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ISSUE NO. 12
DECEMBER 1962

RECREATIONAL MATHEMATICS magazine

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Hail Britannia!

We wish to announce the appointment of a Book Review Editor for RMM. Generally, only mathematical books will be reviewed and attention will be focused on those books of particular interest to the readers of RMM. All correspondence relating to book reviews, including submission of books for review, should be sent directly to our Book Review Editor:

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Associate Editor J. A. H. Hunter
Junior Department Editor Howard C. Saar
Book Review Editor Dmitri E. Thoro

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RECREATIONAL MATHEMATICS MAGAZINE is published bimonthly by Joseph S. Madachy. Second-class postage paid at Kent, Ohio. Subscription rates (worldwide) $3.25 per year, 75¢ per copy. All correspondence concerning subscriptions, changes of address, and advertising should be sent to The Editor - RMM, Box 35, Kent, Ohio. See inside front cover for manuscript and other mailings. See back cover for back issue sales. RECREATIONAL MATHEMATICS MAGAZINE is printed by the Commercial Press, Inc., Kent, Ohio.

FROM THE EDITOR

The extreme delays in publication of this issue of RMM were completely unforeseen. At the time of publication of the October 1962 issue it was believed that all necessary equipment would be on hand to enable us to put out this issue at least two months ago.

Alas, such was not the case. Rather than make a long story out of it, it may be best to stop and go on to finish this issue and get started on the February 1963 issue.

The Editor-Publisher of RMM hopes that impatient subscribers are taking it out on *him* and *not* on his Associate Editor (Mr. Hunter), Junior Department Editor (Mr. Saar), or his new Book Review Editor (Dr. Dmitri E. Thoro). The responsibility for getting RMM published on time rests with the Publisher. The other editors are doing *their* jobs admirably!

* * * * *

Regular readers of RMM will note the change in type-face. We have switched from Linotyping to Varityping. Naturally, the appearance is not quite the same - but content is the important thing.

* * * * *

Manuscripts continue to come in from everywhere. This special British issue of RMM, the continual appearance of foreign bylines, and the partial listing of the contents of the coming February 1963 issue below attest to the international circulation of RMM.

* * * * *

Coming in the February 1963 issue of RMM:

Geometric Magic Squares *by Boris Kordemskii* (U.S.S.R.)
How to construct magic squares in which the products of the numbers in each row, column, and main diagonal form the magic constant.

Multigrades *by D. C. Cross* (England)
The following equation is true for n=1, 2, 3:

$$1^n + 5^n + 8^n + 12^n = 2^n + 3^n + 10^n + 11^n$$

Mr. Cross explains how multigrade equations are formed and how higher order equations can be devised.

Soup, Fish, & Finite Geometries *by Ronald Read* (West Indies)
We'll let the title whet your appetites!

Multiperfect Numbers *by Alan L. Brown* (U.S.A.)
The first in a series of short articles about these interesting numbers.

Protean Shapes With Flexagons *by William R. Ransom* (U.S.A.)
How to form boats and buoys, baskets and pyramids with an ordinary flexagon.

And more international representation from Canada, Denmark, Holland, Ireland, and Japan in other articles and in our regular departments.

30 March 1963 J. S. M.

THE LOST CHORD AGAIN

by W. H. Cozens
Ilminster, England

One line of experimentation leads to another. We seem to have established that if the two ends of a chord chase one another round the circumference of a circle, one end going n times as fast as the other, then the chord will envelope a symmetrical curve with $n-1$ cusps, inscribed in the circle. (RMM No. 10, August 1962, pages 17-19)

But this is by no means the only trick in the repertoire of the lost chord. While one end plods steadily round with its unit speed the other end is free to wander. What if it moved *in the opposite direction*? This inspiration must be followed up immediately.

The numerical worker may like to number his points twice over, with outer and inner rings going clockwise and counter-clockwise respectively, like the double markings on a protractor. But it is sufficient for our first experiment to let the ends go in opposite directions, one twice as fast as the other. (Figure 1)

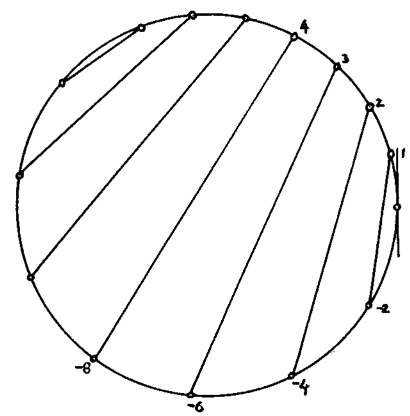


Figure 1

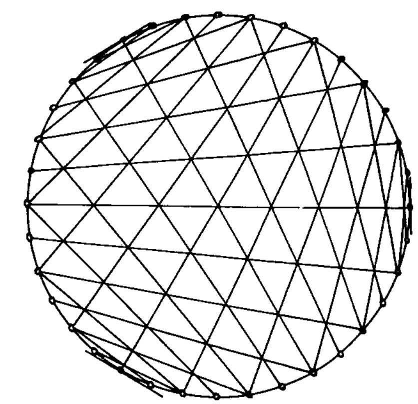


Figure 2

The first few lines are not very promising, and even when we continue until the slower end has completed a revolution the result is most disappointing – no sign of a curve, let alone a cusp. (Figure 2) But before dismissing this experiment as a failure it is worth taking a closer look at Figure 2. Those chords are in groups which seem to converge in three general directions – to the left, the top right, and the bottom right. *What if they were produced and allowed to meet outside the circle?* Imagination leaps at the possibility, and now with renewed energy we draw another circle, divide it into 20° intervals, and triumphantly produce Figure 3. Circumscribed to the circle we now have a figure with three symmetrical outward-turning cusps. This is the *deltoid*, so named – from its resemblance to the Greek letter *delta* – by Euler in mid-18th century.

Now there is no stopping us. The next experiment of course suggests itself: let the ends rotate in opposite directions but with speeds in the ratio 3:1. We may even make so bold as to forecast what the result may be. It is shown in Figure 4 and as we may have anticipated is similar to the deltoid but with one more cusp. This is the *astroid*, the star-curve.

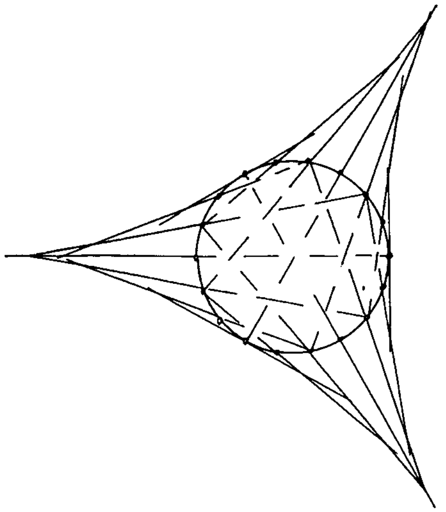


Figure 3

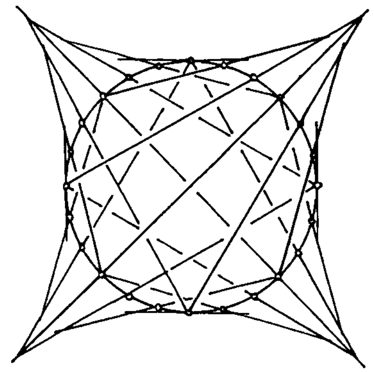


Figure 4

It now begins to appear that when one end moves n times as fast as the other, but in the opposite direction, the chord envelopes from the inside a curve which lies entirely outside the circle; and the number of cusps appears to be not $n-1$ but $n+1$. This assumption we proceed to test by letting one end move four times as fast as its partner. Now to bring the faster end precisely back to its starting point at every revolution, the circle must be divided so that the number of points is a multiple of four; and since we expect five cusps with the first one coming one fifth of the way round the *semicircle*, it will also be necessary to make it a multiple of ten, if the cusps are to be exactly defined. Forty divisions would fill the bill, so we use 9° spacing. The result is Figure 5, elegantly up to expectation.

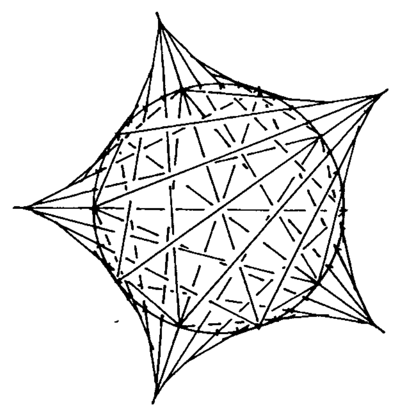


Figure 5

One unsatisfactory thing about these straight-line "curves" is that the *length* of the cusps is not defined exactly. By using larger and larger numbers of smaller and smaller divisions we could approach more and more nearly to the correct point, being limited only by the scale of our work, the quality of our drawing instruments, our skill in using them, and our patience. But since all our speed-ratios and numbers of cusps so far have worked out most pleasingly in simple integers it is worth looking for some similarly straightforward relationship here also.

A comparison of Figures 3, 4, and 5 reveals that in relation to the radius of the parent circle the cusps are certainly getting shorter as they become more numerous. And in the case of the Astroid (Figure 3) it certainly *looks* as if the distance of the cusp out from the cir-

cumference is equal to the diameter. If this is so, does it tie up in any way with the speed-ratio 2:1 which produced that figure? *Yes! The cusp divides the diameter externally in the ratio 2:1.*

Eagerly we measure the Astroid cusp. Yes, it is equal to the *radius* of the circle; *i.e.* it divides the diameter in the external ratio 3:1. And the five-cusp of Figure 5 projects one-third of the diameter, thus dividing it externally as 4:1. Figure 6 will make these relationships clearer.

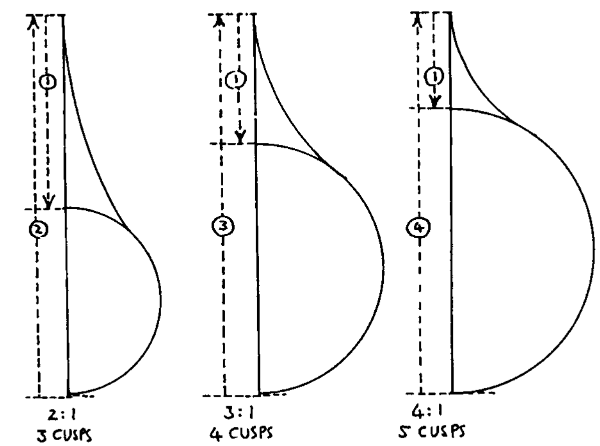


Figure 6

Finally let us forecast the next curve in the series and then draw it as a check. We shall have a speed-ratio of 5:1, and we expect six cusps, each of which should divide its diameter externally in the ratio 5:1; *i.e.* the cusp should project one-quarter of a diameter beyond the circle. Figure 7 confirms our forecast exactly.

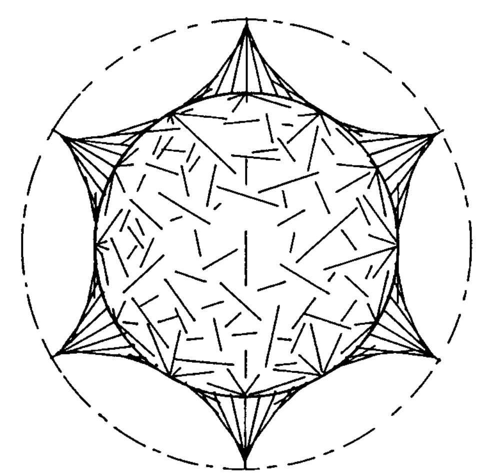


Figure 7

This discovery is a far-reaching one and, as we shall see later, adds great power to our methods.

DIGITAL INVARIANTS

by Max Rumney
London, England

"There are just 4 numbers, after unity, which are the sums of the cubes of their digits: viz.,

$$153 = 1^3 + 5^3 + 3^3, \quad 370 = 3^3 + 7^3 + 0^3,$$
$$371 = 3^3 + 7^3 + 1^3, \quad 407 = 4^3 + 0^3 + 7^3.$$

These are odd facts, very suitable for puzzle columns and likely to amuse amateurs, but there is nothing in them which appeals to the mathematician." So wrote the late *G. H. Hardy* in his book *A Mathematician's Apology.*

It seems surprising that this great mathematician did not apparently notice that those four numbers, which he shrugged off as mere curiosities, are in fact the ultimate sums of the cubes of the digits of most numbers (in our usual decimal notation). This special property entitles them to formal recognition, and here I shall introduce the general class of such numbers for the interest of mathematicians as well as amateurs.

First we consider such numbers, in decimal notation, under cube summation. This will lead to more general applications for higher powers, and other scales of notation.

Say we start with any integer N, of n digits, such that

$$N = a_1 10^{n-1} + a_2 10^{n-2} + \cdots + a_n, \text{ with } a_1 \neq \text{zero}.$$

Summing the cubes of the digits of N, we derive a new number N_1, which with similar cube summing leads to N_2, and so on. The process is continued until we reach a number that either reproduces itself, or that will be a member of a recurring sequence: in either case we shall call the final derived number an *ultimate*, denoting it by N_u. The sequence of successive numbers, formed in the summing process, will be written as:

$$N \to N_1 \to N_2 \to \cdots \to N_m \to \cdots \to N_u,$$

where N_m denotes the derived number at step m, and N_u the ultimate.

For example, we take 4213, 8064, 538, 2168, and 4121.

$$4213 \to 100 \,(\text{i.e. } 4^3 + 2^3 + 1^3 + 3^3) \to 1 \to 1$$
$$8064 \to 792 \to 1080 \to 513 \to 153 \to 153$$
$$538 \to 664 \to 496 \to 1009 \to 730 \to 370 \to 370$$
$$2169 \to 737 \to 713 \to 371 \to 371$$
$$4121 \to 74 \to 407 \to 407$$

In each case we ended up with an *ultimate*, one of them being unity, and the other four being the numbers mentioned by Hardy.

We define a *Perfect Digital Invariant*, of order n, as a number which equals the sum of the nth powers of its digits. Thus, there are five and only five 3rd order Perfect Digital Invariants (PDI) in decimal notation, *i.e.*, 1, 153, 370, 371, and 407.

As stated earlier, these are the ultimates for most numbers. But there are four more which form a second group of invariants of the 3rd order (*i.e.*, for cubes of digits). These are different, in that each characterizes a distinct recurring sequence:

$$133 \to 55 \to 250 \to 133$$
$$136 \to 244 \to 136$$
$$217 \to 352 \to 160 \to 217$$
$$919 \to 1459 \to 919$$

The choice of 133, 136, 217, 919 is arbitrary, since any other in the relevant sequence would qualify just as well. These four numbers we call Recurring Digital Invariants (RDI) of order n (in this case $n = 3$), being numbers that characterize the respective recurring sequences resulting from the summing of the nth powers of their digits. These four comprise all the 3rd order Recurring Digital Invariants for decimal notation.

Using elementary congruence theory it is simple to prove that, for decimal notation, the ultimate resulting from any original number that is a multiple of 3 must also be a multiple of 3. Since 153, a PDI of order 3, is the only such ultimate, it follows that 153 must be the ultimate for one-third of all numbers under cube summation for decimal notation.

For decimal notation, then, all 3rd order Digital Invariants are known. Many Digital Invariants of higher orders have been found, but there must be many more awaiting discovery - a challenge to enthusiastic amateurs. All those that are known, of order 4 (*i.e.*, fourth powers) and higher orders are summarised here: except where noted in parentheses, all have been found by myself. Unity is, of course, a PDI in each case.

