can have a geometry with $n+1$ points to a line when n is a prime number, or a power of a prime number. That is *not* easy to prove, by the way."

"Then I'll take your word for it. Anyhow, the case $n=3$ seems to work, since 3 is a prime."

"Correct."

"And the geometry will have $3{\cdot}4+1$ or 13 points and 13 lines."

"Quite right."

"After that comes 4 - a power of a prime; and then 5, a prime. So those two are OK. How about $n=6$? That is not the power of a prime. Is there a geometry with 43 points and 43 lines?"

"No; 6 is the first value of n not covered by the general result. In point of fact it has been proved that there is no projective geometry for $n=6$. In other words, it is impossible to take 43 objects, and pick out 43 sets of 7 objects each in such a way that the axioms are satisfied."

"How curious! Again, I'll take your word for it. What is the next value of n not covered? If I am right 7, 8, and 9 all come under the general result, so it will be $n=10$. What happens there?"

"The answer to that is that nobody knows," replied Prof. Pennywell. "No one has managed to produce a geometry with 11 points on each line, but on the other hand, no one has been able to prove that it can't be done. That is what Lovelost over there and myself were talking about earlier on. He thinks he has found a proof that a geometry with $n=10$ is impossible, like $n=6$. But I'm still skeptical."

"You always are," grumbled Prof. Lovelace, a little riled at the wanton mutilation of his surname.

"On the contrary," replied Prof. Pennywell, "I am readily convinced by a clearly stated argument. But *your* exposition . . . "

" . . . was crystal clear, as always."

"Hogwash! Take this Galois field GF* that you introduce near the beginning of the proof. It's not at all clear how this has any relevance whatever to . . . "

The Professors were back to their arguing, and back in the realm of higher mathematics where I could not follow them. I retired graciously, and during the rest of the meal (which, incidentally, finished with the finest Coupe Jacques that I have ever tasted) I thought over what Prof. Pennywell had been telling me. I had an idea that it held the makings of a puzzle. Not a very mathematical puzzle - indeed, hardly mathematical at all, and certainly the sort of thing that Prof. Lovelace would snort at in disgust; but possibly an interesting puzzle for all that.

Late that evening I tried my hand at the two proofs that Prof. Pennywell had mentioned, and found (to my surprise) that they *were* fairly easy. The first result was that if, in a geometry, there were $n+1$ points on each line, then each point will belong to $n+1$ lines. My proof went as follows:

Let P be a point. Take a line which doesn't contain P. Call it L. This line L will contain $n+1$ points, say A_1, A_2, A_3, . . , A_{n+1}. By the first axiom we have lines PA_1, PA_2, PA_3, and so on - $n+1$ of them - all through P. What is more, all these lines are different; for if two of them, say PA_3 and PA_5, were the same line, then this line would have *two* points (A_3 and A_5) in common with L, whereas axiom 2 allows them only one point in common.

This gives us $n+1$ lines through P, but we ought to check that there can't be any others. This is easy. If there were another line through P it would have a point in common with L; but we have already accounted for all the points on L.

Since P was supposed to be any point, we have proved what we set out to prove.

I showed my proof to Prof. Pennywell. He grinned and said, "Strictly speaking this isn't correct. You have overlooked the possibility that all the points might lie on the same line. In which case you would not be able to choose a line L which does not contain P."

"But can this happen?" I asked, "If there is only one line in the geometry, and all the points lie on it, surely the axioms could not be satisfied. At least, not the second." I added hastily, realizing that there would be no difficulty satisfying the first axiom, "It specifically mentions *two* lines."

"Yes, it might seem like that," replied the Professor, "but in fact the second axiom would be satisfied in a peculiar, empty sort of way. Look at it like this. The only way in which the second axiom could be violated would be if there were two lines which had more than one point in common. And this couldn't happen, for the simple reason that we couldn't even get the two lines to start off with!"

"It sounds a bit of a quibble to me," I said, "But if what you say is correct then the results you quoted are not, in fact, true."

"Quite so; and the fault is all mine." the Professor admitted, "I should have added an extra condition - another axiom, really - to the effect that the geometry has at least two points and at least two lines.

Then it is not difficult to see that you can indeed have a line L not containing the given point P (you'd better check that some time) and your proof is quite valid."

"Thank goodness for that," I said, "Now will you look at my other proof. You asked me to prove that in a geometry with $n+1$ points on each line there are $n(n+1)+1$ points altogether."

Prof. Pennywell looked quickly through my proof, and OK'd it. It went as follows:

Through any point P there are $n+1$ lines, each of which contains n points other than P. Hence these lines contain $n(n+1)$ points other than P.

The geometry therefore contains at least $n(n+1)+1$ points - counting in P as well. Can there be any others? No; for if A is any point other than P, there is a line containing P and A (first axiom). Hence A, lying as it does on a line through P, will have been counted by the procedure just given.

Then I started thinking about my puzzle. Prof. Pennywell had chosen the letters A, B, C, D, E, F, and G for the points of his 7-point geometry. I began wondering whether it would be possible to pick some other 7 letters of the alphabet in such a way that the 7 lines of the geometry would spell out honest-to-goodness, three-letter English words. After a bit of experimenting I came up with the following:

DRY, *ORE*, *BOY*, *ADO*, *BAR*, *AYE*, *BED*.

I checked that the two axioms were satisfied. They were. Any pair of words has exactly one letter in common, and there is only one word containing any given pair of letters.

So far, so good. But was the same thing possible, I wondered, with the next geometry, the one with 13 points and 13 lines, and with 4 points on every line? It was a question of finding 13 letters from which could be made 13 4-letter words, in such a way that there was just one word containing any two given letters, and that any two of the words would have exactly one letter in common.

This proved to be a much more difficult task, and time passed quickly as I tried combination after combination. At length, at three in the morning, I gave up. I hadn't solved the problem, but I had managed to find the following near solution:

AIRY, PUNY, PITS, LAND, APSE, LION, SLUR,
RENT, DROP, AUTO, DIEU, OYES, TYLD.

The last 'word' is one fly in the ointment.* It looks vaguely Scandinavian, or possibly Welsh, but is certainly not English. The two words before it are a bit suspect, too, but they will do in a pinch. This was the nearest that I could get to a solution. Perhaps some RMM readers can do better. Why not try?

* Editor's Comments: There are other 'flies'. For example, *OYES* and *LAND* have no common letter (also *LAND-PITS* and *APSE-TYLD*). *LION* and *LAND* both have *LN* in common (also *APSE-PITS* with *PS* in common and *APSE-OYES* with *ES* in common)- a total of six additional 'flies' in the ointment. However, a check shows that each letter appears in exactly four words - as should be expected.

The Editor, working from Mr. Read's near solution, submits the following list which has no flaws as far as the two axioms are concerned:

AIRY, PUNY, LAND, PATS, DYES, SLUR, SINO,
DROP, TERN, PILE, AEOU, DITU, LOTY.

However, the last three 'words' constitute *three* flies in the ointment! A different choice of letters appears necessary. (*J. S. M.*)

SUBSCRIPTION RENEWALS - MAILING LABEL CODES

Renewal notices are generally sent out to subscribers whose subscriptions are terminating about a week after each issue is mailed out. We urge all to send in their renewals soon before copies of RMM are depleted - thereby forming a gap in your file of RMMs.

If you would like to know when your subscription ends look at your mailing label. The last three digits indicate the date of the last issue of your current subscription. The last two of these digits indicate the year (63 = 1963, 64 = 1964, etc.), the first of the three digits indicate the issue of that year (1 = February, 2 = April, 3 = June, 4 = August, 5 = October, 6 = December). If your label reads 163, then *this* issue (FEBRUARY 1963) is the last you are due to receive - *send in your renewal notice and payment immediately* ($3.25 per year). If your label reads 263 or 363 you will receive the Combined April-June 1963 issue as the last of your current subscription. Renew *now* so you won't forget later. A code reading 463, 563, 663 or later gives you until the August 1963, October 1963, December 1963 or later issue to renew - but don't keep putting it off.

The first four digits of the mailing label code indicates the mailing zone and state: 1805 means zone 8, California; 1436 means zone 4, New Hampshire. Foreign countries are also indicated by the first four digits: 2104 means Canada, Province of Manitoba; 5400 means India; 3753 means England, Yorkshire county.

The next four digits indicate the city within the state (or country): 1805 4530 means Los Angeles, California; 3732 5000 means London, England; 8200 5000 means Moscow, Russia (yes, RMM goes there, too).

RMM goes to all 50 of the United States, most of the US territories, most of the Canadian provinces, and (as of this mailing) to 32 other countries.

The next six digits on the mailing label indicate the individual customer code number. Each RMM subscriber has a permanent and unique number: 181120 is Alan L. Brown; 412640 is J. A. H. Hunter; 530080 is the Library of New Mexico State University. This customer code number, alone, would enable us to locate the subscription file card of an RMM subscriber. Rarely, these 'permanent' numbers will change: a *Miss* Mary A. Brown (182840) becoming a *Mrs.* Rasputen Q. Prezbylovich (737800); or a subscriber giving only his initials at first (J. A. Jones - 434840) and, later, his full name (Jacques A. Jones - 435190). Occasionally, a subscriber will be given a wrong number and this would be corrected immediately upon discovery.

CATCHING UP - GETTING BACK ON PUBLICATION SCHEDULE

It is hardly a secret to RMM readers that RMM has fallen behind in its publication schedule! There have been a number of reasons which have interfered with normal schedules - all of these reasons seem to have delayed, rather than hastened, publication.

To get caught up one issue at a time *might* take until the end of the year. Therefore, we are going to publish the APRIL 1963 and JUNE 1963 issues *together* in a single 96-page issue. There will be quite a few more articles (see page 2 of this issue for a brief table of contents of the *Combined Issue*) and some of the regular features and departments will be increased in length.

With just reasonable luck we hope to be back on schedule with the AUGUST 1963 and/or OCTOBER 1963 issues.

THE ELUSIVE NUMBER PI

by Emil Fridstein Vaage
New York, New York

The number π (pi), which appears so frequently in formulas used in physics and engineering, has never been evaluated completely. A brief history of the efforts has been given in RMM: *π Has Been Calculated to 100,265 Decimal Places*, RMM No. 8, April 1962, pages 20-21.

One way to estimate the numerical value of π is by means of continued fractions, said to have begun with Bombelli in 1572. Later on, the Japanese mathematician Takebe Kemmei (1722) showed that by using this method the following approximations could be obtained:

$$\pi_1 = 3\tfrac{1}{7} = 3.14\ldots$$
$$\pi_2 = 3\tfrac{15}{106} = 3.1415\ldots$$
$$\pi_3 = 3\tfrac{16}{113} = 3.141592\ldots$$
$$\pi_4 = 3\tfrac{4687}{33102} = 3.141592653\ldots$$

In the above expressions, only the correct number of decimal digits have been given. In order to obtain a measure of the efficiency of these fractions, $\frac{1}{7}$, $\frac{15}{106}$, $\frac{16}{113}$, and $\frac{4687}{33102}$, let us define their *Goodness Factor* as the ratio of the correct number of decimal digits obtained to the number of digits in the denominator of the fraction. Thus:

$$G_1 = \tfrac{2}{1} = 2.0$$
$$G_2 = \tfrac{4}{3} = 1.333$$
$$G_3 = \tfrac{6}{3} = 2.0$$
$$G_4 = \tfrac{9}{5} = 1.8$$

In my spare moments I have enjoyed myself trying to find a better approximation with more correct decimal digits and with a better *goodness factor* than 2.

The next few approximations obtained from the continued fraction method resulted in more correct digits for the number π, but the goodness factor of these expressions did not exceed the value of 2. It was the first after the 20th approximation was evaluated that a better result was obtained:

$$\pi_{20} = 3\tfrac{948881364}{6701487259} = 3.141592653589793238462\ldots$$
$$G_{20} = \tfrac{21}{10} = 2.10$$

The fraction given in π_{20} gives π correct to 21 decimal places, with a goodness factor equal to 2.10. The numerator and the denominator in this fraction can be factored as follows:

$$948881364 = 2\times2\times3\times661\times119627$$
$$6701487259 = 101\times4457\times14887$$

Since all of these factors are prime numbers, and none are common to both numerator and denominator, the fraction in π_{20} cannot be reduced.

After further search, a fraction with a better *goodness factor* has not been found. Does a better one exist?

SIR ISAAC NEWTON, PROBLEMIST

by Maxey Brooke
Sweeny, Texas

If the experts were all asked to name the greatest mathematician of all time, Newton's name would probably appear most often. The co-author of the calculus is usually pictured as a dour and humorless man.

No other mathematician has had so much biographical material written about him. For that reason, only a brief outline of his life will be given.

Newton was born at Woolsthrope, Lincolnshire, December 25, 1642. He was taken out of school at the age of 15 to work on a farm, but his conspicuous lack of ability for this work earned him the right to go back to school. He entered Trinity College, Cambridge, in 1661, graduated in 1665, and took his M.A. in 1668. He was made Lucasian Professor of Mathematics at Cambridge the following year and, in 1671, was elected a Fellow of the Royal Society. He sat in the convention parliament of 1689-1690, was made Master of the Mint in 1699, took his seat in parliament for Cambridge University in 1701, and was knighted in 1705.

In addition to developing the calculus, he discovered the principles of light refraction and invented the reflecting telescope. His law of gravitation is said to have been the greatest scientific contribution of any single man. Most of his life's work was summarized in his classic *Philosophia Naturalis Principia Mathematica*. He died at Kensington, March 20, 1727.

Against this imposing array of accomplishments, his contributions to recreational mathematics seem slight. Indeed they are hardly ever mentioned by his biographers. Here, then, is a new facet in a great man's life and works; a puzzle, a riddle, and a problem.

The puzzle of how to plant nine trees in ten rows of three each has already been discussed in this magazine (RMM No. 6, December 1961, page 51).

The riddle is a bit of doggerel:

Four persons sat down at a table to play,
They played all that night and part of next day.
It must be observed that when they were seated,
Nobody played with them and nobody betted;
When they rose from that place, each was winner a guinea.
Now tell me this riddly, and prove you're no ninny.

And the problem:

Three pastures are covered with grass of equal density that grows at an even rate. The first pasture has an area of 33 acres, the second has an area of 100 acres, and the third 240 acres. On the first pasture twelve oxen can feed for four weeks and on the second twenty-one oxen can feed for nine weeks. How many oxen can feed on the third pasture for eighteen weeks?

LETTERS TO THE EDITOR

Dear Mr. Madachy:

H. E. Dudeney in his *Amusements in Mathematics* declares that he had a solution of the problem of 13 Knights who wish to sit around a table on 66 occasions so that no Knight has the same neighbors more than once. Dudeney appears never to have disclosed his solution.

Can any RMM readers say how it may be done, and how the problem is solved for any number of Knights?

Dublin, Ireland — Victor Meally

* * * * *

Dear Sir:

I would very much like to obtain copies of the JUNE 1961 and the FEBRUARY 1962 issues of RMM. If any readers would like to sell their copies, please state the price to me at the address below. I am also interested in exchanging books on mathematical recreations.

Van Merlenstraat 98 — C. C. Verbeek
The Hague, Holland

Editor's Note: An airmail postcard or postal aerogramme to Holland costs 11¢.

* * * * *

Dear Sir:

In the June 1962 issue of RMM (page 44), Charles W. Trigg quoted the story of Hardy's ride in taxi number 1729 while on his way to visit Ramanujan who was ill in Putney. Ramanujan pointed out that 1729 was the smallest number giving the sum of two cubes in two different ways. Here are some observations on that number to supplement those given by Professor Trigg.

$$1729_{10} = 2331_9 = 1332_{11} \text{ (i.e., bases 10, 9, and 11)}$$
$$1729_{10} = 1001_{12} = 12001_6 = 123001_4$$
$$1729 = 6^2+18^2+37^2 = 8^2+12^2+39^2 = 8^2+24^2+33^2$$
$$= 10^2+27^2+30^2 = 12^2+17^2+36^2 = 18^2+26^2+27^2$$

Representations as the sums of 4 cubes, and of 6 cubes, were given. 1729 is also the sum of 5 cubes, 9 cubes, and 11 cubes:

$$1729 = 1^3+3^3+3^3+7^3+11^3$$
$$= 1^3+2^3+3^3+4^3+5^3+6^3+6^3+7^3+9^3$$
$$= 1^3+1^3+2^3+3^3+4^3+5^3+6^3+6^3+6^3+7^3+8^3$$

When Hardy asked Ramanujan if he knew any corresponding cases for fourth powers, Ramanujan replied that he could think of no obvious example but that the smallest such number must be very large.

The smallest such number certainly seems to be large! Here are three examples, the first being the smallest solution that is known:

$$635318657 = 59^4 + 158^4 = 133^4 + 134^4 = 41\cdot 113\cdot 241\cdot 569$$
$$2094447251857 = 76^4 + 1203^4 = 653^4 + 1176^4$$
$$155974778565937 = 1623^4 + 3494^4 = 2338^4 + 3351^4$$

Finally, I would like to comment on Professor Trigg's remark that 1729 can be expressed as the sum of selected terms of the Fibonacci series. In fact it is not difficult to show that every number has this property, by the following argument.

If all integers up to $(F_n - 1)$ can be expressed using the first $(n-1)$ terms, then all integers up to $(2F_n - 1)$ can be expressed by merely adding F_n to the given expressions. Now, for $n > 5$, we have $2F_n - 1 > F_{n+1}$. Since every integer up to $F_5 = 5$ is expressible, it follows that all those up to F_6, and then similarly up to F_7, and so on, are expressible.

It may be worth noting that the same applies for representation using the powers of 2, with the refinement that each such expression is unique. These unique expressions are the binary-scale numbers!

Knottingley, Yorks, England — Alan Sutcliffe

Editorial Note: Arising from this, J. A. H. Hunter comments as follows:

$$2094447251857 = 17\cdot 2129\cdot 6481\cdot 8929$$
$$155974778565937 = 313\cdot 3217\cdot 5521\cdot 28057$$

Alan Sutcliffe's fourth-power examples stem from the solution developed by Gérardin from Euler's work on the problem. This, the only known solution, gives an infinite number of examples although not necessarily all possible examples:

If $N = X^4 + Y^4 = W^4 + Z^4$,

$$X = a(a^6 + a^4b^2 - 2a^2b^4 + 3ab^5 + b^6),$$
$$Y = b(a^6 - 3a^5b - 2a^4b^2 + a^2b^4 + b^6),$$
$$W = a(a^6 + a^4b^2 - 2a^2b^4 - 3ab^5 + b^6),$$
$$Z = b(a^6 + 3a^5b - 2a^4b^2 + a^2b^4 + b^6).$$

This seems a good opportunity to announce the complete factorization of the complicated algebraical expression for that equal sum of two pairs of fourth powers, recently discovered by myself. So far as we know, this is new.

$$N = (a^4 + 6a^2b^2 + b^4)(a^8 - a^4b^4 + b^8)(a^8 - 4a^6b^2 + 8a^4b^4 - 4a^2b^6 + b^8)$$
$$(a^8 + 2a^6b^2 + 11a^4b^4 + 2a^2b^6 + b^8)$$

e.g., $17332^4 + 529^4 = 17220^4 + 6673^4 = 353\cdot 51137\cdot 65281\cdot 76577$
$= 90{,}239{,}171{,}293{,}339{,}457$

which would defy most attempts at factorization!

The foregoing leads to thoughts on similar treatment for equal sums of pairs of cubes. Ramanujan evolved the semi-general solution for the values of pairs of numbers:

An infinitude of solutions for $N = X^3 + Y^3 = W^3 + Z^3$ is given by:

$$X = 5b^2 + 5ab - 3a^2 \qquad W = 4a^2 + 4ab + 6b^2$$
$$Y = 4b^2 + 4ab + 6a^2 \qquad Z = 5a^2 + 5ab - 3b^2$$

where, approximately, $25b \geqq 10a \geqq 4b$.

I have established the factorization of N for all such cases as: $N = 63(a^2 - ab + b^2)(a^2 + 3ab + 3b^2)(3a^2 + 3ab + b^2)$. So far as we know, this is also new.

Additional Editorial Note (J. S. M.): Interested readers may consult a complete list of 101 solutions to $N = X^3 + Y^3 = W^3 + Z^3$ (for $N < 5{,}000{,}000$) compiled by C. E. Britton and published in *Scripta Mathematica*, Vol. XXV, No. 2 (July 1960), pages 165-166.

* * * * *

Dear Mr. Madachy:

I am interested in any biographical data on Edouard Lucas. Can RMM readers supply any references?

912 Old Ocean Avenue
Sweeny, Texas

Maxey Brooke

* * * * *

Dear Mr. Madachy:

In the October 1962 issue of RMM (pages 9-10) Richard K. Allen presented an article entitled *4,000 Years of Easter*. It reminded me of a formula developed by Gauss to determine the day of Easter Sunday.

Perhaps RMM readers would be interested in the formula as an amazing example of the reduction of a very complicated process to a simple arithmetic form.

For the years 1900 to 2099, $x = 24$, $y = 5$, $N =$ calendar year. a, b, and c are remainders of $N/19$, $N/4$, $N/7$ respectively. d is the remainder of $(19a + x)/30$; e is the remainder of $(2b + 4c + 6d + y)/7$. Easter Sunday is $d + e + 22$ for March and $d + e - 9$ for April.

The formula breaks down if the result is April 25 or April 26. If the result is April 25 the formula does not apply without some extensive recalculations, the details of which are unknown to me. If the result is April 26, Easter Sunday is April 19.

Toronto, Ontario

E. D. Gibb

* * * * *

Dear Sir:

The following are some symbols which I have found quite useful in high school mathematics classes. Perhaps other RMM readers have used similar symbols in their own classes.

$\not\propto$ means *finite* or *not infinite*

$|\!\in\!|$ means *one to one correspondence*

$|\!=\!|$ means *is equal to the absolute value of*

$\nearrow$ means *increasing* $\searrow$ means *decreasing*

$\equiv$ means *identically equal to* in the sense of an equation which is true for all values of the unknown.

$\triangle$ means *perpendicular bisector of*. This is a very frequent and cumbersome expression in geometry.