Order	*PDI*	*RDI*
4th	1634 (*D. Milsom*)	1138
	8208	2178
5th	4150	9044
	4151	9045 (*J. A. H. Hunter*)
	54748 (*J. Romney*)	24584
	93084	
	92727 (*K. Chikawa, et al**)	
	194979 (*K. Chikawa, et al**)	
6th	-	239459 (*J. A. H. Hunter*)
		282595 (*J. A. H. Hunter*)
7th	-	9057586

Alan Bostrum has also drawn attention to some of these numbers, vide RMM No. 10, August 1962, page 42.

In the foregoing list there are found some pairs of what may be called Amicable Digital Invariants. For example, for order 4, we have $2178 \to 6514$ and $6514 \to 2178$ without any intermediate steps.

So far we have considered only decimal notation. Similar concepts apply for other scales of notation. The known Digital Invariants all of 3rd order, for scales of notation other than the decimal scale, are as follows:

*K. Chikawa, K. Iseki, T. Kusakabe, K. Shibamura, "Computation of Cyclic Parts of Steinhaus Problem for Power 5", *Acta Arithmetica* 7 (1962), pages 253-254.

Scale 4.	PDI	1, 20, 21, 130, 131, 203, 223, 313, 332
	RDI	None
Scale 5.	PDI	1, 103
	RDI	14
Scale 7.	PDI	1, 12, 250, 251
	RDI	161
Scale 8.	PDI	1, 134, 205, 660, 661
	RDI	670
Scale 11.	PDI	1, 105, 708
	RDI	366
Scale 13.	PDI	1, 490, 491
	RDI	71, 252, 443, 611, 688, 1523
Scale 19.	PDI	1, 180, 181
	RDI	96*A*, *A*1*B* (*A* represents fifteen, *B* twelve)

Coming back to our normal decimal notation, it has been suggested by *D. St. P. Barnard* that there is a greatest possible Perfect Digital Invariant irrespective of the Order in question. He points out that in no case could it exceed 9^{61}, since the maximum sum of the Nth powers of N digits is $N{\cdot}9^N$. To continue with his reasoning would be rather beyond the scope of *recreational mathematics*, but the point is mentioned as a possible challenge in the more serious aspects of the subject.

Other questions that invite research, and on a more recreational level, are:

1. What connection, if any, is there between the Order (*i.e.*, 3rd, 4th, etc.) and the number of invariants applicable to that order?
2. Are there any prime PDIs?

The various problems that have been suggested in this short paper may prove fruitful when investigated - or they may turn out to be largely trivial. In any case they will provide a challenge for those who may seek to fill some of the many gaps in the framework of invariants as listed here.

My grateful acknowledgments are due to *J. A. H. Hunter* for his help in the preparation of the foregoing.

* * * * *

Symposium on the Exploration of Mars

Denver, Colorado on June 6 and 7 of this year will be the focal point of a lot of space talk. The Symposium on the Exploration of Mars will be held at the Hilton Hotel and specific attention will be directed to the scientific, technical, management, and economic resources necessary for this foreseeable Mars voyage of discovery.

If you plan on being near Denver at that time and would like to know more about the Symposium write to:

Dr. George W. Morgenthaler
General Program Chairman
Symposium on the Exploration of Mars
P. O. Box 766
Littleton, Colorado

before May 15, 1963.

MAGIC KNIGHT TOURS ON SQUARE BOARDS

by T. H. Willcocks
Bristol, England

The problem of moving a knight on a chessboard so that successive moves cover each square once and once only is an old one and is probably familiar to most readers of this article. Solutions by various methods have been given by De Moivre, Euler, Warnsdorff, Jaenisch, Roget, and others and an excellent account of these may be found in W. W. R. Ball's *Mathematical Recreations and Essays.* A harder task is to construct a knight's tour such that, if the successive squares visited by the knight are numbered 1, 2, 3, . . . , 64, the result is a magic square (*i.e.* the sum of each row, column, and long diagonal is equal to the same magic constant). For the ordinary 8x8 chessboard, this magic constant[1] would be 260. It remains an open question as to whether such a tour is possible on the standard chessboard, but we shall see that on certain larger boards such tours may be constructed.

In the first place, consider a board $n \times n$. If n is *odd*, one column of the board will consist of an odd number of black squares and an even number of white; an adjacent column will consist of an even number of black squares and an odd number of white. Since the colors of the squares visited by a knight change with every move, all squares with the same color will contain either odd or even numbers, but not both. Hence there are columns which contain an odd number of odd numbers and an even number of even numbers, *i.e.* they will sum to an odd number; columns adjacent to these contain an even number of odd numbers and an odd number of even numbers, *i.e.* they will sum to an even number. Thus, a magic knight's tour cannot be constructed on an $n \times n$ board if n is odd.[2]

If n is *singly even* (*i.e*, if $n = 4p + 2$, where p is an integer) the magic constant $C = (2p+1)(16p^2 + 16p + 5)$ which is odd. However, one diagonal consists of an even number of even numbers and the other of an even number of odd numbers, *i.e.* the sum of each is an even number. Hence, a magic knight tour cannot be constructed on an $n \times n$ board if n is singly even.

Thus, a magic knight tour, if constructible, must be on a board of side $4w$, where w is an integer.

We begin with a brief résumé of what is known about *semi-magic* knight tours on an 8x8 board (*i.e.* tours in which the rows and columns of the board sum to the magic constant, but the main diagonals do not). The first such tour to be constructed was discovered by William Beverley in 1847 and published by him in August of the following year in *The Philosophical Magazine.* This tour is depicted

[1] The magic constant for a magic square composed of the consecutive integers from 1 to n on an $n \times n$ board is $C = n(n^2 + 1)/2$.

[2] A method is available by which a fully magic square on an odd-sided board can be constructed by knight moves. However, the technique necessitates moves outside the board and the transference of the resulting positions to corresponding cells within the board. For details see *Mathematical Diversions* by J. A. H. Hunter and Joseph S. Madachy (D. Van Nostrand, Princeton, N. J., 1963), pages 86-87.

in Figure 1. Since that date other investigators have found many other tours. Towards the end of the last century General Parmentier gave a list of 110 such tours, but his list contained many duplicates. A collection of 87 diagrams of semi-magic tours was given by M. B. Lehmann in *Neue Mathematische Spiele* (5th Edition, Wiesbaden 1932, pages 301-339) and these together with ten which have subsequently been discovered (one by Lehmann, nine by H. J. R. Murray) form the writer's list. How the early tours were discovered we are not told – probably by trial and patience – and it was not until 1936 that the late H. J. R. Murray, in an article in *The Fairy Chess Review*, revealed the plan behind Beverley's tour. Beverley contented himself with describing a single tour, but Murray exploited the method of construction and was able to construct some new tours.

282

1	30	47	52	5	28	43	54
48	51	2	29	44	53	6	27
31	46	49	4	25	8	55	42
50	3	32	45	56	41	26	7
33	62	15	20	9	24	39	58
16	19	34	61	40	57	10	23
63	14	17	36	21	12	59	38
18	35	64	13	60	37	22	11

210

Figure 1

264

38	35	62	25	60	23	10	7
63	26	37	34	11	8	59	22
36	39	28	61	24	57	6	9
27	64	33	40	5	12	21	58
50	29	4	13	48	41	56	19
1	14	49	32	53	20	47	44
30	51	16	3	42	45	18	55
15	2	31	52	17	54	43	46

256

Figure 2

Semi-magic tours may, of course, be classified in a number of ways, but it is convenient to divide them into two major classes: (1) Open Tours, *i.e.* those where the first and last squares are not connected by a knight's move; (2) Closed or Re-entrant Tours where they are so connected. Beverley's square was of the first type; one of the second type by Jaenisch is shown in Figure 2. This square is interesting in that the diagonals sum to 256 and 264, so that a transfer of 4 from one diagonal to the other would convert it into a fully magic square. This square exhibits a property of certain re-entrant tours by which other closed tours may be obtained by renumbering the original tour. For instance, if the square numbered 49 be renumbered 1, 50 as 2, 51 as 3, . . . , 64 as 16, and 1 be renumbered as 17, 2 as 18, . . . , 48 as 64, we shall have an *arithmetically* different tour. Given a closed semi-magic tour, we can test it to see if arithmetically different tours may be derived from it in the following manner:

Suppose there are N numbers up to and including $65-p$ in each row and column of the squares; then, if p and N are as in the table below, square 1 may be renumbered as p.

p	N	$65-p$
9	7	56
17	6	48
25	5	40
33	4	32
41	3	24
49	2	16
57	1	8

It should also be noted that the reverse tour, open or closed, is an arithmetically different tour.

We have already said that it is extremely doubtful whether any fully magic tour, open or closed, is possible on an 8x8 board.

On a 12x12 board, the writer knows of no fully magic tour. He has, however, constructed a few semi-magic tours where *one* diagonal sums to the magic constant 870. One such tour, where the "wrong" diagonal summed to 748 was published in *The Fairy Chess Review* of December 1955. A somewhat closer approximation is given in Figure 3 where the "wrong" diagonal is 888.

870

144	75	38	33	42	31	138	79	44	29	136	81
37	34	143	76	139	78	43	30	137	80	45	28
74	39	36	141	32	41	84	133	26	47	82	135
35	142	73	40	77	140	25	48	83	134	27	46
72	3	110	105	68	5	132	85	50	23	130	87
109	106	71	4	113	104	49	24	131	86	51	22
2	111	108	69	6	67	90	127	20	53	88	129
107	70	1	112	103	114	19	54	89	128	21	52
64	9	116	101	66	7	126	91	56	17	124	93
117	100	65	8	115	102	55	18	125	92	57	16
10	63	98	119	12	61	96	121	14	59	94	123
99	118	11	62	97	120	13	60	95	122	15	58

888

Figure 3

On a 16x16 board a semi-magic tour was published in *The Fairy Chess Review* of August 1942. This was due to H. J. R. Murray. Starting with a semi-magic tour on an 8x8 board, composed entirely of Roget's squares and diamonds, he inserted the moves connecting their terminals on the central 8x8 of the 16x16 board. He then replaced the squares and diamonds by circuits of squares and diamonds round the quarter of the 16x16 board in which the terminals lay. Some years later this method was re-examined by H. E. de Vasa of Paris and the writer, and almost simultaneously the former constructed a completely magic closed tour and the latter an open tour.

de Vasa took as his starting point the 8x8 tour of Figure 2. This tour has many interesting properties, *e.g.* it possesses a high degree of symmetry and it has equal semi-diagonals. These properties allow of a large number of variations on the method just described. If we select one corner of the large square at will we can replace a square or a diamond in the small central square by any combination of squares and diamonds and still obtain a semi-magic square provided symmetrical operations are performed in the three remaining quarters. By a judicious selection of the extensions available, a fully magic square may be obtained. de Vasa's result is shown in Figure 4 and a study of the diagram will probably disclose the method better than any attempted description.

2056

184	217	170	75	188	219	172	77	228	37	86	21	230	39	88	25
169	74	185	218	171	76	189	220	85	20	229	38	87	24	231	40
216	183	68	167	222	187	78	173	36	227	22	83	42	237	26	89
73	168	215	186	67	174	221	190	19	84	35	238	23	90	41	232
182	213	166	69	178	223	176	79	226	33	82	31	236	43	92	27
165	72	179	214	175	66	191	224	81	18	239	34	91	30	233	44
212	181	70	163	210	177	80	161	48	225	32	95	46	235	28	93
71	164	211	180	65	162	209	192	17	96	47	240	29	94	45	234
202	13	126	61	208	15	128	49	160	241	130	97	148	243	132	103
125	60	203	14	127	64	193	16	129	112	145	242	131	102	149	244
12	201	62	123	2	207	50	113	256	159	98	143	246	147	104	133
59	124	11	204	63	114	1	194	111	144	255	146	101	134	245	150
200	9	122	55	206	3	116	51	158	253	142	99	154	247	136	105
121	58	205	10	115	54	195	4	141	110	155	254	135	100	151	248
8	199	56	119	6	197	52	117	252	157	108	139	250	153	106	137
57	120	7	198	53	118	5	196	109	140	251	156	107	138	249	152

2056

Figure 4

It will be appreciated that having found a tour with correct diagonals it may then be possible to vary a part of the tour not affecting the diagonals and (with the necessary adjustments in the other quarters) obtain a different tour. Again there are a number of different 8 x 8 tours from which to initiate the construction, so that the number of possible fully magic tours on a 16 x 16 board is considerable.

As regards larger boards, de Vasa constructed tours on boards 16 x 16, 32 x 32, 48 x 48, and 64 x 64. For instance, from a fully magic tour on a 16 x 16 board he derived first a semi-magic tour on a 24 x 24 board and from this in turn a fully magic tour on a 32 x 32 board. In a somewhat similar fashion, the writer constructed a completely magic closed tour on a 24 x 24 board and from this a fully magic tour on a 40 x 40 board. It would seem that the method is applicable to any boards of the form $(8n)^2$, but fails for those of the form $(8n+4)^2$.

The writer has constructed a fully magic open tour on a 20 x 20 board but this case requires a different approach and might perhaps form the subject of further articles.

If any readers wish to try their hand at constructing tours, two outstanding questions are:

1. Are there any semi-magic tours on the 8 x 8 board still to be discovered?
2. Is there a fully magic tour on a 12 x 12 board? Can any reader find one or a closer approximation than that of Figure 3?

* * * * *

"YOICKS!" "TALLEYHO!" SHADES OF KING ARTHUR!

SIR MOEBIUS RIDES AGAIN!

(Annual reminisences of Sir Moebius, K.G., K.C.B., K.C.V.O., K.G., K.T., at the exclusive Sts. Farthing and Pence Golf Club - well-known to all Scotsmen.)

by Dorman Luke
West Palm Beach, Florida

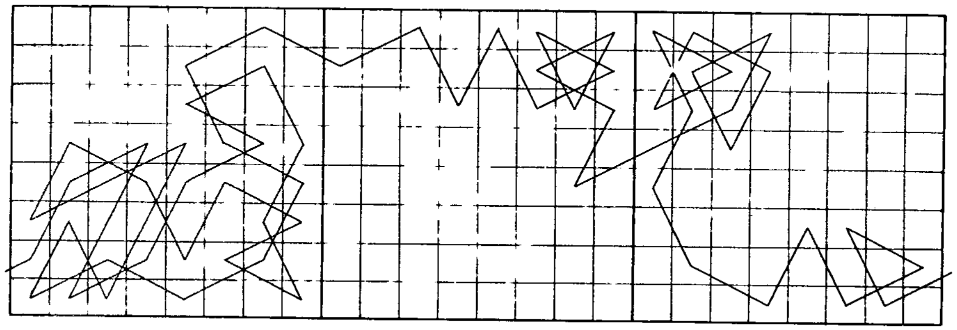
"During the third crusade, King Richard and the first Sir Moebius had learned chess from the Arabs, and other Moslem masters of that ancient game. The little-known "Moebius Gambit" was derived from certain attacks and retreats noted during action in battle, and transferred to the chessboard. Sir Moebius had been knighted on the field of battle for deeds of derring-do.