San Mateo, California

Lloyd A. Walker

BOOK REVIEWS

Edited by Dmitri E. Thoro
San Jose State College

Mathematics: The Man-made Universe. By Sherman K. Stein. W. H. Freeman and Co., San Francisco, 1963, xiii + 316 pages, \$6.50.

Rarely can a book be recommended to both the amateur and advanced mathematician - but Stein's unique blending of scholarship, lucid style, and wit should be a joy to the most lethargic reader (or sophisticated teacher).

RMM readers will be delighted with the rich assortment of recreational mathematics topics: The Highway Inspector and the Salesman; Magic Squares; Map Coloring; The Fifteen Puzzle; Tiling; and especially the nearly two dozen unsolved problems.

In the words of the author, "This book grew out of a college course designed primarily to give students in many fields an appreciation of the beauty, extent, and vitality of mathematics." Since many chapters use only grammar school arithmetic, this is an excellent source of enrichment material for high school students.

Some areas represented are number theory, topology, set theory, geometry, and combinatorial analysis. Normally routine excursions take on the spirit of an exciting adventure. The reader is encouraged to make numerous conjectures and generalizations.

Many of the problems are ingenious. You are invited to use Dirichlet's "pigeonhole principle" (if there are $k+1$ pigeons in k holes, at least one pigeonhole has two or more pigeons) to prove such results as "If 17 pins are stuck into a piece of cardboard in the form of an equilateral triangle of side 2, then at least two of the pins are within a distance ½ of each other"; or, "If $p > 2$ is a prime, then there exists a positive integer A such that $2^A - 1$ is divisible by p."

Twenty-five pages on map coloring constitute one of the best elementary introductions to a subject investigated by Moebius, de Morgan, Cayley, Descartes, and Euler - to name a few. A similar statement would be valid for the chapter on "Memory Wheels".

Professor Stein's originality is often reflected in the seventeen chapters and three appendices. In addition, I feel that some RMM readers would be willing to buy *Mathematics: The Man-made Universe* for the 118 references alone!

D. E. T.

* * * * *

The Fibonacci Quarterly Journal. \$4.00 per year.

The first issue of this newcomer to the ranks of mathematical periodicals appeared in February 1963. It is the official journal of The Fibonacci Association which has emerged from the chrysalis of the Pro-Fibonacci Group formed by a core of enthusiasts at San Jose State College, California. This publication is devoted to the study of integers with special properties, with particular emphasis upon Fibonacci and other recurrent sequences.

The avowed purpose of the quarterly is to encourage mathematics teachers to engage in research and to offer them publication opportunity, to provide ideas which may be adapted to classroom projects, and to present material suitable for high school study.

The dual objective of research and education is well exemplified in the first 75-page number. Approximately half of the issue is devoted to research papers, which show that the activity is not purely recreational. Part II is pointed toward the beginner, the amateur, and the high school student. The material in it is of an expository nature, clearly written, and should encourage the mathematical hobbyist to delve further into the mysteries of arithmetic. There is an advanced problem section and there is also an elementary problem section. Each section offers a number of well-chosen, interesting problems.

The magazine is printed by offset from typewritten copy, in a typesize which is easily read. The organization is good, despite the fact that headings and sub-headings do not stand out as clearly as they would if a larger variety of type faces were available. It would be well to emphasize the division into two parts more clearly in the table of contents.

The editorial board and the list of those cooperating present a nice balance of professional mathematicians, problem addicts, and mathematical recreationists. The well of imagination, ability and enthusiasm represented here coupled with their sincere invitation for active reader participation indicates that the Quarterly will have a long and interesting life.

If anyone finds it difficult to conceive of a periodical devoted solely to Fibonacci and related numbers, let him satisfy his curiosity and discharge his skepticism with a trial subscription. Send it to Brother U. Alfred, St. Mary's College, Moraga, California. It will be a good investment.

Charles W. Trigg
Los Angeles City College

* * * * *

Pillow Problems and a Tangled Tale. By Lewis Carroll. Dover Publications, Inc., New York, 1958, xx + 152 pages, $1.50 (paper).

In *Pillow Problems and a Tangled Tale*, the creator of the Mad Hatter, the Red Queen, the March Hare and the other assorted inhabitants of the World of Alice offers a panacea for all those afflicted with insomnia or weighted down with the unsolvable problems of the mundane world. His solution to the beforementioned problems does not involve the prescription of tranquilizers, sleeping pills or the counting of sheep but rather an escape from it all by an excursion into the world of mathematics. Becoming occupied in an absorbing mathematical problem will immediately push all else to the background of one's consciousness contends Carroll and it is his opinion that "an hour of calculation is much better . . . than half-an-hour of worry."

The book is divided into two sections. The first presents seventy-two "pillow problems", all of them solved by Mr. Carroll either "in the head while lying awake at night" or "while taking a solitary walk". The bulk of these problems may be classified as algebraic, geometric of two dimensions or trigonometric of two dimensions. The very strength of this book lies in the fact that the problems are challenging and imaginative and yet do not require an extensive mathematical background in order to comprehend the problems or solve them successfully. It is my opinion that those problems concerned with the geometry of the plane are the most interesting but certainly this is a purely subjective matter. A good problem is not terminal in that it leads one beyond the solution by suggesting further problems and leads the solver to generalize and determine those conditions under which a given problem has a solution. Such are the problems posed by Lewis Carroll.

A good many of Carroll's geometrical problems involve the possibility of the inscribing of one figure inside of another as in the following: "In a given triangle place a hexagon having its opposite sides equal and parallel and three of them lying along the sides of the triangle and such that its diagonals intersect in a given point". Certainly by drawing a proper diagram and applying a strong intuitive approach one is readily able to see that a solution is possible yet the heart of the problem rests in proving the opposite sides of the hexagon are in fact parallel and equal. Possibly still more provocative is the question raised by the selection of the given point. Can the point at which the diagonals of the hexagon concur be selected randomly at any location in the inner region of the triangle? If not, what restrictions must be placed on the selection of this point so that the hexagon can be successfully inscribed within the triangle? What happens to the hexagon as this point in question is moved toward the base or the apex of the triangle? Are there degenerate cases, i.e., where the hexagon becomes a parallelogram, a line or just a point? In fact, the point must be selected from inside a certain triangular area within the original triangle if the hexagon is to to be properly inscribed. The nature of this very special interior triangle is also a possible area of further exploration.

As previously implied there are many problems to satisfy the number theorist, the algebra buff and those persons who, like myself, enjoy dabbling in the field of probability. There are several fine problems involving the random selection of objects under varying conditions, the most interesting and one of which Carroll himself was most proud is the following: "A bag contains two counters, as to which nothing is known except that each is either black or white. Ascertain their colors without taking them out of the bag."* Carroll's solution is ingenious and the problem itself is properly classified under transcendental probabilities.

The second half of the book, *A Tangled Tale*, is a delightful story in the traditional Carroll style which mixes good sense with nonsense. The tale is constructed in ten knots rather than chapters, each of which contains a mathematical problem which must be abstracted from its strange surroundings and put into some form which is easily understood and attacked. It tests one on two accounts, the ability to abstract from a verbiage as foreign as Carrollese and ingenuity in eventual solution. The knots can be resolved to algebraic and geometric problems. As Carroll states in his preface, his intent in the writing of the tale was, "for amusement and possible edification" and he succeeds on both scores.

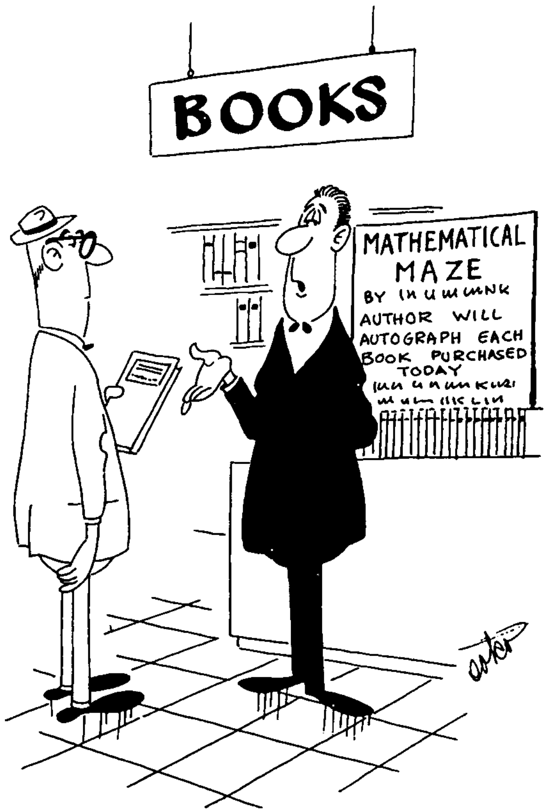
---

* See *Mad Mathematics* by C. Stanley Ogilvy in RMM No. 9, June 1962, page 22.

In conclusion I should like to say that one need not be an insomniac nor a worry wart in order to enjoy Mr. Carroll's book. Nor, may I add, need one be confined to the pillow or committed to the solution of these problems without pencil or paper. Solutions are shown for all problems including the knots of the *Tangled Tale*, some of which may be considerably shortened by a clever reader. I find the symbolism and notation used in solution of these problems to be a bit archaic but this shortcoming is easily hurdled. Rather than be committed to an abyss of arithmetical agony, if a solution is not forthcoming the reader need only refer to the rear of the book - hence one who is not an insomniac is in no danger of becoming one for want of a solution.

Robert H. Gurland
New York University

* * * * *



"THERE WILL BE A SLIGHT DELAY -

THE AUTHOR GOT LOST."

NUMBERS, NUMBERS, NUMBERS

NUMBERS WITH DISTINCT DIGITS OF THE FORM (M−1)M(M+1)

by Charles W. Trigg
Los Angeles City College

There are forty-eight numbers with distinct digits which are the products of three consecutive integers. Five pairs of these, those marked with asterisks (*), are permutations of the same sets of digits.

M	*N*	*M*	*N*	*M*	*N*
1	0	29	24360	111	1367520
2	6	30	26970*	130	2196870
3	24	31	29760*	140	2743860
4	60	33	35904	170	4912830
5	120*	34	39270	183	6128304
6	210*	39	59280	196	7529340
8	504	44	85140	205	8614920
9	720	60	215940	270	19682730
11	1320	65	274560	333	36925704
13	2184	66	287430	341	39651480
14	2730	71	357840*	380	54871620*
17	4896	86	635970	409	68417520*
18	5814	90	728910	429	78953160
19	6840	91	753480*	454	93576210
20	7980	97	912576	1268	2038719564*
21	9240	108	1259604	1333	2368591704*

There are no values of *N* with exactly nine digits. There are two pairs of values of *N*, 24-78953160 and 2184-635970, which together contain the ten digits. The digits of *N* are all even for $M = 1$, 2, 3, 4, and 19. The digits of *N* are permutations of consecutive digits for $M = 5$, 6, and 11, and if the ten digits are considered to be arranged consecutively in a closed cycle, for $M = 20$, 90, 130, 170, and 270.

The frequency of the ten digits in the various values of *N* is given in parentheses following the digits: 0(42), 1(23), 2(33), 3(21), 4(27), 5(22), 6(25), 7(25), 8(23), 9(24). The seven 8-digit values of *N* are the solutions of problem 154, *School Science and Mathematics*, June 1938, page 711. The two 10-digit values of *N* are solutions of problem E 338, *American Mathematical Monthly*, May 1939, page 298.

* * * * *

NUMBER CURIOSITY - SOLUTIONS

In RMM No. 10, August 1962, page 34, *Donald L. Vanderpool* proposed an interesting problem. He mentioned that Dudeney had found 139,854,276 and 923,187,456 as the smallest and largest squares ($11{,}826^2$ and $30{,}384^2$, respectively) that contain all nine digits (excluding 0). It was required to find squares containing each of these nine digits twice, thrice, etc.

Harry L. Nelson of Livermore, California has supplied us with some interesting results along these lines. He programmed an IBM 7030 (STRETCH) computer to solve the problem and found the following:

a. The smallest and largest 10-digit cases:

$32043^2 = 1026753849$ and $99066^2 = 9814072356$

b. The smallest and largest 18-digit cases:

$335180136^2 = 112345723568978496$ and
$999390432^2 = 998781235573146624$

c. The smallest and largest 27-digit cases:

$10546200195312^2 = 111222338559598866946777344$ and
$31621017808182^2 = 999888767225363175346145124$

Mr. Nelson writes that it took the computer 30 seconds to find each of the results in the 10- and 18-digit cases; and 73 seconds to find each of the results in the 27-digit cases.

In connection with this problem, we quote from RMM No. 8, April 1962, page 34, the following result by *David B. Hollander*:

$$(246913578)(987654312) = (493827156)^2$$

where each factor and the squared term contains each of the nine digits.

Merely as *oddities*, and without any bearing on the problem, *J. A. H. Hunter* has found all the 18-digit squares that end with Dudeney's 139,854,276:

$$460{,}949{,}326^2 = 212{,}474{,}281{,}139{,}854{,}276$$
$$539{,}050{,}674^2 = 290{,}575{,}629{,}139{,}854{,}276$$
$$749{,}988{,}174^2 = 562{,}482{,}261{,}139{,}854{,}276$$
$$750{,}011{,}826^2 = 562{,}517{,}739{,}139{,}854{,}276$$

* * * * *

NUMBER CURIOSITIES

$$4^2 + 3^3 = 43 \qquad 6^2 + 3^3 = 63$$
$$1 + 3^2 + 5^3 = 135 \qquad 1 + 7^2 + 5^3 = 175$$
$$5 + 1^2 + 8^3 = 518 \qquad 5 + 9^2 + 8^3 = 598$$

– *J. A. H. Hunter*

* * * * *

$$101 \cdot 101 = 10201$$
$$11 \cdot 10201 \cdot 11 = 1234321$$
$$10101 \cdot 10101 = 102030201$$
$$11 \cdot 102030201 \cdot 11 = 12345654321$$
$$1010101 \cdot 1010101 = 1020304030201$$
$$11 \cdot 1020304030201 \cdot 11 = 123456787654321$$

– *J. A. H. Hunter*

* * * * * *

$$\frac{14}{7} - \frac{7}{14} = \frac{147}{7 \cdot 14}$$

– *J. A. H. Hunter*

VARIATIONS ON THE 1963 THEME

$$19 + 63 + 96 + 31 = 13 + 69 + 36 + 91$$
$$19^2 + 63^2 + 96^2 + 31^2 = 13^2 + 69^2 + 36^2 + 91^2$$
$$196 + 319 + 963 + 631 = 136 + 369 + 913 + 691$$
$$196^2 + 319^2 + 963^2 + 631^2 = 136^2 + 369^2 + 913^2 + 691^2$$
$$1963 + 3196 + 6319 + 9631 = 1369 + 9136 + 6913 + 3691$$
$$1963^2 + 3196^2 + 6319^2 + 9631^2 = 1369^2 + 9136^2 + 6913^2 + 3691^2$$

– *Clifford R. Dickinson*; Camas, Washington

AND MORE

$$\begin{aligned}1963 &= (1+1)(1111-111) - 111/(1+1+1)\\ &= (9+9)(99+9+.\dot{9}) + .\dot{9}\\ &= (666-6-6)\sqrt{(6/.\dot{6})} + 6/6\\ &= 3!(333) - 33 - 3 + 3/3\\ &= (1+9+6+3)(1+96+3) + (1+9)6+3\end{aligned}$$
$$(6-1)^{9/3} = -1^2 + 9^2 + 6^2 + 3^2$$

– *Fr. Victor Feser, OSB*; Richardton, North Dakota

* * * * *

"I'M AFRAID, MONSIEUR FERMAT, THIS IS OUR WIDEST MARGIN."

Editor's Comment: *The study of numbers yields endless discoveries. In past issues we have dealt with prime, Perfect, and Robinson numbers. In future issues we shall deal with amicable, sociable, and other types of numbers. In this note,* Alan L. Brown *introduces a series of short articles about Multiperfect Numbers which will be continued in later issues of RMM.*

J. S. M.

MULTIPERFECT NUMBERS - COUSINS OF THE PERFECT NUMBERS

by Alan L. Brown
Montclair, New Jersey

A Perfect Number is one that equals the sum of its divisors, including unity but excluding the number itself (6 and 28 are the first two Perfect Numbers). So the sum of *all* the divisors of a Perfect Number must be twice that number, and it is this latter type of relation that we consider in defining the various classes of Multiperfect Numbers: for example, a Triperfect Number is one that equals three times the sum of *all* its divisors.

About 550 Multiperfect Numbers have been discovered so far. The simplest of these is 120*, a Triperfect Number: its divisors are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, and 120, totalling 360.

As other examples of Multiperfect Numbers, 30240 is a Quadriperfect, 523776 is a Triperfect, 14182439040 is a Quinqueperfect. The largest known Multiperfect Number contains 264 digits, and of course there must be many more still greater that await discovery.

This 264-digit Multiperfect is puny compared with the giant Perfect Numbers that are known**; but in some ways it is far more complex. Every *even* Perfect Number (no *odd* Perfect Numbers have been discovered) is the product of one single prime and a power of two: the factors of this 264-digit Multiperfect involve 46 different primes, and every Multiperfect must contain more than two different primes in its factors.

Much work has been done on Multiperfect Numbers, but we still know very little about the theoretical considerations that are involved. All known Perfect Numbers conform to one formula, but no rule or formula has yet been discovered that will yield Multiperfects in general or even any particular class of Multiperfects (i.e., Triperfects, or Quadriperfects). For this reason, Multiperfects provide a fine field for study and experiment for amateurs as well as for professional mathematicians.

* Fr. Marin Mersenne, in 1631, noted that 120 was a Triperfect Number.

** A complete tabulation of the 20 known Perfect Numbers can be found in RMM No. 4, August 1961, pages 56-59 (*The First 18 Perfect Numbers*) and in RMM No. 8, April 1962, pages 29-31 (*The 19th and 20th Perfect Numbers*). The 20th Perfect Number contains 2663 digits!

A WORLD OF DIFFERENCE

by J. A. Lindon
Surrey, England

I always maintain that my friend Skewling is a genius, but I admit it was a near thing. He emerged from the table, under which he had plunged with a howl on my entrance, and rose accusingly to his feet.

"The greatest loss," he began, his cold eye following the swirl of papers, "ever sustained—"

The door slammed. I looked under his hat on the ledge, but there was nothing: only the false teeth Skewling had made himself out of a pastry-cutter.

"Beer-germ?" I queried, noting the microscope.

"A universe," replied Skewling. "An entire universe."

I raised my eyebrow. (The other had got sliced off the previous day when Nora threw the meat-axe.) "In that case," I said, "there's nothing much to worry about. It'll expand, and in a few thousand million years perhaps, if you advertise—" A thought struck me. I looked in the bread-bin: only the patent moth-mousetrap with the supersonic whistling bait. "How big is it?"

"One point," said Skewling grimly.

I felt peeved.

There's nothing worse
For building a universe,
Where you need parts and joints,
Than points.

Was he pulling my podium? "Look—" I began.

"Man," interrupted Skewling with animation, "it was Trichronotopia, the only genuine 4-D universe with three time-dimensions and one space-dimension ever discovered. And since the Trikes move along their space-axis, as we do along our temporal one, this means that the whole of their space at any given 3-D moment is a single point. It's unique! Wonderful! Fascinating! And you have lost it," he finished coldly.

"One point?" I mused. "Rather cramping. Have to keep your elbows in and all that sort of thing."

"Do we find our moment of current time cramping?" retorted Skewling.

"Oh, well, that's different," I said. "I mean, we have plenty of space to move about in, haven't we?"

"The Trikes have plenty of *time*," Skewling explained. "At any given point you find them sprawled about all over it. And it's three-dimensional, of course."

I frowned. "I don't quite understand—"

"I was just inquiring about that," replied Skewling, "when you blew it away. So far as I could make out, there are three time-axes: real time, imaginary-possible time, and imaginary-impossible time. It's very curious."

I thought about Nora throwing the meat-axe. That was unimaginable-certain time. "You have forgotten—"

Skewling interrupted again. "There's a world of difference between our universe and that of the Trikes. Yet there are striking similarities, too. For example, most of the Trikes believe that distance is infinite, though a few of the more religious-minded talk about the Point of Judgment and the End of the Line. Life is measured in spatial terms. It's no good your asking a Trike how long a pencil is; he would answer that no one could possibly say until it had ceased to exist. In their world things die when they come, literally, to an end. Ephemera live only a few inches: elephants, on the other hand, may stretch for miles."

"You're thinking of ducks," I said. "Clearly, the most ductile—" I was always a good ducker.

"Space in Trichronotopia is a great metaphysical problem," Skewling went on, his ruler poised for a second attempt. "You see, they can perceive only a single point of it. They feel vaguely that each point leads to the next, and that this is somehow connected with the idea of causation, but a full analysis defeats them. Their History is that of time at a particular spot. Thus they speak of the Plague Mile, and of the Bloody Hectometre, when all the members of a reigning family across the Time Channel had their Royal Lines severed. Their calendar is all chains and furlongs."

"Simultaneity—" I began.

Skewling waved his ruler. "Trichronotopian relativity would be beyond you, my poor fellow," he said. "Though it is worth noting that events which occur at different times, but in the same place, are said to be 'punctaneous' or 'spotaneous'. Actual collisions, of course, with them as with us, occur only when two 4-D point-events coincide. Times can overlap quite freely along some axes and not others, just as an aeroplane can fly over a house, or two Cadillacs pass safely on opposite sides of the road. You may find it a bit confusing at first."

"Motorists who are not law-abiding would seem to present a problem," I suggested. "A single-point prison—"

"The time-pen," clarified Skewling. "They call it 'doing space', because they are prevented from moving about in time. Many of their phrases have a familiar ring. You hear the Trikes, when they are bored, talk of 'killing space'; their fairy-tales traditionally begin 'Once upon a spot'; when they return from their holidays – which take them well away from the rigours of real time and out into imaginary realms – people ask 'Did you have a good length?' and they will perhaps answer 'Darling, we had the length of our stretches!' Children often accost adult Trikes with a request to be told the 'right distance', they ask 'How far are we?' or 'What's the space, please?' and the adult will pull out and consult his automatic tape-measure."