"At a later period, by royal decree, the land of Scotland was to be surveyed. Two groups of engineers did the surveying, one group starting at Mucka Flugga and the other at the border. They were using the Scotland Yard (2 feet, 10 inches), and, as a result, each group incorrectly placed its final stakes for the other group to find; but these stakes were *never* located by the opposite groups. At the Inn of The Four-Sided Triangle - owned and operated by Miss Polly Gons, and her cousins, Miss Polly Hedra and Miss Polly Topes (their mutual ancestor being dear old Polly Grams - and so-called by all the children in the neighborhood) - one Priscilla Roby, a bonnie lass, overheard one of the surveyors commenting on this error of omission. Miss Priscilla contacted her fiance, one John Alexander Macdonald, who immediately went to Sir Moebius, and imparted this information. The old boy checked up, and found the missing "strip" to be located in the rolling, folded hill and dale contours he rode and loved so well, under the crags and over the low, flat leas and meadows. Sir Moebius claimed the land, and presented his claim to the King. His Majesty, after consulting his Privy Council, said, 'There are 10 counties, with 64 hundreds in each county, so-placed as to form 8 x 8 squares. If you can make a compleat Knight's Tour of all 10 counties as a single re-entrant tour I will grant this land to you and your heirs, in fee simple, etc., etc.' (Figure 1* indicates the re-entrant tour used by this Sir Moebius. All 10 counties are identical tours, with one entry and one departure each.)

"Some generations later, during The Hundred Years' War, the Sir Moebius of that day made a wager with his King, wagering his lands against a higher title or two. To win, he was to devise a more complex form of the Knight's Tour.

"He made many 'trial and error' tours, but to no avail. One night, his old charger, Doodles, who could smell an apple or a carrot a mile away, got out. When they finally caught up with him, he had found, and heavily partaken of, some fermented mash; and had staggered through three counties, ending up entirely drunk.

"Sir Moebius rode out the next morning, and made a drawing of of old Doodle's trail. He found that 19 'hundreds' were crossed in the first county, 12 were crossed in the second county, and 33 were crossed in the third county - a total of 64 'hundreds', with no duplications. (These are shown in Figures 2a, 2b, 2c; and are 'stacked' in Figure 3 to indicate one compleat county.) By continuing these tours through all 10 counties, Sir Moebius found his answer. He had gone into each county three times, each entry of a county being at a new point; the final entry and exit making the trip compleat for each county. (This re-entrant tour is shown in Figure 4.*)



a *b* *c*

Figure 2

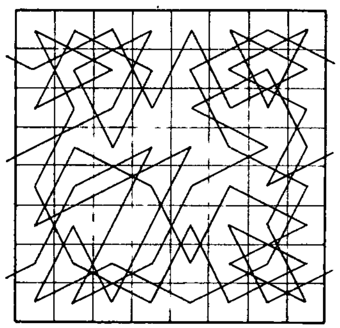


Figure 3

"By the time they had brought blankets, heating coils and other articles of equipment, and had settled down to revive old Doodles, they had also, unknowingly, run off a batch of excellent Scoth whiskey, having gotten mixed up in the fermented mash. A fine distillery was established at this spot, and *Doodles* became the trademark for the 'Liquor For The Gods'.

* See Center Spread in this issue.

"The earnings were immense, and formed the foundation for the *Moebius Millions*. Somewhere about this time, the famous country estate, *Mullioned Mansion*, often called *Moebius Mullions*, was erected at Burping-On-The-Bottle - a beautiful river, the old Bottle.

"Sometime through the years, a mathematician from the continent, while on a camping trip, found the unusual curved properties of the 'Moebius Strip', and imparted this information to his own field of science, but he never included the different Knight's Tours. The 'edge' has caused a bit of feuding among the neighbors, due to fence troubles.

"By this time, our family funds had so greatly increased, that we founded our own bank, The Farmers and Distillers Bank and Trust Company, Ltd., with liquid assets. Several years later, a branch office was opened in Paris, France, called The Left Bank; its medium of exchange being the well-known old Latin Quarter.

"And thus Sir Moebius won his wager and his several titles; Duke of Upsan Downs, Marquis of Insandouts, Earl of Hillandale, Viscount Under Andover, and Baron Lowlea. And old Doodles spent his remaining years roaming the 'Strip', beloved of all who knew him and his deeds.

"And so, gentlemen, we are assembled here at our club to propose a toast. The only toast my people have is to the reigning head of the royal family, and we rise to say, *'God bless her Majesty the Queen!'* "

Author's Note: To any bonnie Scot who may read this: my first ancestors in the U. S. were John Alexander Macdonald and Priscilla Roby, coming from Edinburgh in 1800; but I have never seen the old home. I am of the 5th generation here, thus I have chosen ancestral names. And as my dear Mother says, 'You can't kill a Scotsman with a hatchet!'

* * * * *

CURIOSITIES

D. C. Cross, of Birmingham, England, has sent us a neat sums-of-squares identity that uses the 15 integers, 1 to 15:

$$1^2+2^2+3^2+4^2-(5^2+6^2+7^2+8^2+9^2+10^2+11^2+12^2)+13^2+14^2+15^2=0$$

Not to be outdone, *J. A. H. Hunter* gives us the next two *bigger* cases on precisely similar lines:

Using the 4895 integers, 1 to 4895:

$$1^2+2^2+\cdots+1511^2+1512^2-(1513^2+1514^2+\cdots+3959^2+3960^2)+3961^2+3962^2+\cdots+4894^2+4895^2=0$$

Using the 1,576,239 integers, 1 to 1576239:

$$1^2+2^2+\cdots+487084^2-(487085^2+487086^2+\cdots+1275204^2)+1275205^2+1275206^2+\cdots+1576238^2+1576239^2=0$$

CHINESE ARITHMETIC

by G. E. Ashley
Chelmsford, England

Our children spend long hours mastering the intricacies of the multiplication table. Could those hours be put to better use? It is not really necessary to know all the various times tables in order to multiply - if the two-times table is known any multiplication whatsoever may be worked.

This particular method (or trick if you prefer that word) has been quite often called *Chinese Arithmetic*, but there is no proof that the Chinese were the actual originators, and in fact I have seen it called Ethiopian and Dutch as well as Chinese. The outstanding thing about it is that it is unusual, but let me hasten to say straightaway that actually it is merely a mathematical curiosity, as the ordinary arithmetical methods of multiplication (not to mention such aids as the slide rule) are so much shorter and quicker. Yet this method has a certain fascination - on encountering it for the first time we are intrigued to discover why it should work - and, believe you me, it does work.

The essential basis of the method is that no table other than the two-times is used and yet the product of *any* two numbers can be obtained. It is obtained as a result of the three operations: multiplying by 2, dividing by 2, and adding. The two numbers to be multiplied together are placed side by side, one is divided by 2 successively until we get to 1 (fractional parts of answers simply being ignored), and the other is multiplied by 2 successively until there is a number against every number we wrote down in dividing. The column in which we have the numbers obtained by multiplication is then added and the sum is our required answer. As an example I have set down below, side by side, 72 times 63 done by the Chinese method and by ordinary multiplication. You will notice the answer is the same, whichever method is used.

63	72
31	144
15	288
7	576
3	1152
1	2304
	4536

```
  72
  63
 216
432
4536
```

Now, this unusual method need not be confined to the two-times table. Here is a case where the three-times table is used in just the same sort of way, dividing one number by 3 (ignoring fractions) and multiplying the other by 3. The answer obtained by adding the multiplication column is the same as that obtained by ordinary arithmetic.

72 times 40

40	72
13	216
4	648
1	1944
	2880

```
  72
  40
2880
```


And now for good measure we will show examples using all the tables as far as ten-times: (Omitting the ordinary arithmetic forms.)

Four-times: 85 times 72

85	72
21	288
5	1152
1	4608
	6120

Five-times: 156 times 72

156	72
31	360
6	1800
1	9000
	11232

Six-times: 43 times 72

43	72
7	432
1	2592
	3096

Seven-times: 57 times 72

57	72
8	504
1	3528
	4104

Eight-times: 73 times 72

73	72
9	576
1	4608
	5256

Nine-times: 91 times 72

91	72
10	648
1	5832
	6552

Ten-times: 1111 times 72

1111	72
111	720
11	7200
1	72000
	79992

Now, before you start trying some examples on your own and find they do not work properly I will at once hasten to add that the numbers used in these examples have been very carefully chosen. But with just one small extra rule the method with the two-times table can be made to work for *all numbers*, though with the other tables the required adjustments are quite complicated.

To make sure of the two-times table method of Chinese Arithmetic working one has, before performing the addition, to look at the column of figures obtained by division. Wherever an even number occurs, cross out (in the column obtained by multiplication) the number opposite the even number. It is only in the division column that even numbers cause crossing out in the multiplication column - we are bound to have lots of even numbers in the other column.

Thus: 99 times 99

99	99	99
49	198	198
24	~~396~~	
12	~~792~~	
6	~~1584~~	
3	3168	3168
1	6336	6336
	9801	9801

It may perhaps be easier to avoid adding in the crossed out figures if you write down afresh the numbers that have to be added, as has been done above.

Thus, you see, when we are using the two-times table we must not count in for addition any figures opposite a number that is itself in the two-times table (or, in other words, is divisible exactly by two, as, of course, all even numbers are). This same proviso can be made to work with the other tables also, as we see from the following examples, where working with the three-times table we cross out all numbers opposite any division column number divisible exactly by three, and when working with the four-times table all numbers opposite those divisible exactly by four.

81 times 97:

Three-times table

81	~~97~~
27	~~291~~
9	~~873~~
3	~~2619~~
1	7857
	7857

Four-times table

81	97
20	~~388~~
5	1552
1	6208
	7857

However, although it has been made to work in these cases, with tables higher than the two-times still more adjustments are needed to make sure it works every time - sometimes, for example, we have to count in certain lines twice, or even more times. So, therefore, it is only in connection with the two-times table that we find Chinese Arithmetic usually employed - and, as I have said earlier it will work *every time*, (provided, of course, there are no slips in the working) if we divide successively in the one column, multiply successively in the other, cross out all numbers opposite even numbers in the division column and then add up the remaining numbers.

Now, what mathematical principle is at the root of the whole method? A mathematician would tell you straightaway it is an application of what in his text-books is called 'scale of notation'. Our ordinary method of writing down numbers is the decimal scale of notation, and this has come into general use because of the undeniable fact that humans have ten fingers. There are other scales of notation in use, for example in the building industry the duodecimal system (counting in twelves) is by no means unknown. In our Chinese Arithmetic we have been using (albeit unconsciously) the binary system, or scale of two, the system that we human beings could be conceived as having evolved for general use if Nature had decided that we had only two fingers apiece in all. But before you dismiss this binary scale as being just some theoretical nonsense of the mathematicians remember that such modern devices as electronic computers use it. Thus if a computer records 11111 it does not mean eleven thousand, one hundred and eleven $(10^4+10^3+10^2+10+1)$ but, rather, it means $2^4+2^3+2^2+2+1$, or thirty-one. So if we want to multiply by 31 we *can* do it by multiplying by $(1+2+4+8+16)$, and that is in fact what we do if we set the sum down as:

31	72	A
15	144	B
7	288	C
3	576	D
1	1152	E
	2232	

Line A gives us once 72, line B gives us twice 72, line C gives us four times 72, line D eight times 72, and line E sixteen times 72. Adding up we have the full thirty-one times 72.

Connected with this business of the binary scale is the fact that if you have weights of 1, 2, 4, 8, 16 pounds, etc., you can weigh out any whole number of pounds up to the sum of your set of weights. For example all the weights up to and including the 16-pound one will counterbalance 31 pounds. To weigh out 29 pounds, we simply remove the 2-pound weight; 5 pounds can be weighed out by using the 4- and 1-pound weights; and so on. With weights arranged 1, 3, 9, 27 we shall not find it as easy - 12 pounds are easily counterpoised by a 9- and 3-pound weight. But how are we to weigh out 11 pounds? We just cannot do it with these weights if we are obliged to have weights in one pan and the article (or commodity) being weighed in in the other pan. If, however, we are allowed to place weights in the same pan as the article being weighed we can do it - 3-pound and 9-pound weights in one pan to balance the 11-pound article and a 1-pound weight in the other.* But just attempt to try obtaining any specified weight with a set of weights running 1, 4, 16 pounds. For a start, how would you weigh out 2 pounds? The answer is you simply cannot with just those weights - though if you were allowed two sets of the weights the story would be different.

Now to return to our Chinese Arithmetic. If we want to multiply by 29 we are actually desirous of multiplying by $16+8+4+1$. So in this case we do not want the twice times. Before, in multiplying by 99 $(=64+32+2+1)$, we did not want the four, eight or sixteen times. How are we to know which multiplications are to be omitted? That is the function of the even number in the division column - it just tells us we are not to add the corresponding numbers in the multiplication column. Let us try one more problem, say 29 times 85:

29	85	A
14	~~170~~	B
7	340	C
3	680	D
1	1360	E
	2465	

Line A gives us once 85, line C gives us four times 85, line D eight times 85, and line E sixteen times 85. Adding these up we have 29 $(=16+8+4+1)$ times 85, which is exactly what we wanted.

* The startling power of this method was pointed out by *Martin A. Brewer* of Salt Lake City, Utah. Referring to J. H. Sweitzer's problem in RMM No. 7, February 1962, page 25, we are asked what five weights are necessary to make integral weighings from 1 to 121 pounds on a 2-pan balance. The answer is that weights of 1, 3, 9, 27, and 81 pounds each are required. These weights are 3^n for values of n from 0 to 4. It follows that all integral weighings from 1 to 364 pounds can be made using only six weights - the five just given plus a 243-pound weight. The maximum weight that can be weighed out is the sum of the individual weights.

The value of 3^n increases rapidly as n increases and it becomes almost unbelievable that, for example, only 10 weights (1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19683) are needed to make all integral weighings up to 29,524 pounds; or that only 21 weights (n from 0 to 20) are all that are required to make integral weighings from 1 to 5,230,177,601 pounds!

Of course, as we often do in recreational mathematics, we ignore the questionable practicality of handling multi-million-pound weights or constructing a 2-pan balance of sufficient strength! *(J.S.M.)*

PUZZLES AND PROBLEMS

1. *Flowers for the Girls*

Mr. Jones was staying at the Hotel Splendide with his famous dancing troupe. Flowers were on sale in the lobby at the following rates: orchids, 50¢ each; roses, 40¢ each; carnations, 30¢ each.

During the eight days they were there, Mr. Jones bought one flower each day for each of the girls, spending \$6.00 daily on them. On no two days did he buy the same assortment of flowers.

How many girls were in the troupe?

(*H. E. Tester*; Ilseworth, England)

* * * * *

2. *Four Against Four*

"On one pan of this balance," said Mr. Serang's son, "I've placed a single cube, exactly balanced by four cubes, all different, on the other pan."

"Very interesting," commented his father, "Presumably all five are of uniform density?"

"Of course," replied the boy, "And, if you remove two cubes from that one pan you can restore the balance by replacing them with two other cubes each of a new size."

Mr. Serang found the solution, involving the smallest possible whole numbers. Can you do so? (*D. C. Cross*; Birmingham, England)

* * * * *

3. *Down to Earth?*

On the surface of a sphere draw the two smallest possible isosceles triangles with integral sides and identical perimeters such that the sum of the squares on the three sides of one triangle will equal the sum of the squares on the three sides of the other triangle.

(*D. C. Cross*; Birmingham, England)

* * * * *

4. *Did the Butler do it?*

"Where are those new pennies I left on the table this morning, James? I put them in a square array and now there are only two left. You didn't take them, did you?"

"Well, sir," replied the butler, "shortly after you left, three burglars came in. They divided the pennies equally among themselves, but left these two because they could not divide them equally."

Is the servant telling the truth?