I laughed. "I'm getting beyond my term," I said. "Fancy refereeing a Trike boxing-match: 3-foot rounds, with 2-foot rests, in a ring 20 seconds square with imaginary time-ropes! Or athletics: trying to run the 4-mile minute, and being timed—no, *spaced* with a micrometer! Oh, dear!"

Skewling looked supercilious. "Such matters are elementary," he said. "The Trike I was talking to had just quarrelled with his wife. As she hadn't been at the moment he was supposed to meet her at he had gone awhile, and she reckoned he ought to have stayed then longer. So where he returned she declared she had been then the whole space. 'When in hades have you been?' he demanded angrily. 'You weren't now a few inches away.' 'Ooh, I was!' she flamed. 'It's now, precisely now, that I've been and nowhen else. When were you, if it comes to that? I saw you, over in Nut Thursday, with that little flirt who lives next-day to us. Come on, out with it here. What were the pair of you imagining?'"

"She must have had telescopic vision," I said, with a grin. "*Time*-telescopic," I amended hastily. "No doubt her jealousy magnified the occasion. How, by the way, do you construct a time-telescopic?"

Skewling pushed the table aside and went down on his hands and knees. "I think," he said, "that, as you are so interested, we had better try to find Trichronotopia again. Kindly pass me three clocks ticking at right-angles and a pair of headphones."

I looked in the whisky-trough, but no: only Skewling's underground bicycle and the vacuum crash-helmet in case you ran into something that wasn't there.

"Look," I said. "I can't wait. I've got to buy a meat-axe before the shops shut. A foam-rubber one. See you in a few yards space." I popped my head back round the door. "With padded walls and a lock on the outside."

That was when I lost my other eyebrow.

* * * * *

ROYAL V. HEATH

The Editor of *Recreational Mathematics Magazine* has obtained the papers and manuscripts of the late *Royal V. Heath* - a puzzlist well-known for many years to puzzle lovers. In preparation for the publication of a series of his works, the Editor would like to obtain as complete a list as possible of his previously published works. Heath published material ranging from small notes to articles in a variety of mathematics journals, magicians' publications, pamphlets, and books.

Would RMM readers do a bit of research for RMM by submitting any references to anything by Royal V. Heath that has been published in any *book, journal, magazine, pamphlet, etc.?*

If at all possible list:

- Title and date of the publication and where published.
- Title of the article or note published.
- A brief abstract of the contents.
- Any errors subsequently noted.

If anyone can actually supply the original publication, the Editor will make a copy and immediately return the original. If only partial information can be supplied, please submit it - it may be the *completion* of someone else's partial information.

Heath published many numerical recreations and the actual numbers and observations (or at least enough to distinguish the work from others) should be sent.

GEORGE CANTOR (1845-1918)

by Ali R. Amir-Moéz

"What an interesting but simple mathematical idea!" said George to his teacher. "We can make an arithmetic using sets."

"What do you mean?" asked the teacher.

Then George Cantor explained his idea: Suppose we take sets of points in a plane. We call these sets A, B, C, . . . Then we say $A+B$ is a set which has all the points of A and all the points of B (Figure 1). The shaded area is $A+B$. Then we call the part which is in both A and B the product of A and B. The shaded area in Figure 2 is $A \times B$.

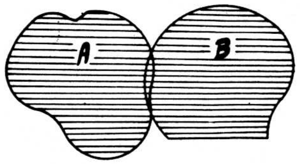


Figure 1

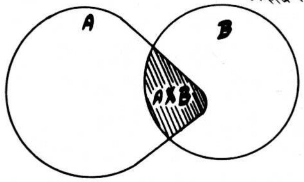


Figure 2

"What happens if there is no common part of A and B?" asked the teacher.

"We say $A \times B = 0$." answered George.

"This is ridiculous! How can a product be zero without one of the factors being zero?" said the teacher, while laughing. "You are wasting your time."

George Cantor did not give up his ideas but went on to develop his set theory. He taught at Halle from 1869 until 1905 and had started publishing his mathematical works in 1870.

It is well known among mathematicians that Kronecker, a mathematician during the nineteenth century, did not like Cantor. After first making quite a bit of money, Kronecker settled in Berlin in 1855, teaching at the University without having a professorship. Finally, in 1883, he became a professor.

Kronecker thought that the work of Cantor was going to ruin mathematics. But it did not take long before the work of George Cantor on set theory became one of the most important parts of mathematics; particularly when topology became an essential part of mathematics. In 1901, Lebesque introduced a new idea about measure which made Cantor's set theory ideas even more important. The arithmetic of sets is very easy, particularly for all the sets which are in a plane inside a rectangle.

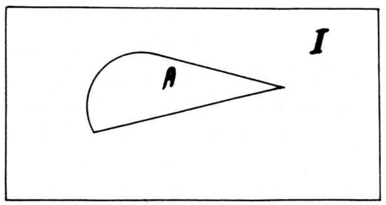


Figure 3

We suppose the set of all points in a rectangle is called I (Figure 3). Then for any set A inside I we see that

$$I \times A = A.$$

This means that the common points of I and A are all in A. Here I really works as if it were 1.

We shall not continue the discussion here, but give only a few simple ideas in order to arouse the reader's curiosity. We are sure that the reader will find some of these ideas in most high school books and there are books which now teach set theory to even younger children.

George Cantor's theory of transfinite cardinal numbers is a great contribution to mathematics. Many mathematicians refused to accept the infinite except as a process. Cantor's most serious opponent was Kronecker who held opposite ideas. He wanted to arithmetize mathematics. Today's controversy between the formalists and the intuitionists is a continuation of the controversy between Cantor and Kronecker.

* * * * *

PROTEAN SHAPES WITH FLEXAGONS

by William R. Ransom
Tufts University

Proteus was a minor deity who could change his shape at will. A hexagonal Proteus is made from a strip of tough paper, about twelve times as long as it is wide, folded into a succession of equilateral triangles (Figure 1). If you follow the directions *very carefully* until you are satisfied that the shapes in Figures 2 to 11 are possible, then with an unmarked Proteus you have a very intriguing puzzle to attain them again. The Proteus is generally called a *flexagon* and the name *Proteus* is applied here only to indicate the recreations possible with a flexagon.

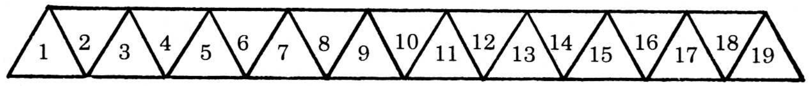


Figure 1

Number the triangles in the strip 1, 2, 3, . . 19, with the letters A, B, C, . . S on the back, with the A and 1 back-to-back and so forth up to S and 19 back-to-back. To make the flexagon, fold: B against C; D against E; 2 against 5; F against G; H against I; J against K; 8 against 11; L against M; N against O; P against Q; 14 against 17; A against R; and then paste S against 1.

Check: your flexagon will have 4, 7, 10, 13, 16, 19 on one side and 3, 6, 9, 12, 15, 18 on the other.

Now you immediately pass from the plane hexagon shape to a series of 3-dimensional figures.

1. *BOAT*: Fold the flexagon along the diagonal that divides 4 and 16 from 7 and 13, with these sides outside. Bringing 3 and 6 together hold 4-7 in the right hand with the fold at the top. Bring the corner of 13 that touches 7 toward you and bring 17 against 8 and 14 against 11. Now, with the long side up, open the boat separating I from H (Figure 2).

2. *BUOY*: Bring bow and stern of the boat together so that the upper edge of 7 touches the upper edge of 13 and the upper edge of 4 touches the upper edge of 10 (Figure 3).

3. *SQUARE CUP WITH HANDLE*: Return to the boat, and hold 10-13 in the left hand. Push the corner of 4 that touches 10 towards you and down and you have a square cup, with 3, 6, 9, 18 inside and its handle in your left hand (Figure 4).

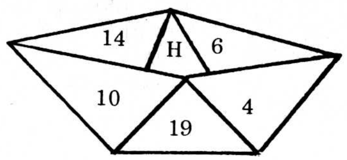


Figure 2

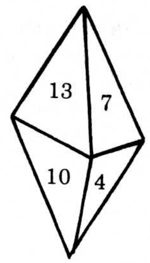


Figure 3

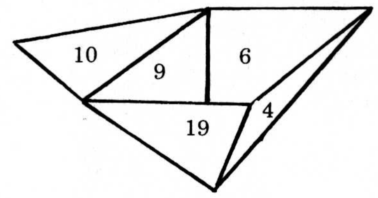


Figure 4

4. *TWIN CUPS*: Flatten the square cup, bringing 3 against 6 and, with the long side at the bottom, pull out the upper corner of 4-19. Then keeping R and 10 in one plane and the cup (A, F, G, R inside) up, pull out the upper corner of 13-16. This gives the twin cups that have A, F, G, R and J, K, P, Q inside them (Figure 5).

5. *PENTAGON CUP*: Pull on 7 so as to make 10-J slide in between Q and R. Now you have a pentagonal cup with a triangular cup attached. Their insides are A, F, G, 9, R and K, P, Q (Figure 6).

6. *TRIANGULAR CUPS WITH HANDLE*: Fold A against F. This makes a handle with two cups, G, 9, R inside one and P, Q, K inside the other (Figure 7).

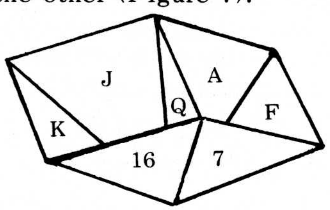


Figure 5

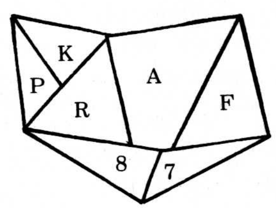


Figure 6

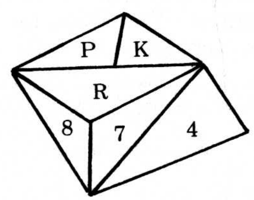


Figure 7

7. *CORNERED CUPS*: Return to the pentagon cup and fold A against R. This gives two triangular cups, corner to corner, without a handle (Figure 8).

8. *SIDE-BY-SIDE CUPS*: Bring 4 against 19. This gives the same two cups side-by-side (Figure 9).

9. *THREE PYRAMIDS*: Swing the cups apart until 8 and 16 almost touch, keeping F and Q in the same plane. Lift the corner of 4-19 at the bottom of the cups and you have a square pyramid with 4, F, Q, 19 outside and 3, 6, 10, 18 inside, and two triangular pyramids attached to the F and Q sides (Figure 10).

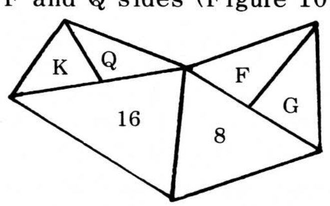


Figure 8

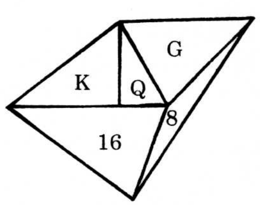


Figure 9

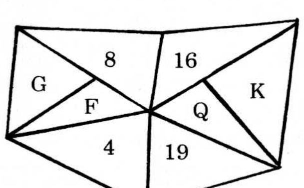


Figure 10

10. *FLOWER BASKET*: Bring the sides 3 and 18 together and you have something to hang up with flowers in the two cups (Figure 11).

Now you will wish to get back to the original hexagon. Open the 4-19 and flatten 3-18 against 6-10, and you can open out and get the pentagon cup figure. Push 8-9 through between Q and R arriving back to the twin cups. Flatten A to R and G to F and flatten P to Q and J to K. Hold F against G and raise the corner of 13 that touches 7 and flatten 8 against 11 and 14 against 17. Now you have the original hexagon when you unfold along the 7-13 diagonal, opening out the edges at 10 and 19.

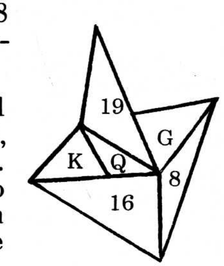
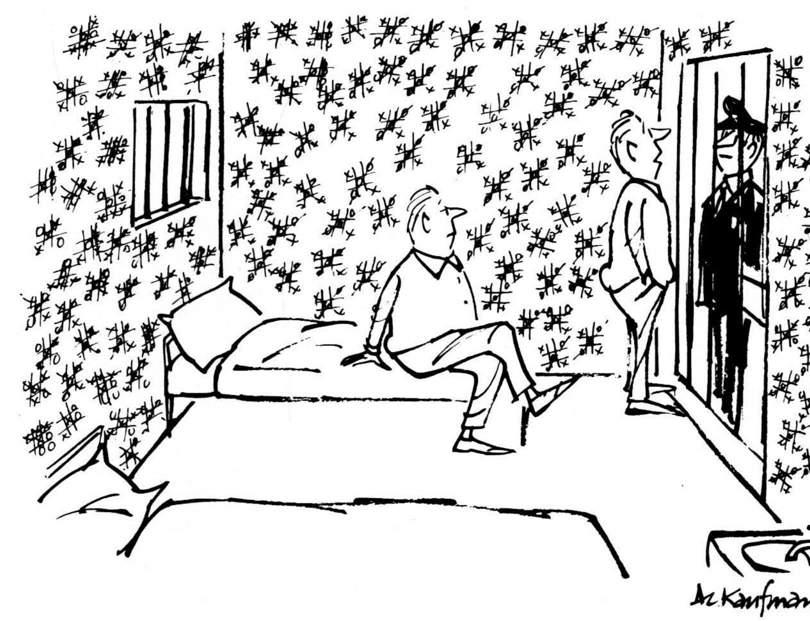


Figure 11

There are a great many other shapes, not easy to describe, that can be made by folding and unfolding Proteus. In your struggles to get the ten listed here, you will come across many of them maddeningly.

It is so easy to go astray and not be able to get back to where you were, that you will undoubtedly find it necessary - before becoming an "expert" - to unfasten the ends, straighten out the strip, and refold it into the original hexagon.

* * * * *



"ANY CHANCE OF BEING MOVED TO A NEW CELL?"

PUZZLES AND PROBLEMS

1. *Cigarette Selling*

A cigarette factory sells cigarettes in two different packs: packs of 12 and packs of 25 cigarettes. Mr. Smith, our accountant puzzlist, has noted that sales of a given number of cigarettes would involve some problems. 37 could be sold without breaking any packs and also 36 - but not 38. He also noted that, for example, 573 cigarettes can be sold without breaking any packs (21 packs of 25's and 4 packs of 12's); likewise 574 and even 929 cigarettes (29 packs of 25's and 17 packs of 12's). He wonders what might be the greatest number of cigarettes that can *not* be sold without breaking open any packs. That is, *any* greater number of cigarettes can be sold without breaking open any packs.

As an accountant, he naturally wonders what the general solution to this problem would be for any two different size packs consisting of L and M cigarettes. (*C. C. Verbeek*: Den Hague, Netherlands)

* * * * *

2. *The King's New Banquet Hall*

"Forsooth!" said the King one medieval day, "My banquet hall is getting rather shabby. I must build another."

With the help of his architect he worked out the cost. Then he went to see his mathematician and tax-collector, interrupting him half-way through the calculation of a recurring decimal.

"How many people, counting barons, knights, and yeomen, appear on my list for taxes?" the King asked.

"One hundred and forty-two, your Majesty."

"My architect tells me I shall need 1136 crowns in order to build my new banquet hall. You must impose a capitation levy."

"That works out at just eight crowns per taxable person."

"That won't do at all," said the King. "I want you to soak the barons. They're getting much too powerful - and, besides, they'll be using the hall more than anyone else. They carouse too much. I suggest we tax them eighty crowns each. That way the knights and yeomen will have to pay only a little."

After a few calculations, the tax-collector decided to accept the King's suggestion and taxed each baron eighty crowns. Then by taxing each knight four crowns, and each yeoman one crown, the tax-collector obtained altogether the required amount of exactly 1136 crowns.

How many barons, knights, and yeomen were there?

(*D. C. Cross*; Birmingham, England)

* * * * *

3. *Wasted Inches*

"I thought you were playing Patience," said Susan. "Why all the cussing?"

"The table's too small," Len laughed. "Or maybe the cards are too big, even though they're regular size, 2¼ by 3½ inches."

"So the furniture's wrong!" his wife smiled. "What's the game?"

"A new one. I have to lay out the whole deck of cards, edge to edge without overlapping or projecting beyond the table," Len told her. "But there's not enough room, and I'm left with a few clubs and the same number of spades. Unfortunately, it's the only rectangular table we have."

"It's a funny shape," Susan commented. "One inch longer than it is wide."

"I know." Len was jotting down some figures. "Arranging the cards the best way I can, I've still got eleven and a quarter square inches of wasted space."

What were the dimensions of that table top?

(*J. A. H. Hunter*; Toronto, Ontario)

* * * * *

4. *Some Like it Wet*

Bill had almost given up hope when at last a taxi responded to his calls and drew into the curb. His feet barely touched the drenched sidewalk as he made the open door from the shelter of the porch, but even so the deluge soaked him.

"All right for ducks, and I guess for cabbies too," he said. "City Hall, please."

The driver grinned. "Only time this job pays off. Last month I averaged exactly thirty-four bucks a day for the days it rained."

"And did it rain!" exclaimed his passenger. "Most days, it seemed to me."

"Sure did," the driver agreed. "It's funny, but I averaged an exact number of dollars for the fine days, but for the whole month my average was just three times that."

Bill's mind works fast. "Didn't you take any days off?" he asked.

"Not last month, I didn't," the man replied. "It was too good a break to waste."

It must have been a wonderful month for him. How much had he taken? (*J. A. H. Hunter*; Toronto, Ontario)

* * * * *

5. *How Many Oranges?*

A fruit dealer has a number of oranges for sale. To the first customer she sold half of her oranges plus half an orange. To her second customer she sold half of the remaining oranges plus half an orange. To her third customer she sold half of the remaining oranges plus half an orange. To her fourth customer she sold half the remaining oranges plus half an orange, thereby selling out her entire stock.

How many oranges had she started with?

Selling in a similar manner, and never cutting any oranges in half, how many oranges would she require for N sales?

(*Dorman Luke*; West Palm Beach, Florida)

* * * * *

6. *A Number Problem*

In how many ways can M^2 be expressed as the sum of N different squares, $N > 1$?

For $M = 20$ we have only one way for $N = 2$ $(16^2 + 12^2)$, none for $N = 3$, and many (how many?) ways for $N = 5$ (e.g.: $17^2 + 9^2 + 5^2 + 2^2 + 1^2$).

The problem may be generalized: in how many ways can M^n be expressed as the sum of N different nth powers, $N > 1$?

(*U. Clid*; Cleveland, Ohio)

* * * * *

7. *A Famous Digit Problem*

The nine digits, 1 to 9, are to be used, in order and in reverse order, with plus or minus signs, only, to equal 100. For example: $12+3+4+5-6-7+89=100$ and $9+8+76+5+4-3+2-1=100$. The *Junior Department* of RMM had featured part of this problem as a good exercise for students (October 1962). The reader is also referred to RMM No. 1, February 1961, pages 39-42 for 113 solutions to the problem using *any* mathematical symbols. The 11 solutions to the first part of the problem (the digits used in order) appear in that list. The additional 15 solutions to the second part of the problem appear in the January 1963 issue of *Scientific American*, page 10.

It is interesting to note that none of these 26 solutions start with -1 or -9. Yet there are four solutions. We leave the reader to discover them.

Ira Doak, of Pasadena, California, points out that the sum of these nine digits is 45. Hence we suggest the same problem, with the same condition (using plus or minus signs only), with two minor modifications: the sum is to be 45 and we are to use the *minimum* number of plus and minus signs.

* * * * *

8. *A Factorial Conjecture*

Fr. Victor Feser, of Richardton, North Dakota, submitted this interesting little factorial oddity:

$$6!\cdot 7! = 10!$$

where $n! = 1\cdot 2\cdot 3\cdot\,\cdot\,\cdot\,\cdot n$.

Further work produced two more examples, admittedly trivial, where the product of the factorials of two consecutive numbers equals a factorial: $(0! = 1)$

$$0!\cdot 1! = 1! \quad\text{and}\quad 1!\cdot 2! = 2!$$

It is not difficult to find a number of examples if non-consecutive numbers are used: $4!\cdot 23! = 24!$, $2!\cdot 4!\cdot 47! = 48!$, $2!\cdot 3!\cdot 4!\cdot 287! = 288!$.

Of more particular interest is whether a factorial can be formed by the product of three or more factorials of numbers in arithmetic progression. For example: $3!\cdot 5!\cdot 7! = 10!$.

J. A. H. Hunter conjectures that if we define $F(x,n)$ as the product of n factorials, $N!\cdot(N+x)!\cdot(N+2x)!\cdot\,\cdot\,\cdot\,\cdot(N+nx-x)!$, with a "common difference" x, then $F(x,n)=M!$ can have only a finite number of solutions for any particular values of x and n. We have given the three solutions for $F(1,2)=M!$. For $F(2,3)=M!$, we have only $1!\cdot 3!\cdot 5! = 6!$ and $3!\cdot 5!\cdot 7! = 10!$.

Unproven conjectures always offer a challenge to the professional mathematician as well as to the amateur. Another teasing problem here, of course, is to find numerical examples for given values of x and n in $F(x,n)=M!$.

* * * * *

In the October 1962 issue we expressed the hope of publishing a dozen puzzles and problems in each issue. However, the supply has suddenly slowed down and we urge readers to try their hand at composing original material for this department. All submitted work will be examined carefully for possible publication.

Please, do *not* submit previously published work unless the original source is given.

ANSWERS FOR THE DECEMBER 1962 ISSUE OF RECREATIONAL MATHEMATICS MAGAZINE.

The list of puzzle solvers will be found on Cover III.

ALPHAMETICS (Page 24 - December 1962 RMM)

(1) MATH CLUB MEETS = 1345 8697 10042 (for the largest value of CLUB).

(2) ALFRED ÷ E = NEUMAN — 704836 ÷ 3 = 231572 (base-9). *Harry L. Nelson*, of Livermore, California, notes that, in base-9, ALFRED·E = NEUMAN — 164057·5 = 852318.