(*Sidney Kravitz*; Dover, New Jersey)

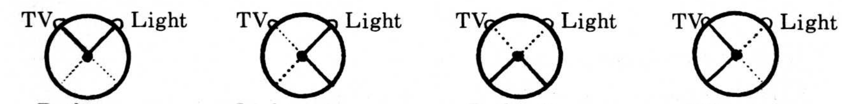
* * * * *

5. *The Seven Fortunes*

Seven kids each own a different number of pennies. The ratio of any kid's fortune to the fortune of every kid poorer than himself is an integer. In total the combined fortunes of the seven kids is 2879 pennies. Find the fortune of each of the seven kids. There is a unique answer. (*Sidney Kravitz*; Dover, New Jersey)

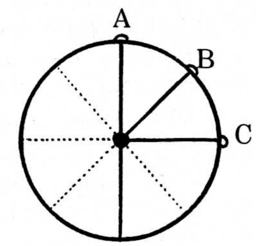
6. *Electrical Switching*

It is not a difficult problem to design a rotary switch which will do one of four things for each 90° turn of the switch. These things may be (1) turn on the TV set and the light, (2) turn on the TV set only, (3) turn on the light only, (4) turn off both the TV set and the light. The proper design is shown below (the center of the circle represents a source of current).



Both on Light only Both off TV only

A rotary switch for three appliances so that all eight combinations can be found by 45° turns, is shown at the right.



You are asked to find a design for a 16-position switch for all combinations of 4 appliances, requiring 22½° turns.

(*Sidney Kravitz*; Dover, New Jersey)

* * * * *

7. *Can You Couple the Couples?*

Of three men - Andy, Bert, and Chris - two are married, their wives being Eva and Fay, and the other is a bachelor. The younger of the wives has a daughter. Eva does not know Chris. Andy, who is 15 years older than his wife, graduated from high school in the same year as Chris and Eva.

Which couples belong together, and who is the mother of the daughter? (*Gerard Mosler*; Forest Hills, New York)

* * * * *

8. *Rather Precious Antiques*

Looking around in an antique shop, John spotted a couple of very early desk calculators, both of which seemed old enough to have been rejected by Babbage as obsolete.

"Those new models sell at \$245.00," said the dealer, nodding towards another table. "They're the latest thing."

"I'm more interested in those two old calculators," John told him. "How much are they?"

"I'm sorry, but if you want to buy one of those you'll have to buy this new model, too. I won't sell them otherwise."

"Okay," replied John, somewhat taken aback. "But what are the prices?"

"If you buy this one with a new machine, the total cost would be one-third the cost of the other. But if you have that one with a new machine, altogether you'd pay four times the cost of this one."

Those two valuable antiques are still collecting dust where John saw them, but what were their prices? (*U. Clid*; Cleveland, Ohio)

* * * * *

ANSWERS FOR THE OCTOBER 1962 ISSUE OF RECREATIONAL MATHEMATICS MAGAZINE.

The list of puzzle solvers will be found on Cover III.

ALPHAMETICS (Page 8 - October 1962 RMM)

(1) NOEL BELLS = 9387 18774
(2) THREE SEVEN NINE TWELVE = 13244 84546 6964 104754 *
(3) This turned out to be solvable both as an *addition* and as a *subtraction* alphametic.
(a) SQUARE + DANCE = DANCER DANCER = 915736
(b) SQUARE - DANCE = DANCER DANCER = 574280
(4) GREEN GREY FAWN YELLOW = 89007 8901 4357 102265
(5) This was solvable *only* as a subtraction alphametic (many readers *assumed* it was an addition alphametic and remarked that it was not solvable).
CANADA - UNITED = STATES 920272 - 408137 = 512135
(6) HAPPY . . . DAYS AHEAD = 29661 . . . 3910 92893

* It was pointed out by *Harry L. Nelson* of Livermore, California that $3+7+9=12$ in base-17 also - and he submitted all 340 distinct solutions in base-17, e.g.:

$$1\,2\,R\,3\,3 + (15)\,3\,4\,3\,(14) + (14)\,I\,(14)\,3 = 1\,0\,3\,L\,4\,3$$

with 21 choices for R, I, L with $R<I$. This, alone, gives 42 solutions while similar results for other values gave the total of 340 solutions!

PUZZLES AND PROBLEMS (Pages 31-33 - October 1962 RMM)

1. *A Solid Dissection Problem*: Three straight cuts transform a 24 x 9 x 8 block into four pieces to form a cube (a variety of 4-piece dissections were submitted):

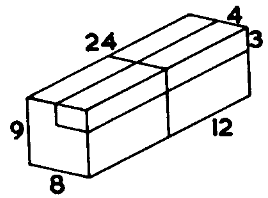


FIG. I

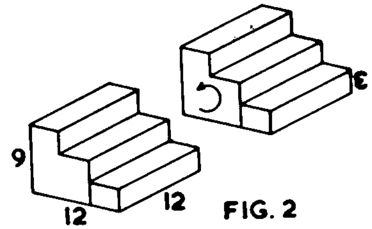


FIG. 2

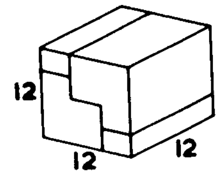


FIG. 3

2. *The Yacht Races*: Lord Ambling owned *Gazelle*; Mrs. Beetle owned *Irene*; Captain Clam owned *Firefly*; Admiral Dreery owned *Hydra*.

3. *Singletree Farm*: The sides of the park are 119, 145, 145, 167 yards.

4. *The Postmaster's Problem*: Three stamps can be selected in 40 ways, four stamps in 78 ways, five stamps in 147 ways.

5. *Here You Meet Resistance*: This was an open problem. Some of the solutions submitted will be published in a future issue of RMM.

6. *Area Duplication Puzzle*: The conditions of the puzzle were such that the son's solution could only have been squares with sides 1, 6, 11, 11, 11 while the father's solution could have been one of the following three: 1, 5, 5, 5, 18; 5, 5, 5, 6, 17; 5, 5, 5, 10, 15.

Although not solutions to the puzzle, many readers pointed out that if three or more squares are to be equal, then the sums of the following sets equal 20^2 (400): 1, 1, 1, 6, 19; 2, 2, 2, 8, 18; 3, 3, 3, 7, 18; 6, 6, 6, 6, 16; 8, 8, 8, 8, 12; 6, 9, 9, 9, 11; 6, 8, 10, 10, 10. Still confining ourselves to the sums of five squares, the Editor would like to point out that there are many other solutions with two squares the same (e.g., 1, 1, 3, 10, 17; 2, 3, 7, 7, 17) or with no duplication (e.g., 1, 2, 5, 9, 17; 2, 5, 9, 11, 13),

7. *A Problem in Logic*: The house numbers are given below, the names of the residing couples are given, and the tradesmen whose names are the same as the residing couples are given:

1	2	3	4	5
Brown	Smith	Green	Cook	Jones
Grocer	Butcher	Coalman	Milkman	Baker

8. *Weatherman's Report*: Yesterday was fair - no rain.

* * * * *

On Setting Alphametics (Article by Alan Sutcliffe, RMM No. 10, August 1962, pages 8-10):

DUNKED A FILLING = 925189 4 3700756
MEN OF SOUND SENSE = MENSA
(a) 518 94 29387 21821 = 51820
(b) 564 91 29347 26426 = 56428

In (a) 4-7 and 1-8 are interchangeable; in (b) 1-7 and 4-6 are interchangeable.

ALPHAMETICS

Successful meetings are anticipated since we will make the CLUB as large as possible.

(*Donval R. Simpson*; College, Alaska)

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 MATH
 CLUB
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MEETS
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* * * * *

Readers of RMM are likely to be *MAD* readers and ALFRED E. NEUMAN should be familiar to many. This *mad* little alphametic is in scale-9 notation. (*C. R. J. Singleton*; Sheffield, England)

ALFRED ÷ E = NEUMAN

* * * * *

There can be no argument about the total here.

So what do you make of THREE?

(*Richard L. Breisch*; Royersford, Pennsylvania)

```
 ZERO
  ONE
  TWO
-----
THREE
```

* * * * *

```
  RMM
  HAS
 ----
 xxxx
xxxx
RMM
-----
MOVED
```

One of the reasons for the late issues these last few months.

(*Paul M. Nemecek*; Riverside, Illinois)

* * * * *

Although not quite an alphametic - unless it be taken as a result of alcoholic ebullience mixed with a touch of the sun - this cryptarithmic monster certainly has its points.

For this interesting puzzle we are indebted to *Steven R. Conrad* of Clayton, Missouri.

```
       SUN
      LOSE
     UNTIE
    BOTTLE
   ELISION
  NINETEEN
 NONENTITY
 EBULLIENT
 INSOLUBLE
----------
NEBULOSITY
```

* * * * *

MATH TEACHERS! – Don't forget the NCTM Summer Meeting in Eugene, Oregon, August 22-24, 1963. For Registration - Write to Scott D. McFadden, 2489 Emerald, Eugene, Oregon.

RECREATIONAL MATHEMATICS magazine

INDEX TO ISSUES 6 THROUGH 12
FEBRUARY 1962 THROUGH DECEMBER 1962

Issue and page references are listed as 8:17, meaning Issue No. 8 (April 1962), page 17. Titles of the individual Puzzles, Problems, and Alphametics are listed under these categories in the subject index. Trivial references to persons or topics are omitted.

Issue numbers correspond to the following dates: Issue No. 7 - February 1962; Issue No. 8 - April 1962; Issue No. 9 - June 1962; Issue No. 10 - August 1962; Issue No. 11 - October 1962; Issue No. 12 - December 1962.

Author Index: Titles of the authors' works are listed in order of appearance, an (A) denoting an article. Only the initial appearance of puzzles or problems is given - answer references can be found in the subject index under **Puzzles** or **Alphametics.**

Title Index: The titles of all articles and significant notes are listed alphabetically.

J	J	J	J	J	J	J	J
K	K	K	K	K	K	K	K
L	L	L	L	L	L	L	L
M	M	M	M	M	M	M	M
N	N	N	N	N	N	N	N
O	O	O	O	O	O	O	O
P	P	P	P	P	P	P	P
Q	Q	Q	Q	Q	Q	Q	Q
R	R	R	R	R	R	R	R
S	S	S	S	S	S	S	S
T	T	T	T	T	T	T	T
U	U	U	U	U	U	U	U
V	V	V	V	V	V	V	V
W	W	W	W	W	W	W	W
X	X	X	X	X	X	X	X
Y	Y	Y	Y	Y	Y	Y	Y
Z	Z	Z	Z	Z	Z	Z	Z
A	A	A	A	A	A	A	A
B	B	B	B	B	B	B	B
C	C	C	C	C	C	C	C
D	D	D	D	D	D	D	D
E	E	E	E	E	E	E	E
F	F	F	F	F	F	F	F
G	G	G	G	G	G	G	G
H	H	H	H	H	H	H	H
I	I	I	I	I	I	I	I
J	J	J	J	J	J	J	J
K	K	K	K	K	K	K	K
L	L	L	L	L	L	L	L
M	M	M	M	M	M	M	M
N	N	N	N	N	N	N	N
O	O	O	O	O	O	O	O
P	P	P	P	P	P	P	P
Q	Q	Q	Q	Q	Q	Q	Q
R	R	R	R	R	R	R	R
S	S	S	S	S	S	S	S
T	T	T	T	T	T	T	T
U	U	U	U	U	U	U	U
V	V	V	V	V	V	V	V
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B	B	B	B	B	B	B	B
C	C	C	C	C	C	C	C
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J	J	J	J	J	J	J	J
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0	0	0	0	0
A	K	U	E	O
A	K	U	E	O
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A	K	U	E	O
A	K	U	E	O
B	L	V	F	P
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B	L	V	F	P
B	L	V	F	P
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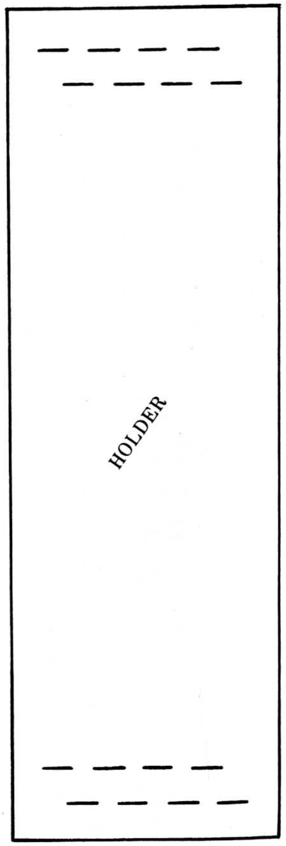
C	M	W	G	Q
C	M	W	G	Q
D	N	X	H	R
D	N	X	H	R
D	N	X	H	R
D	N	X	H	R
D	N	X	H	R
E	O	Y	I	S
E	O	Y	I	S
E	O	Y	I	S
E	O	Y	I	S
E	O	Y	I	S
F	P	Z	J	T
F	P	Z	J	T
F	P	Z	J	T
F	P	Z	J	T
F	P	Z	J	T
G	Q	A	K	U
G	Q	A	K	U
G	Q	A	K	U
G	Q	A	K	U
G	Q	A	K	U
H	R	B	L	V
H	R	B	L	V
H	R	B	L	V
H	R	B	L	V
H	R	B	L	V
I	S	C	M	W
I	S	C	M	W
I	S	C	M	W
I	S	C	M	W
I	S	C	M	W
J	T	D	N	X
J	T	D	N	X
J	T	D	N	X
J	T	D	N	X
J	T	D	N	X
0	0	0	0	0

1	2	3	4	5	6	7	8
T	T	T	T	T	T	T	T
U	V	W	X	Y	Z	A	B
V	X	Z	B	D	F	H	J
W	Z	C	F	I	L	O	R
X	B	F	J	N	R	V	Z
Y	D	I	N	S	X	C	H
Z	F	L	R	X	D	J	P
A	H	O	V	C	J	Q	X
B	J	R	Z	H	P	X	F
C	L	U	D	M	V	E	N
D	N	X	H	R	B	L	I
E	P	A	L	W	H	S	Q
F	R	D	P	B	N	Z	Y
G	T	G	T	G	T	G	G
H	V	J	X	L	Z	N	O
I	X	M	B	Q	F	U	W
J	Z	P	F	V	L	B	E
K	B	S	J	A	R	I	M
L	D	V	N	F	X	P	U
M	F	Y	R	K	D	W	C
N	U	B	I	P	W	D	X
O	W	E	M	U	C	K	F
P	Y	H	Q	Z	I	R	N
Q	A	K	U	E	O	Y	V
R	C	N	Y	J	U	F	D
S	E	Q	C	O	A	M	L
T	G	T	G	T	G	T	T
U	I	W	K	Y	M	A	B
V	K	Z	O	D	S	H	J
W	M	C	S	I	Y	O	R
X	O	F	W	N	E	V	M
Y	Q	I	A	S	K	C	U
Z	S	L	E	X	Q	J	C
A	U	O	I	C	W	Q	K
B	W	R	M	H	C	X	S
C	Y	U	Q	M	I	E	A
D	A	X	U	R	O	L	I
E	C	A	Y	W	U	S	Q
F	E	D	C	B	A	Z	Y
G	G	G	G	G	G	G	G
1	2	3	4	5	6	7	8

R 1	R 2	R 3	R 4
G	G	G	G
F	E	D	C
E	C	A	Y
D	A	X	U
C	Y	U	Q
B	W	R	M
A	U	O	I
Z	S	L	E
Y	Q	I	A
X	O	F	W
W	M	C	S
V	K	Z	O
U	I	W	K
T	G	T	G
S	E	Q	C
R	C	N	Y
Q	A	K	U
P	Y	H	Q
O	W	E	M
N	U	B	I
M	F	Y	R
L	D	V	N
K	B	S	J
J	Z	P	F
I	X	M	B
H	V	J	X
G	T	G	T
F	R	D	P
E	P	A	L
D	N	X	H
C	L	U	D
B	J	R	Z
A	H	O	V
Z	F	L	R
Y	D	I	N
X	B	F	J
W	Z	C	F
V	X	Z	B
U	V	W	X
T	T	T	T
1 R	2 R	3 R	4 R

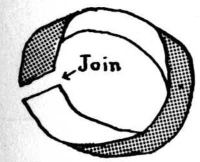


Let your English Department know about word shifts. Extra copies of this center spread may be purchased for only 25¢ each (stamps, coin, check, or money order) - but no billing or invoicing on orders of less than 25 sets. Send all orders and payment to RMM, Box 35, Kent, Ohio.

Sir Moebius Rides Again!
(See article by Dorman Luke in this issue)

These strips are printed on both sides so that the continuous single path on Strip 1 and the three paths on Strip 2 become continuous and joined when:

(a) Each strip is cut out and
(b) The ends of each strip are joined after a half-twist is made. e.g.:



Word Shifts
(See article by J. A. Lindon in this issue)

Cut out each of the 25 strips (the sets are in groups - see text of article for details) and the holder. Make slits in the holder as marked.

Up to 8 strips can be inserted in the holder as indicated below:

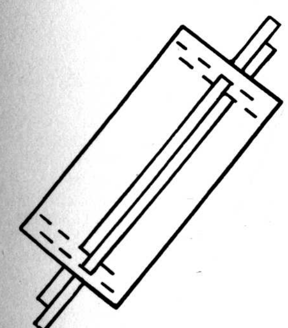


Figure 1

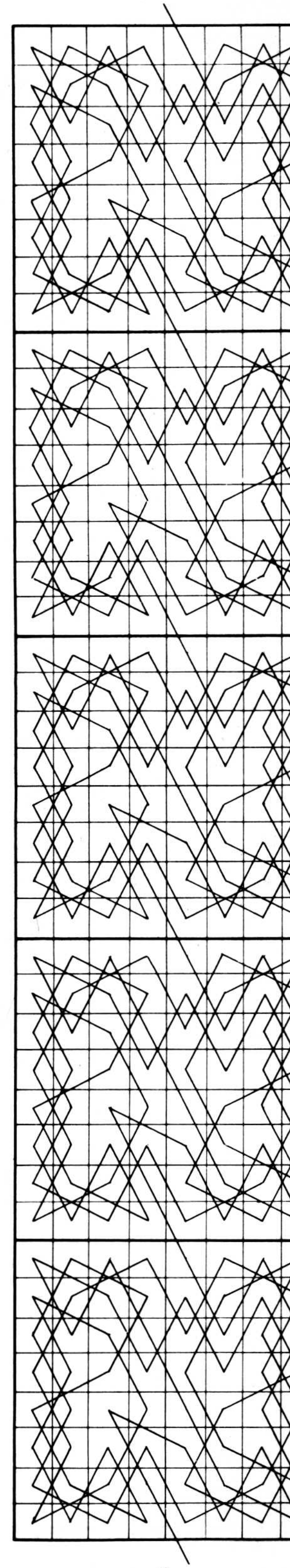
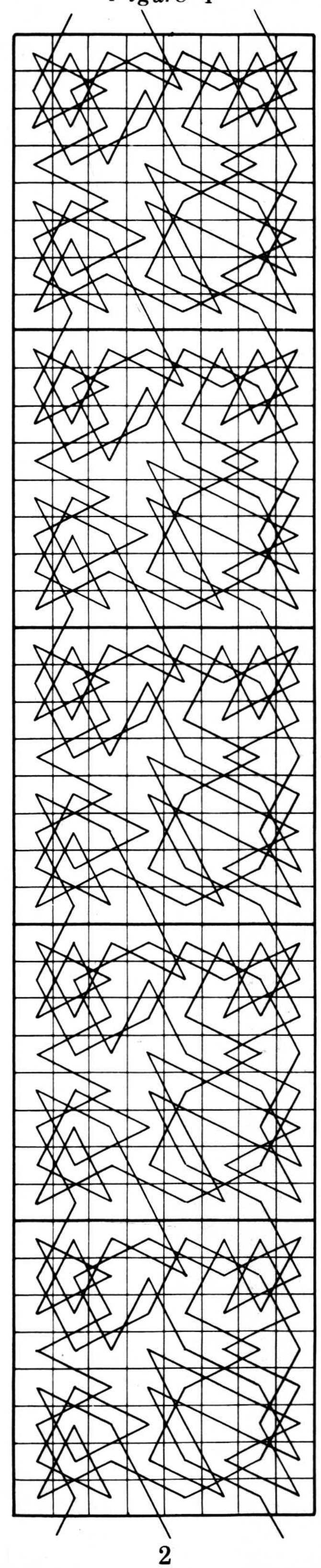
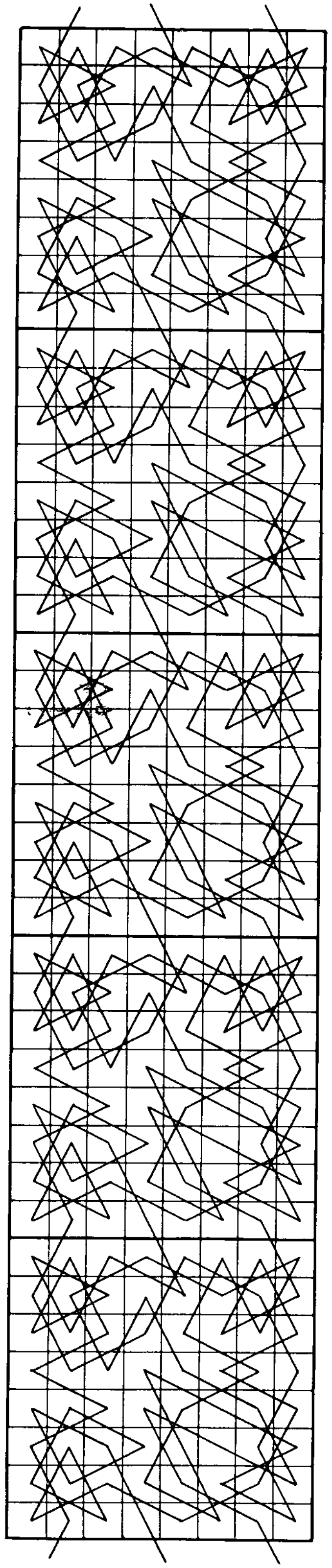
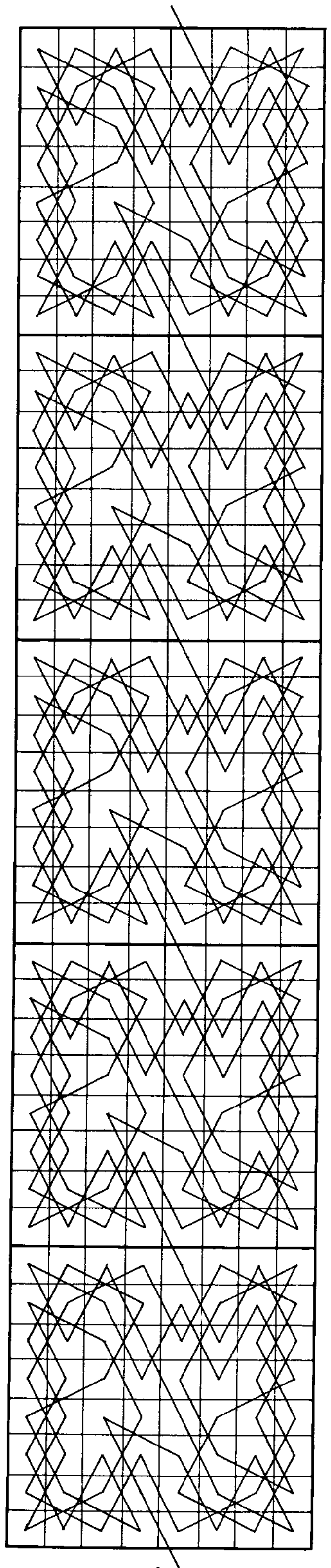


Figure 4





2



1

Subject Index: Major Departments are listed here.

ERRATA FOR ISSUES 7 THROUGH 12

Issue 7 - February 1962

Page 31: A '5' should be written in at the lower right corner of the bottom figure.

Pages 37 & 38: Equations 39 & 74 - "$\sqrt{9!}$" should read "$\sqrt{9}\,!$"

Page 39: 16th line from bottom - "Palindromic primes" should read "palindromic pairs".

Page 45: Top equation should read: $\frac{a}{b} + \frac{a^3}{10b^3} + \frac{a^2}{5b^2}$

Page 47: 6th and 4th lines from bottom - interchange 73 and 79.

Page 49: Paragraph (2) - 12749 = (11)(19)(61) and is, therefore, *not* a prime.

Page 56: 5th line from bottom - "123" should read "128".

Issue 8 - April 1962

See Errata: 9:52, 10:42.

Page 38: Problem 13 - "3' ½"" should read "3½' ".

Page 47: Puzzle 3, 7th line from bottom - end of line should read "old as Anne as Anne is years".

Issue 9 - June 1962

See Errata: 10:42.

Issue 10 - August 1962

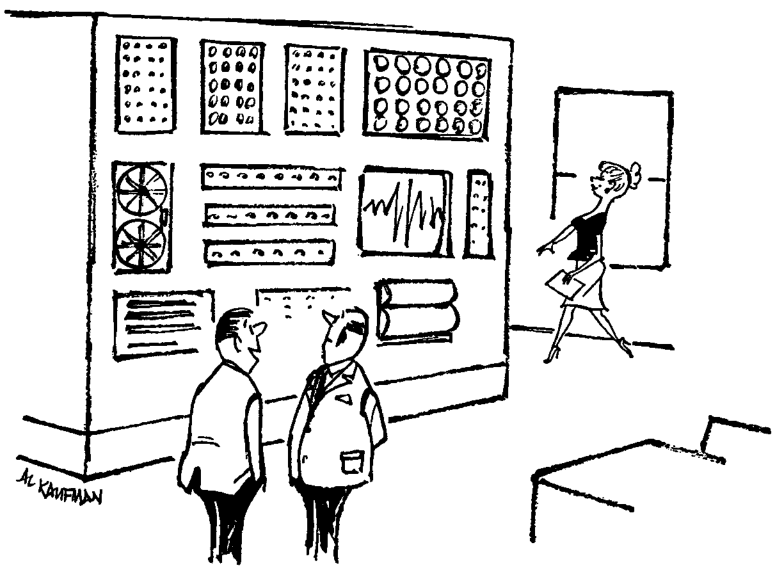
Page 5: Last paragraph, first line - "Herodian" should read "Heronian".

Page 13: 5th line: "circle moves" should read "line moves".

Page 35: Problem 5 - The question should have read:
Of how many children was the second grandfather the father?
Page 46: 7th and 16th lines - for "Karhi" read "Karkhi".

Issue 11 - October 1962
Page 11: Line 3 - Tertary should read Tertiary.
2nd paragraph, line 7 - Kabby should read kabbby.
2nd paragraph, line 10 - "I add" should be in italics (*I add*)
2nd paragraph, line 11 - *uzhifa* should read *vzhifa*.
Second sample sentence (bottom of page) - *de sgifdi* should read *be sgifdi*
Page 12: 1st paragraph, line 9 - for "opphagite" read "oophagite".
Page 12: 3rd line from bottom - *hhca-dedd* should read *hhac-dedd*.
Page 13: In the *Debibi* paragraph, line 4 - *doxi* should read *dolxi*.
Page 29: The last 50 digits of F_{521} belong to F_{522} and vice versa.

ADDENDA
Issue 10 - August 1962
Page 29: The two free years of RMM go to *Ben R. Ezzell, III* for choosing the name *BAROQUE* for Mr. Abbott's new game.



"IT REPLACED TWO HUNDRED AND FIF....ER... TWO HUNDRED AND FORTY-NINE PEOPLE."

Editor's Comment: *Ever since RMM dropped its* Word Games *department a year ago, many readers have asked for at least* some *word recreations. As many readers have indicated that a recreational* mathematics *publication is no place for word games. There is a virtual tie - pro and con. So, the Editor has excercised his own judgement to bring Mr. Lindon's entertaining work to RMM readers. The article should supply many hours of recreation for everyone. Why not bring it to the attention of the English Department in your school? Extra sets of strips (see the Center Spread in this issue) may be obtained for 25¢ (stamps, check, coins, money order - but no invoicing on orders of less than 10 sets) by writing to* RMM, Box 35, Kent, Ohio. *(J.S.M.)*

WORD SHIFTS

by J. A. Lindon
Weybridge, England

"Bulls? Well, shift 'em along a bit, will you?
I'm looking for my little boy."
–What is the child's Christian name?

It was Leigh Mercer who drew my attention to this particular gimmick. CHEER (he said), shifted along the alphabet, becomes JOLLY. An intriguing discovery.

There is nothing new in the process. The trick of forming a cipher by shifting each letter *n* places along the alphabet is known to every schoolboy. The novelty, for us, lies in a further shift, that of focus of interest. Decoding gibberish we leave to others. We want words that transform into words.

We have two methods of attack. We can either choose our word and try all possible shifts, or we can choose our shift and try all posible words. (Well, some of them.) For the latter investigation we may use a set of cards, one for each shift, bearing on the same side both the positive shift, e.g.

$$+4\text{:}\ \left[\begin{array}{cccc} A & B & C & \cdots & Z \\ E & F & G & & D \end{array}\right]$$

and the negative shift, e.g.

$$-4\text{:}\ \left[\begin{array}{cccc} A & B & C & \cdots & Z \\ W & X & Y & & V \end{array}\right]$$

This gives the same information twice over, but saves time and muddles in the end. An interesting alternative is to stick new letters on the keys of your typewriter, and so type the transforms directly. One soon learns the new positions, and whole word-lists can be shifted automatically by this means.

To save the reader much work, RMM has drawn up a full set of the strips to be described here. A holder is also included. The center spread of this issue contains the makings for this article. All you need to do is cut and assemble.

Suppose, for example, that the word to be tested is PET. Putting three strips side by side and horizontally aligning the letters P E T, we see that a shift of –4 puts PET into LAP (very appropriately), and that a shift of –15 (or +11) turns him into APE. A fourth strip enables us to add an S, turning PET into PETS and showing that APE becomes APED. LAP becomes LAPO, not a word, but we may notice that it is the reverse of one, as is PETS, so we have M_{-4}STEP/OPAL, using an obvious notation. A further strip, on the left hand side, shows that by prefixing an N to APE we get M_4NAPE/RETI (familiar view of a chess-player), and M_{10}RETI/BODS (perhaps the odd people watching). Well, anyway, you get the idea.

How many shifts are there? It soon becomes clear that a shift of n letters (either way) is equivalent to an additional shift of 26 letters (again, either way). Numerically, we need not take n beyond 13. $M_{13} = M_{-13}$, but only for $n = 13$. We have, for instance:

M_{13}IRAQ/VEND and M_{-13}IRAQ/VEND

M_6BOG/HUM and M_{-6}BOG/VIA

There are only thirteen essentially different shifts, and positive and negative ones can, for many purposes, be lumped together, since all word-transforms that belong to M_n belong, in reverse order, to M_{-n}, and no others do.