(3) ZERO ONE TWO THREE = 9635 546 185 10366 with 4 and 8 interchangeable.

(4) RMM HAS MOVED = 599 162 97038.

(5) NEBULOSITY = 1234567890.

PUZZLES AND PROBLEMS (Pages 20-22 - December 1962 RMM)

1. *Flowers for the Girls*: We derive the equation $5x+4y+3z=60$, which has 37 positive non-zero integral solutions. The maximum number of these solutions, giving the same sum for $x+y+z$, is 8: the sum being 15. Hence there were 15 girls in the troupe.

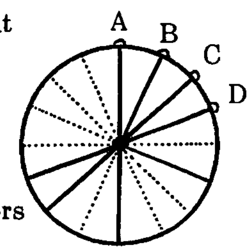
2. *Four Against Four*: $x^3+y^3+a^3+b^3=z^3+w^3+a^3+b^3=A^3$, so $x^3+y^3=z^3+w^3$, for which the smallest integral solution is well known to be: $1^3+12^3=9^3+10^3$. Hence, $a^3+b^3=A^3-1729$. By quick trial, $13^3-1729=468=5^3+7^3$, so the required solution is: $1^3+5^3+7^3+12^3=5^3+7^3+9^3+10^3=13^3$.

3. *Down to Earth?*: The two smallest triangles are those with sides 4, 4, 1 and 2, 2, 5. However, it was pointed out that two congruent triangles with sides 1, 1, 1 fulfill the stated conditions.

4. *Did the Butler do it?*: Yes, he did. He lied when he said that three burglars left two pennies after dividing the rest evenly. Any square number divided by three leaves a remainder of 0 or 1, but never 2.

5. *The Seven Fortunes*: The fortunes of the seven kids are 1, 2, 4, 8, 16, 32, and 2816 pennies making a total of 2879 pennies.

6. *Electrical Switching*: The diagram at the right shows one possible solution.



7. *Can You Couple the Couples*: Andy and Fay, the mother; Bert and Eva; Chris is single.

8. *Rather Precious Antiques*: The two calculators cost \$980 and \$3675.

ERRATA FOR THE DECEMBER 1962 ISSUE OF RMM

Page 7: A 4th order Perfect Digital Invariant was omitted. Please insert 9474 ($=9^4+4^4+7^4+4^4$) in the table after 8208.

Page 25: The *Index* covers Issues 7 through 12, not *6* through 12.

NOTE: In reference to the article *Fermat's Last Theorem* in the December 1962 *Jr. Dept.*, it has been reported that the theorem has been proven true for all values of n less than 4001. There are no integral solutions to $x^n + y^n = z^n$ for values of $4001 > n > 2$.

* * * * *

Miscellaneous

Alphametics (RMM No. 10, August 1962, page 11): Dudeney's subtraction "Verbal Arithmetic" has a multitude of solutions:

EIGHT − FIVE = FOUR:

12348 − 6291 = 6057; 12375 − 6281 = 6094; 12780 − 6231 = 6549;
13870 − 6341 = 6529; 14820 − 7461 = 7359; 15230 − 7541 = 7689;
16725 − 8631 = 8094; 16743 − 8651 = 8092; 16905 − 8671 = 8234;
17936 − 8791 = 8245; 17054 − 8761 = 8293; and U and V are interchangeable.

Puzzle No. 8 *Curious Number Relationships* (RMM No. 9, June 1962, page 49). Some additional curiosities: $10^2 + 1^2 = 101$ and $1^3 + 2^3 + 3^3 + 4^3 = 100$.

Dale Kozniuk of Delburne, Alberta gives us this interesting form:

$$10^n + 0^n_{n-2} + 0^n_{n-3} + \cdots + 0^n_2 + 0^n_1 + 1^n = 1000\cdots001$$

* * * * *

IF YOU HAVE MOVED OR ARE GOING TO MOVE

Please let us have your new address as soon as possible. Always include your old address with the new one. If you have your old address label copy the IBM code numbers (see page 17 in this issue for explanation of the label code) - this will tell us just about everything we need to know to find *your* card.

Much time is lost - as well as magazines - when address changes are delayed or not reported.

If you have any questions about your RMM subscription don't hesitate to write - including your full name, address, and code numbers with the questions.

Send all address changes and information and subscription informa- requests to **RMM, Box 35, Kent, Ohio.**

ROYAL V. HEATH FANS –
See page 33.

RECREATIONAL MATHEMATICS magazine

JUNIOR DEPARTMENT

All correspondence and material relating to the *Junior Department* should be sent to:

Howard C. Saar
1014 Lindell Avenue
Petoskey, Michigan

All articles, problems, solutions, ideas, etc., should be submitted to the *JD* editor typewritten or neatly written in ink on 8½ x 11 paper (except those items obviously too large to fit those specifications).

RMM reserves the right to edit all manuscripts accepted for publication to conform to reasonable literary standards. If we plan any major alterations you will be notified. All materials submitted to us for consideration become our property and will not normally be returned.

* * * * *

A number of readers noted that Mr. Dickinson's problem *Sums of Primes* (problem 4 in the December 1962 *Jr. Dept.*, page 55) led to another problem.

First, we must drop the idea that 1 is a prime. Also, 2 *is* a prime - the only even prime, of course.

Primes were found equal to the sum of three smaller primes in only one way (e.g., $19 = 3 + 5 + 11$; $23 = 5 + 7 + 11$). Many other primes were found to be equal to three or more different groups of three primes (e.g., $29 = 3 + 7 + 19 = 5 + 7 + 17 = 5 + 11 + 13$; 47 is the sum of eight different groups of three different smaller primes.

As an elementary problem we ask: can any prime greater than 2 be expressed as the sum of 2 plus two different odd primes? If so, give an example. If not, prove it.

* * * * *

We hope teachers find the *Jr. Dept.* of some value. Obviously, not *all* teachers will find *all* the material in any one *Jr. Dept.* useful in their own particular circumstances. However, the rest of RMM can certainly be used effectively. The *Alphametics* offer an excellent source of fun and application of mathematical principles. Very probably, alphametics constitute one of the very few types of mathematical problems requiring no more than the knowledge of the basic operations of arithmetic and which can still offer considerable challenge to the graduate student of mathematics.

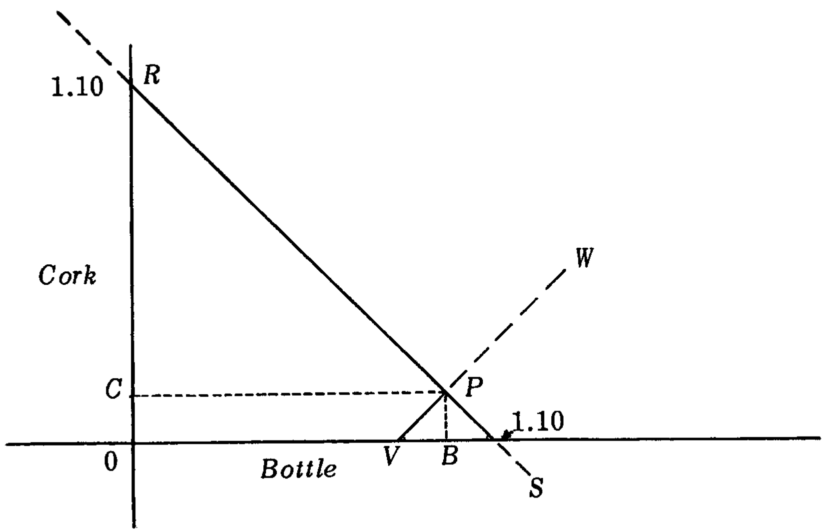
In this issue, for example, we refer teachers to *The Elusive Number Pi* (page 18), *Sir Isaac Newton, Problemist* (page 19), and *Protean Shapes with Flexagons* (page 35) for material for classes and mathematics clubs.

A BOTTLE AND A CORK

by Robert H. Scott
Dunbar High School Mathematics Department
Dunbar, West Virginia

A BOTTLE and a CORK
Cost one DOLLAR and a DIME.
The bottle cost one dollar
MORE than the cork.
What did the cork cost?

Most persons answer "ten cents" and of course they are wrong. This problem can be solved a dozen different ways but our geometry class got a "big bang" by solving it graphically, using rectangular coordinates. It is not necessary to plot points to an exact scale or to introduce any of the theorems of analytic geometry.



If the cork cost nothing the bottle would be \$1.10 and *if* the cork cost \$1.10 the bottle would cost nothing. However, these statements are true for one of the conditions of the problem, but violate the other condition. Thus, the bottle has a domain between 0 and 1.10 inclusive, while the range of the cork is from 1.10 to 0 inclusive. This is shown on the graph by a negative slope from R to S. Now, violating the first condition of the problem, but not the second, *suppose* the cork cost nothing. Then the bottle would cost \$1.00. If the cork cost 1¢, the bottle would cost \$1.01, and so on. Thus, the increment of the bottle (ΔB) would equal c, the cost of the cork. This is shown on the graph by a positive slope from V to W. The point of intersection, P, is that point which correctly fulfills *both* conditions of the problem. Perpendiculars from P to B and C intersect the axes at *Bottle* = \$1.05 (midway between \$1.00 and \$1.10) and at *Cork* = \$0.05. This gives us the answer to the problem.

RECURRING DECIMALS

by Peter Farrell, Age 15
Birmingham, England

I read *The Magic of One Ninety-Seventh* by 12-year-old Avner Ash in the October 1962 issue of RMM and worked on recurring decimals with some results which I believe will be of interest to fellow readers of the *Jr. Dept.*

The recurring pattern of $\frac{1}{7}$ is 0.142857 . . . This may be obtained:

1. by writing down the 7 — 7
2. multiply this by 5, write the last digit of the product down to the left of this, and carry the other digit(s) — $_{3}57$
3. repeat by multiplying the 5 by 5, adding on the 3, etc. — $_{2}857$

The whole recurring pattern may be obtained this way. This may appear rather trivial, or a freak example, but it is in fact a particular example of a general case.

The General Case

It is required to find the recurring decimal pattern of $\frac{1}{10x-3}$ using the above method.

1. It always ends in 7, write this down first.
2. Repeat the above method, using $7x-2$ as the multiplier.

(In the first example, 5 was a particular case of the multiplier $7x-2$.)

To clarify this further I will give another particular example.
$\frac{1}{17} = 0.0588235294117647$ . . .

This is an example of $\frac{1}{10x-3}$ where $x=2$. The recurring pattern ends in 7 as noted earlier. The multiplying factor, $7x-2$, where $x=2$, is 12.

The 7 is written down	7
And multiplied by 12. The 4 is written down and the 8 is carried.	$_{8}47$
The 4 is multiplied by 12, the 6 written down, and the 5 is carried. Etc.	$_{5}647$

The process is continued until the recurring pattern becomes evident.

Notation

To conserve space in further general cases I have devised a notation.

[The recurring pattern of $\frac{1}{10x-3}$ may be obtained by starting with 7 at the right hand end and successively multiplying by $7x-2$]

may be symbolized by $\frac{1}{10x-3} = X\,(7x-2) \longleftarrow 7.$

The following are also found to be true.

$$\frac{1}{10x-1} = X\,x \longleftarrow 1; \quad \frac{1}{10x+1} = X\,(9x+1) \longleftarrow 9; \quad \frac{1}{10x+3} = X\,(3x+1) \longleftarrow 3$$

There is one recurring decimal in the form of $\frac{1}{10x-1}$ which is of particular interest to me. It is the case when $x=5$, i.e., $\frac{1}{49}$.

$\frac{1}{49} = 0.020408163265306122448979591836734693877551 \ldots$

This may be formed by taking the 1 with which it ends, as noted in the symbolized formula above, and working backwards as described earlier by multiplying by 5.

This number may also be found another way by taking the powers of 2, shifting the last digit of each successive power of 2 to the right, and performing an indefinite addition as shown below:

```
02
  04
    08
      16
        32
          64
           128
             256
               512
                1024
                  2048
                    4096
                      8192
                       16384
                           etc...
020408163265306122448979...etc...
```

Similar properties are possessed by the recurring pattern of $\frac{1}{97}$.

$\frac{1}{97} = 0.010309278350515463917525773195876288659793814329896907216494845360824742268041237113402061856 7 \ldots$

This may be obtained by adding successive powers of 3 in the same manner as above, thus:

```
01
  03
    09
      27
        81
         243
           729
            2187
              6561
               19683
                 59049
                     etc...
010309278350515463...etc...
```

There is so much that could be said about this one curiosity. I don't propose to go into it any further, but I would be very interested if any reader could generalize it, using algebra.

To get back to $\frac{1}{49}$ - Consider it written out on a cylinder, the 1 at one end coming next to the 0 at the beginning. This number may be multiplied by any number (excluding multiples of 7) between 2 and 48 just by starting at a different point along it. The following shows the period of $\frac{1}{49}$, showing the point along it at which one must start to multiply it by the number in the circle.

(10) (20) (40) (31) (13) (26) (3) (6) (12) (24) (48) (47) (45) (41) (33) (17) (34) (19) (38) (27) (5)

0 2 0 4 0 8 1 6 3 2 6 5 3 0 6 1 2 2 4 4 8 9 7 9 5 9 1 8 3 6 7 3 4 6 9 3 8 7 7 5 5 1

(2) (4) (8) (16) (32) (15) (30) (11) (22) (44) (39) (29) (9) (18) (36) (23) (46) (43) (37) (25)

I will clarify this with an example. Suppose it is desired to find $\frac{26}{49}$. Find the number 26 in the circle. Start reading off the period from this point, going to the right, and upon reaching the end go back to the beginning. Therefore, the recurring pattern of $\frac{26}{49}$ is 0.530612244897959183673469387755102040816326 . . .

The reader may notice a pattern existing in the series of numbers in the top and bottom rows of circles.

It is worth examing the period of $\frac{1}{19}$ under the same conditions.

$\frac{1}{19} = 0.052631578947368421 \ldots$

This number may be multiplied by any number between 2 and 18 in much the same manner as $\frac{1}{49}$.

(10) (5) (12) (6) (3) (11) (15) (17) (18) (9) (14) (7) (13) (16) (8) (4) (2)

0 5 2 6 3 1 5 7 8 9 4 7 3 6 8 4 2 1

I have dealt with $\frac{1}{49}$. What about $\frac{1}{343}$? This number has properties much the same as $\frac{1}{49}$. Here it is:

$\frac{1}{343} = 0.0029154518950437317784256559766763848396501457725947521865$
$889212827988338192419825072886297376093294460641399416909620991$
$253644314868804664723032069970845481049562682215743440233236151$
$603498542274052478134110787172011661807580174927113702623906705$
$53935860058309037900874635568513119533527696793 \ldots$

The rest of the *work* is up to *you*!

* * * * *

PROBLEM CORNER

The *Problem Corner* is a bit small this time in order to bring *Peter Farrell's* most interesting observations about recurring decimals to our readers.

1. *What's His Number?*

"You got a very special car plate number this time, Dad," said Andy. "I've been figuring it out."

"Yes, it's never easy to get a 3-figure number." agreed his father. "I guess that's what you mean."

The boy grinned. "More than that. The digits add up to 24, and the first digit of its cube is a four."

That did make it special! What was the number?

(*Dale Kozniuk*; Delburne, Alberta)

* * * * *

2. *Another 1963 Digit Game*

Using only the digits 1, 9, 6, 3, and in the same order, form expressions equal to the integers from 0 and up. For example: $1 \cdot 9 - (6+3) = 0$; $19 - (6 \times 3) = 1$; $(1/9)(6 \cdot 3) = 2$; etc.

(*Clifford R. Dickinson*; Camas, Washington)

3. *Area Doubling*

A rectangle has a perimeter of 440 feet. Its area can be tripled by increasing its perimeter to only 480 feet.

What are the dimensions and area of the original rectangle?

* * * * *

ANSWERS TO THE DECEMBER 1962 JUNIOR DEPARTMENT *PROBLEMS*

1. *1963 Puzzle*: There are several ways of using the ten digits as requested, e.g.:

$$\begin{array}{r} 201\times 7=1407 \\ 69\times 4=\ \ 276 \\ 35\times 8=\ \ \underline{280} \\ 1963 \end{array} \qquad \begin{array}{r} 120\times 9=1080 \\ 64\times 8=\ \ 512 \\ 53\times 7=\ \ \underline{371} \\ 1963 \end{array}$$

2. *A Train and Tunnel Problem*: We admit there might have been a slight ambiguity here. The train, travelling 60 miles per hour (88 feet per second), takes 3 seconds to enter completely inside the tunnel - so the train is 264 feet long. However, the "30 seconds to pass completely through the tunnel" includes the 3 seconds required to completely enter the tunnel *and* the 3 seconds to completely leave the tunnel. This means that the *complete* train was in the tunnel only 24 seconds. Any given *part* of the train, say the front, spends only 27 seconds in the tunnel - so the tunnel is 2376 feet long.

3. *A Number Puzzle*: The digit a is 3, and the original 4-digit number is 1377.

4. *Sums of Primes*: If we include 1 as a prime, then the smallest primes that are the sums of 2, 3, 4, or 5 different groups of three different primes are:

$$17=1+3+13=1+5+11$$
$$19=1+5+13=1+7+11=3+5+11$$
$$23=1+3+19=1+5+17=3+7+13=5+7+11$$
$$29=1+5+23=1+11+17=3+7+19=5+7+17=5+11+13$$

If we do not include 1 as a prime, then the smallest primes in their respective groups are:

$$23=3+7+13=5+7+11$$
$$29=3+7+19=5+7+17=5+11+13$$
$$31=3+5+23=3+11+17=5+7+19=7+11+13$$
$$37=3+5+29=3+11+23=5+13+19=7+11+19=7+13+17$$

5. *Designing Stunt Section*: 184,756 designs are possible.

* * * * *

No list of solvers was turned in at press time, but we will include it in the next listing in the *Combined Issue* (see page 17 for details about the *Combined Issue*).

* * * * *

In the next issue: Have you ever played with magic squares? If so, you will be quite interested in the work of young *Dale Kozniuk* in this interesting area of recreational mathematics. If not, you will be enlightened by the magical world of magic squares! An original cross-number puzzle particularly for the age 12-or-less set will make its appearance.

PUZZLE SOLVERS: We have listed the solvers of the various puzzles in the December 1962 issue of RMM. See pages 41-42 in this issue for the December 1962 puzzle answers.

ALPHAMETICS

Raymond Aaron, Toronto, Ontario (1, 3); *Merrill Barnebey*, Grand Forks, North Dakota (1, 2, 3, 4); *David R. Barstow*, Wethersfield, Connecticut (1, 3); *D. C. Cross*, Birmingham, England (1, 2, 3); *Clifford R. Dickinson*, Camas, Washington (1, 3); *Harry M. Gehman*, Buffalo, New York (1, 2, 3); *Edward Joris*, Antwerp, Belgium (1, 2, 3, 4, 5); *Prof. Edgar Karst*, Norman, Oklahoma (1, 2, 3, 4, 5); *Felix Kestenholz-Seiler*, Liestal, Switzerland (1, 3, 4); *Jonathan Khuner*, Berkeley, California (1, 3, 4, 5); *Dale Kozniuk*, Delburne, Alberta (1, 2, 3, 4, 5); *Thomas J. Morris*, San Juan, Puerto Rico (1, 3, 4, 5); *Harry L. Nelson*, Livermore, California (1, 2, 3, 4, 5); *Wade E. Philpott*, Lima, Ohio (1); *Stanley Rabinowitz*, Far Rockaway, New York (3); *Thomas A. Roberts*, Lincolnwood, Illinois (1); *R. Robinson Rowe*, Naubinway, Michigan (1, 2, 3, 4, 5); *C. W. Sweitzer II*, San Diego, California (3); *Alfred Vasko*, Swanton, Ohio (1, 3, 4); *Anneliese Zimmermann*, Bad Godesberg, West Germany (1, 2, 3, 4, 5).

PUZZLES AND PROBLEMS

Anneliese Zimmermann, Bad Godesberg, West Germany (1, 2, 4, 5, 7, 8); *Alfred Vasko*, Swanton, Ohio (1, 2, 3, 4, 5, 6, 7, 8); *Graham C. Thompson*, Binghamton, New York (1, 2, 4, 5, 6, 7, 8); *C.W. Sweitzer II*, San Diego, California (1, 2, 4, 7, 8); *R. Robinson Rowe*, Naubinway, Michigan (1, 4, 5, 6, 7, 8); *Thomas A. Roberts*, Lincolnwood, Illinois (1, 4, 5, 6, 7, 8); *Stanley Rabinowitz*, Far Rockaway, New York (1, 4, 5, 7, 8); *Wade E. Philpott*, Lima, Ohio (1, 4, 6, 7, 8); *Clarence R. Perisho*, North Mankato, Minnesota (6); *Harry L. Nelson*, Livermore, California (1, 3, 4, 5, 6, 7, 8); *Thomas J. Morris*, San Juan, Puerto Rico (1, 2, 4, 5, 7); *Dale Kozniuk*, Delburne, Alberta (1, 2, 3, 4, 5, 6, 7, 8); *Jonathan Khuner*, Berkeley, California (1, 2, 4, 5, 6, 7, 8); *Felix Kestenholz-Seiler*, Liestal, Switzerland (1, 4, 6); *Edward Joris*, Antwerp, Belgium (1, 2, 4, 5, 6, 7, 8); *R. S. Johnson*, Town of Mount Royal, Quebec (1, 2, 4, 5, 6, 7, 8); *Harry M. Gehman*, Buffalo, New York (4, 5); *Ronald L. Enyeart*, Los Angeles, California (1, 4, 6, 7, 8); *Clifford R. Dickinson*, Camas, Washington (1, 4, 6, 7, 8); *Dermott A. Breault*, Waltham, Massachusetts (4, 6, 7, 8); *David R. Barstow*, Wethersfield, Connecticut (1, 2, 4, 5, 6, 7, 8); *Merrill Barnebey*, Grand Forks, North Dakota (1, 2, 4, 5, 6, 7, 8); *Raymond Aaron*, Toronto, Ontario (1, 4, 8).