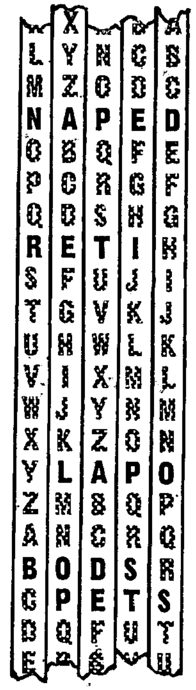


Figure 1

What is the criterion of a good shift? Obviously, that it should transform a good many words. If a common sequence of letters like ION transforms into KQP, that is a bad sign. It is equally bad if KQP transforms into ION. If at first you find this surprising, you will soon see why it is so. A good shift transforms common letters into common letters, uncommon ones into uncommon ones. M_6, which transforms the four vowels I, O, U, Y into O, U, A, E, might be expected to be better than M_7, which transforms them into V, P, B, F; and in fact my list of 4-letter transforms includes some two dozen for the first shift and only about half a dozen for the second. As a very tentative merit-order, I suggest:

Increasing merit ⟶

Shift: 5 2 8 9 3 7 1 11 10 12 4 13 6.

Now for some examples. A typical shift will give quite a few 3-letter words and a fair number of 4-letter words, but shifts with 5-letter words are rare and longer ones are almost unknown. Here is a generous selection (including proper nouns) of the commoner 4- and 5-letter word-pairs that transform in the various shifts:

M_1 ADDS/BEET ANNA/BOOB ANTS/BOUT OHMS/PINT STAR/TUBS ADMIX/BENJY SNEER/TOFFS

M_2 AMIC/COKE GRAF/ITCH BY-LAW/DANCY SLICE/UN-KEG

M_3 ARAB/DUDE COLD/FROG CROP/FURS DOLT/GROW ELIA/HOLD FOLK/IRON PELT/SHOW PIXY/SLAB ROAR/URDU PERK/SHUN COBRA/FREUD

M_4 DANK/HERO (D,L,P,S)AWN/(H,P,T,W)EAR FANG/JERK HEWN/LIAR LANE/PERI LEAN/PIER OPEN/STIR WHAT/ALEX BANJO/FERNS MODEL/Q-SHIP PECAN/TIGER RAVEN/VEZIR

M_5 AVON/FATS ODIN/TINS FIZZY/KNEED

M_6 ACHE/GINK BOMB/HUSH BUFF/HALL CLEM/IRKS DIES/JOKY DO DO/JU-JU FOAM/LUGS GULF/MARL HYMN/NEST JOEY/PUKE JOHN/PUNT LION/ROUT LOAM/RUGS LOCH/RUIN WOLF/CURL BULLS/HARRY CHAIN/INGOT FILLS/LORRY JIMMY/POSSE

M_7 BELT/ILSA IBEX/PILE LIN(G,K)/SPU(N,R) WHIR/DOPY ZEBU/GLIB LATEX/SHALE TIMER/APTLY WHEEL/DOLLS

M_8 HAWK/PIES SULK/ACTS TALK/BITS TASK/BIAS VANS/DIVA DADDA/LILLI HALLS/PITTA

M_9 BYRE/KHAN CRIB/LARK DRIP/MARY GIRD/PRAM (S,T)IRE/(B,C)RAN SLIP/BURY TREK/CANT JERKY/SNATH SLEEP/BUNNY

M_{10} BEEF/LOOP (D,R,S,W)EED/(N,B,C,G)OON TOUT/DYED DUMBO/NEWLY UREDO/EBONY

M_{11} EDDA/POOL (E,I)TCH/(P,T)ENS HATS/SLED HIDE/STOP PATH/ALES SPAT/DALE STUN/DEFY THEN/ESPY DRIPS/OCTAD

M_{12} BOAS/NAME CUSS/OGEE DOFF/PARR GIFT/SURF HIPS/TUBE JO(B,G)S/VA(N,S)E ROBS/DANE (P,T)OUCH/(B,F)AGOT

M_{13} BALK/ONYX CRAG/PENT ENVY/RAIL GNAT/TANG PHON/CUBA REEF/ERRS UREA/HERN CLERK/PYREX NERVY/ARIEL SNORE/FABER TERRA/GREEN WHARF/JUNES

Long words that transform are unknown, the best to date having only seven letters:

M_{13} ABJURER/NOWHERE and M_3 PRIMERO/SULPHUR

The former - the only shift, apart from Mercer's CHEER/JOLLY, that I have had from another person - was communicated to Martin Gardner by Dmitri Borgmann. Using hyphens, we might perhaps ask: Why not QUALIFY for a WAG-ROLE, or try CLEATED RAT-PITS? Well, anyway, EX-BEATS can certainly be PIMPLED! Good 6-letter transforms are almost equally hard to find. My best are:

M_1 STEEDS/TUFFET and M_6 FUSION/LAYOUT

but there are numerous runners-up:

M_1: SNOOK'D/TOPPLE M_4: GANJAH/KERNEL M_6: GULCHY/MARINE

M_{11}: HIDETH/STOPES HIDDEN/STOOPY

M_{13}: FARREL/SNEERY GREENY/TERRAL PURREL/CHEERY

We can extend our lists very considerably if we include the more doubtful cases: WHITE/DO, PAL; IN-LAW/PUSH'D; GIANT/TV NAG; HI, BUG!/NO HAM; SUSY B./CECIL; A HINT/BIJOU. All sorts of oddities turn up, and knowing where to draw the line is not easy. MODEL/Q-SHIP is obviously good. But F-TWANG/MAD HUN? (A crazy German with a guitar is needed to make sound sense of this!)

In fact, such lower-grade material should not be rejected out of hand, for a very good reason: we may need it when we come to construct whole sentences or phrases that shift. Here we follow the example of the palindromist, who certainly never demands that every word he uses should itself be reversible. With such a restriction (word-for-word transforms) we can find only a very few phrases that shift, e.g.

BOMB? I.E. BY GUS?/HUSH! O.K., HE MAY

I MUNCH BUN/O SATIN HAT!

BY BOX, MAX - WHEEL IT/IF I'VE THE DOLLS, PA

I HYMN MOLLY'S MUFFS/O NEST, SURREY SALLY!

Writing more 'bittily', however, we can cook up extended passages like

WE, SEMI-MUTE, MEET CUD-HERD EVEN SECURER/
GO, COWS! WED, O WOODMAN! ROB NO FOX. (COME, BOB!)

Sheer nonsense, of course, but then we are only beginners, with next to no material to draw on. There is an enormous field for investigation and development here.

It is often good fun to give transforms verbally, so as to link the words by meaning. Leigh Mercer, for example, passed on Borgmann's 7-letter triumph to me in some such way as this: "Never back a horse called ABJURER. It can shift, but it'll finish NOWHERE." In like manner we might say: Never trust a cowboy named JIMMY. He'll be away with the POSSE after him. MELON can be CUBED, of course. A STAR goes into TUBS, a FROG can be COLD or a ROAD become DAMP, NAVY blue comes from ANIL, a KEPI may be lost in a GALE, examples are endless. 'Bitty' results may often be conveyed to advantage in this manner, e.g.

Their B. A. OPERA was once FESTIVE.
INFANTS (at two in the morning) produce a JOG-BOUT.
Move RADIALLY, dear, and show your "LUX"-CUFFS.
The wild FANDANGO became all JER-HERKS!
The chemist felt DENUDED - NO XENON.
WASH BOB and put on a proper shift, I MET NAN!

What of palindromes, anagrams, reversals? A palindrome always shifts into a palindrome, and plenty of examples exist among common 3-letter words: BIB/NUN, TAT, etc., or EYE/GAG, TNT. Slightly longer examples are ANNA/BOOB; ABBA/DEED, NOON; ALULA/TENET; ANONA/REFER.

Anagrams (word sets containing the same letters - which include reversals as a special case) must be considered in more detail. Suppose we apply the shift M_n to a word of x letters. Then if the first letter y shifts to y' and the new word is to be an anagram of the first, y' must be in the original word and will itself shift to y'', and so on, the final shift completing the letters we started with. We must therefore have

$$nx = 26k$$

where k is any integer. This means in effect that (ignoring trivial solutions):

(a) Any shift will transform a scrambled whole-alphabet into another scrambled whole-alphabet;

(b) Any *even* shift will transform any scramble of either the even-numbered or the odd-numbered half-alphabet into another such scramble; and

(c) The shift M_{13}, which swaps letters in pairs (e.g. turns A into N and N into A), will give anagrams of any group of letters formed solely from such pairs, however scrambled.

Taking (b), we find that the two half-alphabets are respectively

B, D, F, H, J, L, N, P, R, T, V, X, Z and A, C, E, G, I, K, M, O, Q, S, U, W, Y.

The first is obviously of no use for forming words or phrases. The second, which contains all the vowels, can easily be shuffled into words, and might just conceivably be arranged in such a way as to give a transform-pair. The two Q's are awkward, but each can be followed by its U. Perhaps some ingenious shuffler can find a sensible example.

Case (c) is simpler. Our letter-pairs are AN, BO, CP, DQ, ER, FS, GT, HU, IV, JW, KX, LY, and MZ, and we may use any selection of them any number of times each. Thus, from the pairs AN and ER we can form the name of the composer ARNE, which shifts into NEAR. IVAN becomes VINA. ROBE becomes EBOR (naturally a robe becomes an archbishop!) and GNAT becomes TANG – both *reversal*-transforms, you will note. For longer examples we have NET-RAG/ARGENT and INVAR 'E'/VAINER.

One further application before going on to something new. If we restrict ourselves to words taken from either the starting-end or the finishing-end of a particular shift, and construct a word-square or fill in a crossword diagram, we shall find that the resulting criss-cross of of words will shift as a whole, giving a second word-square or a new fill-in for our diagram. This is illustrated below by the top left hand corner of a hypothetical crossword puzzle.

Notice that we do not have to consider the second set of words at all - they interlock automatically. And the same is true for word-squares. Third-order ones are not difficult to construct and two are shown in figures 3 and 4.

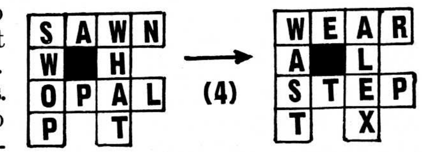


Figure 2

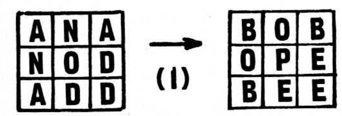


Figure 3

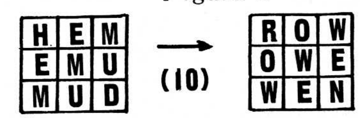


Figure 4

And for a fourth-order example, admittedly using a few uncommon words, we might have figure 5:

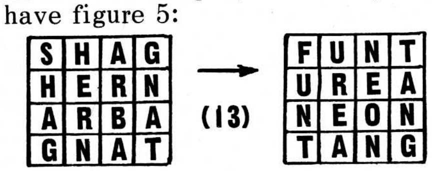
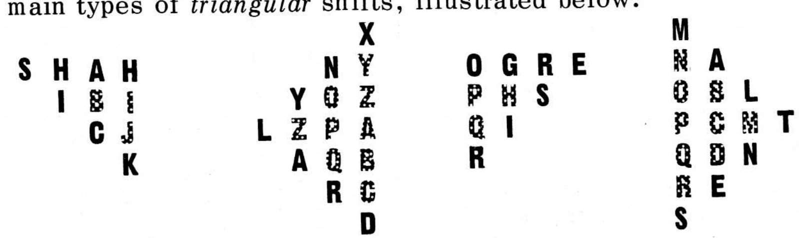


Figure 5

All shifts considered so far have been of linear (parallel) type, in which motion along the alphabet leads to a new word parallel to the first. Of equal interest are the *angular* shifts, in which the first word *rotates* into the second. Choosing the left (L) or right (R) terminal letter as pivot, and using a rotation of 45° or 90°, we get the four main types of *triangular* shifts, illustrated below:



L 45 SHAH/SICK — L 90 LYNX/LARD — R 45 OGRE/RISE — R 90 MALT/SENT

Special strips are advisable for triangular shifts. They are found in the center spread of this issue of RMM. They are numbered from 1 to 8 and are lettered by taking every letter, every second letter, every third letter, and so on according to the number of the strip. It will be noted that on the even-numbered strips the alphabet breaks into two halves.

For the common pivotal letters of our words we use the strips marked with a zero and having each letter repeated *five* times. If now we arrange our strips in order (as shown in the diagram), we can read off, *horizontally and adjacently*, the four possible triangular shifts in each case, 45° and 90° both before and after. The illustration will make clear, where the words being tested are LISLE (with last-letter pivot) and UNDUE (with initial pivot). The first is seen to have a fore-shift of 45° to PLUME and (if we allow an old spelling of town) a further fore-shift of 90° to TOWNE; the second has a 45° back-shift to UMBRA and a 90° fore-shift to UP-HAM. (Not a word? "We were obliged to up-ham and eject the club bore, or he'd have sat talking all night.")

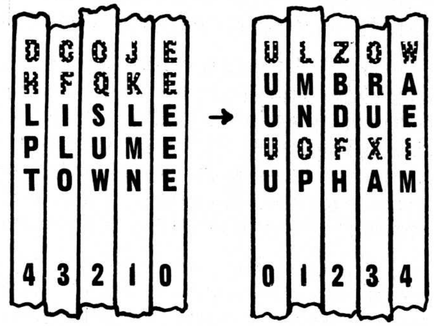


Figure 6

Now let us collect some examples.

L 45 AMY/ANA (a triple with APE), O'ER/OFT, SOW/SPY, SOY/SPA, THE/TIG, SOGS/SPIV, IMBOW/INDRA, SHEEP/SIGHT, SOPRA/SPRUE, WHEEP/WIGHT, ADREAM/AETHER, AONIAN/APPLES (of Discord?)

L 90 ANA/APE, APEX/ARID, DYNE/DARK, LYNX/LARD, LYON/LAST, SAWN/SCAT, HYENA/HAITI, PYLOS/PAPUA, SNELL/SPIRT

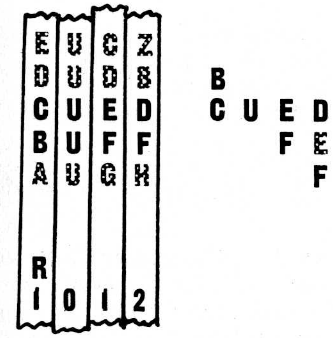
R 45 ACME/DENE, AGUE/DIVE, BUDS/EWES, FADS/ICES, OGRE/RISE, PANT/SCOT, BLATS/FOCUS, BOYLE/FRAME, PLUMS/TOWNS

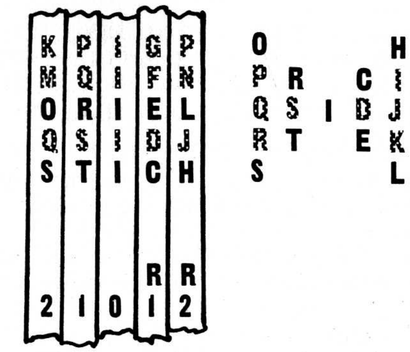
R 90 AWNS/GAPS, BELT/HINT, FELT/LINT, FENS/LIPS, HENS/NIPS, JACK/PEEK, JAYS/PEAS, MALT/SENT, NAYS/TEAS, NERO/TITO, PAPA/VERA, DOORS/LUSTS, FURYL/NAVAL, GREBE/OXIDE, JOINS/RUMPS, KEELS/SKINS, KEENS/SKIPS, KNELT/STINT, TINGS/BORIS, TONES/BURGS, TONKA/BURMA, TUNEE (piano!)/BARGE

For good measure we add a single $2\tan^{-1}2$(R) shift: MONAD/CAVED. Palindromes, anagrams, reversals? Nothing short. But don't forget the dodge of linking both words by meaning. For example:

Per ARROW ad ASTRA (King Harold's last words)
From PAPUA to PYLOS via Triangle Line
We'll shift from HAITI, here comes a HYENA
ALACK! I must turn around, I REEK!
PEG ME to thy shift, and I'll be THINE
Being UNFIT, he had to UP JOB after a single shift at 90°

Angular shifts, however, need not be *tri*angular: the pivot may be at an internal or external point, and in such cases we shall get CROSS-SHIFTS or TRAPEZIUM-SHIFTS respectively. This is all virgin territory, but I can give a few examples. First cross-shifts. The pivot - the common letter occupying the same position in both words - will usually be obvious.

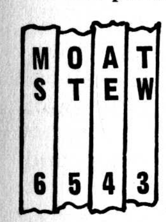




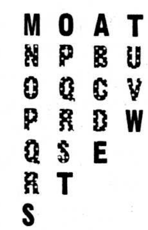
X 45 GONG/HOME, TALC/SAME (a triple with RANG), AGING/CHIME
X 90 TALC/RANG, ISLET/OWNER, RIVET/TITAN, TOP JET/BUTLER

To find cross-shifts we may use our triangular-shift strips and zero strip for the letters on one side up to and including the pivot, while for the other side we must make similar strips (four will do) having the letters *in reverse order*. Our cross-shifts will then, as with triangular shifts, appear horizontally and adjacently as shown in the figure.

For the trapezium-shifts we can, if we wish, use our ordinary parallel-shift strips, and look for words both transversely and at an angle. But it is more agreeable to use once again the triangular-shift strips. Take any few with consecutive numbers, and look for words that are either adjacent or next-but-one. Thus the shift appearing on our strips as

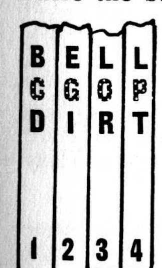


indicates

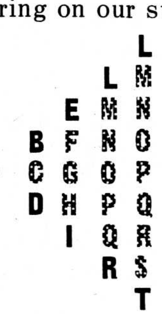


and can be denoted by 3 (R) 45: MOAT/STEW,

while the shift appearing on our strips as



indicates



and can be denoted by 2 (L) 90: BELL/DIRT.

Notice that the linear part of the shift (i.e. that along the shortest side of the trapezium) is given by the lowest number on the strips, doubled if the angle is 90°.

Trapezium-shifts are probably quite common. Here are just a few examples:

L 45 FOGY/ISLE, MENU/PISA, MIME/SPUN, OVAL/SAGS
L 90 BELL/DIRT, HERE/PODS, LEAN/TOMB, PENS/RITA, TIED/DUST
R 45 CAVE/LICK, COGS/GRIT, PUMP/WART, YOGA/CRIB, YOGA/LARK
R 90 CAMP/MIST, CUBA/SINK, DUST/TIED, LASH/BOER

L 45/90: HIED/MOLL/RUST is an example of a triple. I omit the linear-shift figures in the above; the reader can easily find them by counting from one letter to another.

We started with parallel shifts, so it seems only fitting that we should return to them briefly before we close. Shifts in other languages are of course possible, but it must be remembered that foreign alphabets may differ from ours, thereby necessitating special strips. A few examples follow - for expert linguists among RMM readers to howl with derision over!

We will use an alternate symbolism to denote the extent of shift, e.g. CHEER(7)JOLLY is equivalent to M_7: CHEER/JOLLY.

FRENCH: (Same alphabet as for English) The following are all for Shift 13: aniere/navrer, antre/nager, cerf/pres, cher/pure, creve/perir, crepure/percher, encre/raper, entre/rager, grave/tenir.

GERMAN: (Same alphabet as for English, replacing their varied letter-forms and compound consonants by ordinary Roman letters, as is often done.): du(6)ja, ein(6)Kot, nun(6)tat, Tag(7)Ahn, Tau(11)elf, Atem(7)halt, Buch(6)Hain, bunt(6)Hatz, dann(4)Herr, Grab(3)Jude.

The shift *Jude* into *Grab* was certainly known to Hitler.

ITALIAN: (21-letter alphabet, same order as ours, but omitting the J, K, W, X, Y): sei(4)zio, tre(4)avi, cippi(4)gotto, pena(4)tire, nappa(4)rette, pappa(4)tette, pinna(4)torre, villa(4)coppe, ville(4)coppi, pagana(4)temere, pannina(4)terrore, tre(8)eco, stile(8)desto.

SPANISH: (29-letter alphabet, like ours with W omitted and compound consonants *ch*, *ll*, *n* and *rr* coming after *c*, *l*, *n* and *r* respectively): lo(5)os, su(5)ya, baya(5)feche, cuja(5)gane, chazan(5)heder, se(7)al, uses(7)cala, choyas(7)juega, seas(7)alga, uno(9)dux, tu(10)de, ocho(12)ana, asco(12)lema, poyo(12)baja, perro(12)boda, peno(12)boya, cerro(12)moda, chorro(12)nada, toco(12)fama, edad(13)pollo.

I apologise in advance for the many errors likely to be in the above. But foreign-language shifts should not be rejected too lightly. The French ones I have found, using a single shift, already include a 7-er, equalling our English record; and the Italian short high-utility alphabet has enabled me to do as well.

Who is going to be the first to come up with an acceptable non-hyphened word-shift (in *any* language) of 8 or more letters?

BOOK REVIEWS

Edited by Dmitri E. Thoro
San Jose State College

Tomorrow's Math - Unsolved Problems for the Amateur. By C. Stanley Ogilvy. Oxford University Press, New York, 1962, 182 pp., $5.00

Tomorrow's Math is more than a mere collection of over 150 problems classified under the headings: applied problems; problems concerning games; geometrical problems; arithmetical problems; topological problems; probability and combinatorial problems; problems of infinite sets; variational problems; problems of analysis. The author's lucid style and scholarly editing should appeal to both the amateur and professional mathematician (subtitle notwithstanding).

In addition to such classics as Fermat's last theorem, the four-color problem, Goldbach's conjecture, the sphere packing problem and the traveling salesman problem, many topics such as polyominoes and lucky numbers are of more recent vintage.

There are references to several problems proposed in RMM. On page 90 we see the conjecture passed on by Alan Sutcliffe - that there is at least one prime between two consecutive squares (RMM 7:49) - is still unsolved.

It is interesting to note that several problems have been solved since the book went to press. For example, referring to page 65, Free Jamison has shown that there are obtuse triangles which can not be dissected into seven acute isosceles triangles (*Amer. Math. Monthly*, Vol. 69 (1962), page 550).

Two statements on page 76 tend to be misleading; otherwise the language is precise. Abundant illustrations, historical remarks, and over 125 succinct notes add to the reader's enjoyment.

This is unquestionably one of the most stimulating books that I have seen. C. S. Ogilvy (author of *Through the Mathescope*) has been a first-rate problem solver for many years. In *Tomorrow's Math* he succeeds in communicating the excitement found in problem solving.

D. E. T.

* * * * *

Fibonacci Numbers. By N. N. Vorob'ev (translated from the Russian by H. Moss). Blaisdell Publishing Co., New York, 1961, viii + 66 pp., 95¢ (paper).

In 1202 Leonardo of Pisa (better known by the nickname Fibonacci) came across the sequence 1, 1, 2, 3, 5, 8, 13, . . . in connection with a now famous problem concerning the number of descendants produced by a single pair of rabbits in one year. Ever since, these numbers have appeared in the most unexpected places (from pineapples to electrical networks) - particularly in recreational mathematics. (See RMM 2:42, 4:61-62, 7:42, 9:45, 10:5, 11:17, 11:19-30).

This little book is a fairly smooth translation of one of the volumes in the Russian series *Popular Lectures in Mathematics*. According to its author it is "designed to appeal basically to pupils of 16 or 17 years of age in a high school". While this may be too optimistic, I feel that any RMM reader who has a good command of high school algebra can read the majority of sections with pleasure and profit.

Typical of the results established are: Binet's formula for the n-th Fibonacci number; that the greatest common divisor of two Fibonacci numbers is again a Fibonacci number; that at least one number divisible by m can be found among the first m^2 Fibonacci numbers.

Especially well done are the discussions of divisibility (including the Euclidean algorithm for finding the greatest common divisor of two numbers) and continued fractions.

This book concludes with a chapter on Fibonacci numbers and geometry, followed by several unsolved problems.

D. E. T.

* * * * *

Classics in Logic. Edited by Dagobert D. Runes. Philosophical Library, Inc., New York, 1962, 818 pp., $10.00.

One can hardly expect thorough treatment of over 60 philosophies in 818 pages. However, Dr. Runes has picked the meat of most of these. Here is an excellent guide for those whose interests in the study of logic and philosophy are taking hold.

Of particular interest to us is the inclusion of some fifteen or more mathematicians and their contributions to logic. We have Jacob Bernouilli and Laplace on probability, Bernard Bolzano on the paradoxes of infinity, George Boole on syllogisms, Lewis Carroll on the Bilateral Diagram, Euclid and the First Elements, and others. The familiar and unfamiliar are here - Plato and Raimundus Lullus, Bacon and Ludwig Wittgenstein (who uses decimal-numbered propositions), Descartes and Ernest Mach (who did more than supply a name for the measurement of supersonic flight).

The brevity of some of the excerpts is regrettable, but understandable. To have tried to include even the basic outlines of the many ramifications of any one person's philosophy might have reduced this book to Plato and Aristotle - if that many. As it is, I'm sure that even advanced students of philosophy will find some new items here. A better book on logic or philosophy for sheer browsing would be hard to find.

J. S. M.

HOW TO CONSTRUCT HEXAFLEXAGONS

by Sidney H. Scott
Watford, England

Hexaflexagons are structures of folded paper or other material, having the overall shape of regular hexagons made up of six wads each of which is an equilateral triangle. A flexagon has the property that it may be folded into a shape like that of the wings of a dart, and then reopened at what was originally the centre, revealing an entirely new face at the expense of hiding one of the original ones, while the triangles of the original face are reoriented. The required folds are indicated in Figure 1; the continuous lines representing 'mountain' folds and the dotted lines 'valley' folds. Clearly we may attempt this move in four ways, and either two or all four of them will be possible.

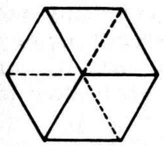


Figure 1

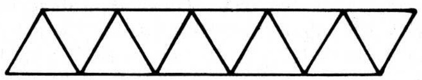


Figure 2

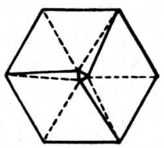
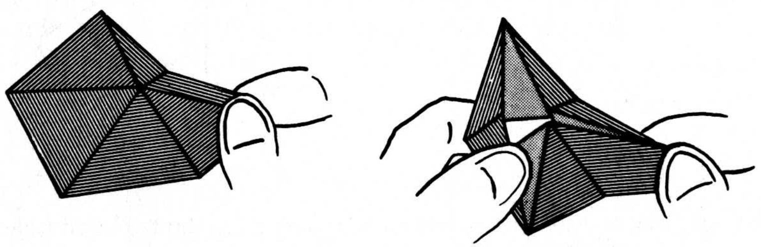


Figure 3



A flexagon may be opened - or flexed *- by taking hold of two triangles and pinching together. An edge of the opposite set of triangles can be lifted up. If, at this point, flexing does not occur, start by pinching together another pair of triangles. Once the process is started, the flexagon will open inside out and show a new 'face'.*

Flexagons originated because, in 1939, Arthur H. Stone, then doing post graduate study in mathematics at Princeton University, had to trim strips from his American note paper so that the sheets would fit the binder he had brought with him from Britain, and he experimented with the strips. The first model, with three faces, he made from a straight strip of ten equilateral triangles (Figure 2). This was folded as in Figure 3 and the flexagon completed by gumming the end triangles together so that the model really consisted of nine triangles - three times the number of faces which result upon 'flexing'. Now if the respective faces of this model are numbered 1, 2, 3, we see that the flexagon may be represented by a *structure diagram* (Figure 4) where each vertex corresponds to a face, and each line segment to a position of the model. The next one discovered had six faces. It is made by taking a straight row of nineteen triangles and folding pairs of adjacent triangles face-to-face to get a strip of ten triangles, nine of them of double thickness. The folds are made so that the strip becomes twisted rather like a flattened spiral. Finally, this

this strip is folded as in the case of order 3, keeping the twist in the same sense throughout the entire operation. It will be found that this model, with appropriate numbering of faces can be represented by the structure diagram shown in Figure 5. Here the correspondence is again: vertex-face and segment-position. Moreover, possible moves are such that we can change from one position to another if and only if the corresponding segments of the structure diagram belong to the same triangle. Figure 5 is obtained from Figure 4 by applying one triangle to each side of Figure 4. In general, the figure obtained by applying a triangle to a side of one given triangle, then applying a second triangle to an outside segment of the resulting figure, and repeating this operation as often as we wish, is a polygon subdivided into triangles by non-intersecting diagonals. This raises the question, answered in the affirmative by Stone and his friends; can we make a flexagon which corresponds in the above way to any such structure diagram? The purpose of this article is to explain how this is done.

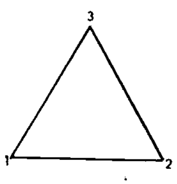


Figure 4

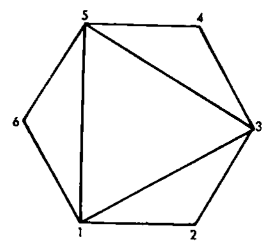


Figure 5

The originators of flexagons developed a method of construction such that the precise instructions for making the model could be derived from the structure diagram. The model thus made exhibited those properties, no more nor less, than implied by the diagram in the sense that has been stated. The instructions given below are a distortion of the original method. The distortion consists of using a different notation for the vertices and sides of the structure diagram, and the introduction of the concept of positive and negative directions when getting the shape of the cut-out strip. Stone's original method is superior if we want to consider the subject more widely than it is treated here, and develop rigorous proofs of the various rules. The method given below is a little quicker for a person actually making flexagons.

I shall deal with the general case of order n, corresponding to a structure diagram in the form of an n-gon subdivided into $n-2$ triangles by $n-3$ non-intersecting diagonals. For purpose of illustration I give the fully worked out plan for one case each of orders 7 and 8, there being a small difference in the routines for even or odd n. Note that, since to each possible structure diagram, there corresponds a flexagon (unique unless we regard right or left handedness as a significant distinction), there are several possibilities if $n \geqslant 6$. e.g. if $n=9$ there are 27 possibilities.

Start by drawing a structure diagram with n vertices. Number the vertices 1, 2, 3, . . . , n in any order you wish. Choose any interior point of each of the $2n-3$ segments (the total of the sides of the polygon and the interior non-intersecting diagonals) - about the middle of each, say. With these points as vertices draw another $n-2$

triangles, so that each triangle of the structure diagram itself contains just one of the new triangles. In Figures 6 and 7 the new triangles are shown in dotted lines. This system of triangles constitutes a closed path which crosses itself on each diagonal. It is, of course, possible to define the path so that it does not intersect itself on a particular diagonal, but it is necessary that you should reject this interpretation for each junction. Every possible structure diagram must contain at least two triangles each having two sides that are also sides of the polygon. Choose one of these triangles. Letter one of its non-diagonal sides 'A', the other 'B'. This must be done so that if you imagine that a point moves from A to B along the dotted path by the direct route the inner triangle is being described in the positive sense (counter-clockwise). Now suppose that the moving point continues along this closed path, crossing itself on each diagonal, until it arrives back at A having described each inner triangles just once. Within each of these triangles write a + or - sign according to the sense in which the moving point described it. Now letter the other sides of the polygon C, D, E, . . . (n letters altogether) in alphabetical order corresponding to the order in which they are reached by the moving point. The resulting diagram (Figures 6 or 7) will be referred to as the *augmented structure diagram*.

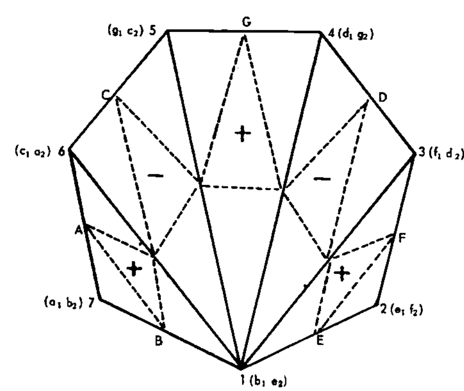


Figure 6

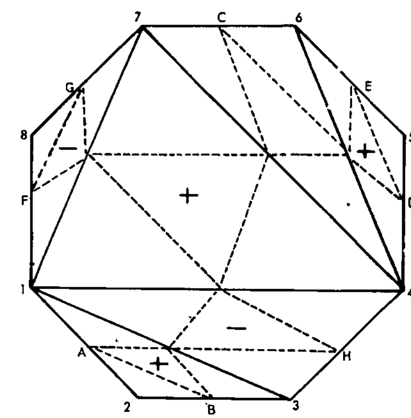


Figure 7

The next step consists of writing out a $4 \times n$ array which will contain all the information necessary to design the strip which is to be folded into a flexagon. For the first row, write down the n letters A, B, C, . . . Each letter of the augmented structure diagram occurs between two numbers on the perimeter of the polygon. Let the number on the positive side of any letter P be p_1, that on its negative side be p_2. By positive side I mean that reached by moving round the perimeter counter-clockwise. Figure 6 is numbered and lettered to show the method. The sequence a_1, b_2, c_1, d_2, e_1, f_2, g_1, . . . is used to determine the second row of the array by writing the number of the corresponding vertex. For the third row, write: a_2, b_1, c_2, d_1, e_2, f_1, g_2, . . . and also write the number of the corresponding vertex.

I have found the best practical method to be that of writing all those with suffix '1' first. The fourth row consists of + and - signs and is easy to write. To determine which sign to put below a particular letter simply look it up from the augmented structure diagram: you will find it inside the triangle of which your letter is a vertex.

The arrays for the selected examples (Figures 6 and 7) are, therefore

A	B	C	D	E	F	G
7	7	6	3	2	2	5
6	1	5	4	1	3	4
+	+	−	−	+	+	+

A	B	C	D	E	F	G	H
2	2	7	4	6	8	8	3
1	3	6	5	5	1	7	4
+	+	−	+	+	−	−	−

Because of a restriction on choice made previously, the fourth row of the array must begin with at least two + signs. If there are n + signs you can omit the next stage, for the strip will be straight.

You are now ready to start work on the actual model. Take a large sheet of fairly stiff paper and cover one side of it with a network of accurately drawn equilateral triangles. The area of paper required increases with n, not only because $3n+1$ triangles are required, but also because if the model is to move satisfactorily, the minimum possible size of triangle is also an increasing function of n. A convenient size for my examples is a triangle of 1½-inch side. It is often more convenient to get an idea of the shape of the strip by using rough paper first, and here it is quite easy to work without drawing an actual network. Start with one equilateral triangle and put an arrow on each of its sides, all pointing round the triangle in the same sense. Call these directions positive, and directions directly opposite to them, negative. The idea is to choose $3n+1$ triangles which make up a strip in such a way that moves along the strip from one triangle to its neighbour are made in a sequence of directions given by the fourth row of the array. Two adjacent triangles do not define a direction, but three do. In Figure 8, the order of triangles (i) (ii) (iii) 'point' in a positive direction as defined by the arrows on (i), while the order of triangles (i) (ii) (iv) 'point' in a negative direction.

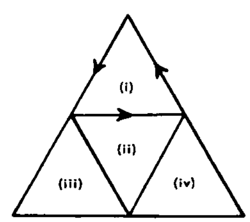


Figure 8

By taking the fourth row of the array three times over you get an ordered sequence of $3n$ signs. You have already selected the first triangle of the strip; it is the arrowed triangle. Associate with the ith sign of the extended sign sequence the direction that the $(i-1)$th, ith, $(i+1)$th triangles must follow in the strip. Thus, the first, second and third triangles point in a + direction based on the arrows in the first triangle. Then for $i=3, 4, 5, 6, \ldots, 3n$ obtain the $(i+1)$th triangle by ensuring that the $(i-1)$th, ith, and $(i+1)$th triangles point in the direction given by the ith sign. Once the first three are chosen this can be done in one way only. The advantage of this method is that if the extended sequence contains k consecutive like signs you can select k triangles together.

Cut out your strip and make creases along all common sides of adjacent triangles.

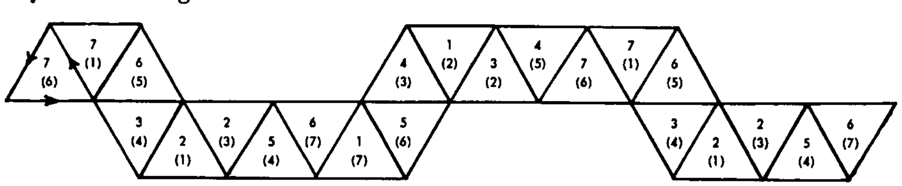


Figure 9

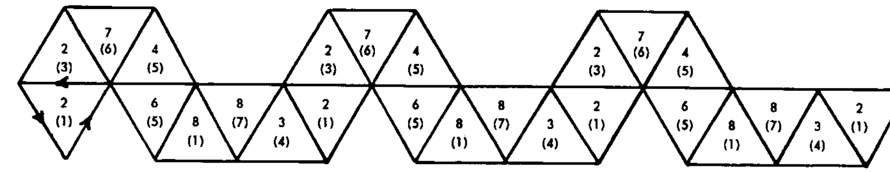


Figure 10

The shapes for the examples are shown in Figures 9 and 10. The second number (in parentheses) on each triangle is meant to be written on the back and you arrive at the numbering in the following way:

(I) n even: Write the numbers of the second row of the array, in order, one on each successive triangle on one side of the strip, starting with the first triangle. Repeat the array twice so that the first $3n$ triangles have been numbered. Write the number $a_1=b_2$ on the last triangle. In like fashion use the numbers of the third row for the other side of the triangles, being careful to start again at the first triangle.

(II) n odd: The ordered sequence of numbers to be put on one side of the strip is made up of: the second row of the array, then the third row, the second row again, and then a_2. The sequence for the other side is made up of: third row of the array, second row, third row again, and a_1. The strip should now bear each symbol six times except for the two which appear on the end triangles. When you have completed the flexagon, these end triangles will become one and the seventh appearance of each of the 'end' symbols will be eliminated.

All that remains for you to do now is to fold and seal. The model will flex more easily if you trim off a narrow ribbon right round the perimeter. As implied previously there must be at least two vertices of the structure-diagram polygon which are not met by a diagonal. One of them is numbered a_1 ($a_1=7$ in the Figure 6 and Figure 9 pair).

Suppose that one of the other such vertices bears a number c ($c=e_1=2$ in the Figure 6 and Figure 9 pair). You will find that the number c occurs on three pairs of adjacent triangles of the strip. Fold the triangles of each pair face to face so the c is hidden everywhere. If you were to paste these faces together (*but do not*) you would get a strip for a flexagon of order $n-1$. Moreover, the diagram for the smaller model is obtained by removing the triangle with vertex c from the diagram of order n. This is a hint towards Stone's inductive proof of his process.

Hence the rule for folding: Find a symbol which occurs on pairs of adjacent triangles, but does not occur on the first triangle. Eliminate that symbol (number) by folding together each pair of such triangles. Repeat this operation on the reduced strip thus obtained. Repeat on the further reduced strip, and so on until only a_1 and a_2 remain. Paste together these two end triangles which have come together into the correct position, so that they become one triangle. You then have a flexagon showing a_1 six times on one side and a_2 six times on the other side.

A possible difficulty may arise. When finding the shape of the strip, you may find that one triangle of the network gets selected twice, i.e. the strip crosses itself. The model can still be made, but not from a single sheet of paper. You have to paste on an overlapping strip, somewhat like a Riemann surface. The method will be obvious when you meet the problem. In particular this happens for 'street' flexagons (mentioned later) when $n \geqslant 7$.

Disregarding right or left handedness, the correspondence between structure diagrams and flexagons is one-to-one, but a particular strip may possibly fold into two or more different models. The earliest (smallest n) example is the seventh order strip of Figure 9. This strip can be renumbered to give a flexagon which does not correspond to Figure 9. I offer as an easy exercise for the reader to make the other model from the same strip and draw the appropriate diagram.

Apart from their mathematical significance, flexagons have a novelty appeal even for non-mathematicians. People not in the know about their structure find it a puzzle to find all the faces, if n is sufficiently large. Once the flexagon is finished, the original symbols may be erased and suitable patterns substituted. The snowflake motif should provide endless possibilities for those with talent for design.

One special type is that which has been given the apt name of 'street' flexagon. Its faces may be so numbered that the symbols 1, 2, 3, . . . , n may be made to appear in uninterrupted order. There is one such flexagon for each n and in the structure diagram the diagonals form an unbranched zig-zag path. Street flexagons, with letters instead of numbers, can spell out, say, a child's name. Also, I would like to suggest to the very artistic and industrious the idea of a Christmas card in the form of a twelfth order street flexagon giving the twelve days of Christmas in Order. Each of the unique flexagons of orders three, four and five is, of course, a street flexagon. Furthermore, they also belong to the same class as the order seven example (Figure 6) where the diagonals all arise from one vertex of the structure diagram. Such models can be worked so that one face seems to remain, while the other one changes.

Since straight strips of equilateral triangles are quickly made without any drawing beforehand, the case where there are n + signs in the array is worthy of special mention. It is easy to see, from the rule from which you get the signs, that if you take a structure diagram of order n, and convert it into one of order $n+1$ by adding on another triangle, either a + is replaced by – –, or a – is replaced by + +. It follows that + becomes + + + +, if you apply the side 15 of the figure 12345 (part of Figure 5) to any positive side of a structure diagram. Hence, if you do this to an nth order structure diagram of a 'straight' flexagon, you will have drawn the diagram for another straight model, this time of order $n+3$. A second method of conversion, but from order n to $2n$, is by building a triangle on every side of the original diagram (compare Figures 4 and 5). The n + signs become $2n$ – signs and this is again a condition for a straight strip (the practical method of making models has been formulated with some loss of generality). It is easy to see, then, that the order of a straight model is always a multiple of 3. The first flexagons ever made are members of a subclass of straight-strip ones. Here n is of the form $n = 3 \cdot 2^q$ (q is any non-negative integer). The strips can be folded quickly by an extension of the short method outlined in the second paragraph, i.e. by

successive windings each of which halve the length of the strip. The structure diagrams for this class are got by using the second (n to $2n$) method of conversion, as many times as necessary, starting from order three. However, any structure diagram of a straight flexagon can be developed by using only the first (n to $n+3$) method of conversion. In this way you can satisfy yourself that there is only one straight flexagon of each of the orders 3, 6, 9; while there are four of order 12, and fourteen of order 15.

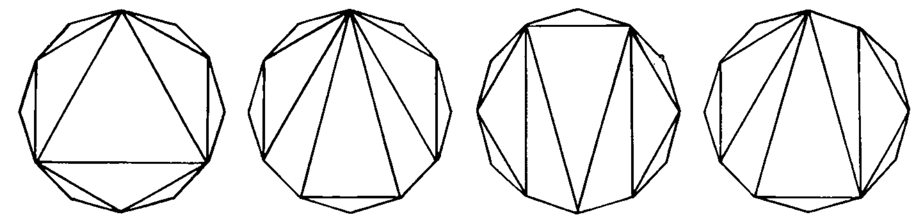


Figure 11

The structure diagrams of straight flexagons for $n = 12$ are shown in Figure 11, from which it may be seen that the diagonals make up $(n-3)/3$ discrete triangles which may be chosen in any way so that they touch vertices of the polygon, and each other, only with their own vertices.

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* * * * *

NUMBERS, NUMBERS, NUMBERS

A NUMERICAL FAREWELL TO THE YEAR 1962

by Harold S. Tribe
Surrey, England

Playing around with the digits of each year is a favorite pastime of recreational mathematicians. RMM has devoted some space to the project: RMM No. 7, February 1962, pages 37-40; RMM No. 8, April 1962, page 33; RMM No. 11, October 1962, page 18.

The most striking is $987 + 654 + 321 = 1962$ where the digits are in numerical sequence and the three 3-digit numbers are in *arithmetic progression*. Switching the digits in the first and last of these numbers preserves both the sum and the arithmetic progression:

$$981 + 654 + 327 = 927 + 654 + 381 = 921 + 654 + 387 = 1962$$

The questions naturally arise: (a) How often can this happen? (b) When will be the next occasion? *Answers*: (a) Not very; (b) The year 2016.

Certain facts emerge. The sum of the nine digits, 1 to 9, is equal to 45 which is divisible by 9. Any year which can be formed in this way is also divisible by 9. The middle of the three numbers in arithmetic progression must be one-third of the year, and is therefore divisible by 3.

It seems clear (though I am open to correction on this point) that the hundreds digits, the tens digits, and the units digits of the three numbers must form three separate arithmetic progressions. A tabulation of all possibilities follows: 1-2-3, 2-3-4, 1-3-5, 1-4-7, 2-4-6, 3-4-5, 1-5-9, 2-5-8, 3-5-7, 4-5-6, 3-6-9, 4-6-8, 5-6-7, 5-7-9, 6-7-8, and 7-8-9.

Therefore the middle number can contain only the digits 2, 3, 4, 5, 6, 7, or 8. Moreover, since the number is divisible by 3, only restricted combinations need to be considered, namely: 234, 237, 246, 258, 267, 345, 348, 357, 456, 468, 567, and 678. Some of these may be discarded. For example, if the middle number were 234, this would require the hundreds digits of the other two numbers to be 1 and 3, which would lead to a 3 being repeated. Similar considerations lead to the elimination of all but the following: 258, 267, 348, 357, and 456. Any permutation of these gives us the middle number, leading to 30 years only! In chronological order they are:

774	1071	1395	1692	1962	2286
801	1125	1449	1719	2016	2475
828	1152	1584	1746	2178	2502
855	1314	1611	1881	2205	2529
1044	1368	1638	1935	2259	2556

This list would seem to be exhaustive. If the condition of the arithmetic progression is omitted, the possibilities of using the nine digits, 1 to 9, to form a year-sum would be increased enormously, and I shudder at the prospect of working them out.* If 0 is included, I give up!

Editor's Note

The *Wiskunde Post*, 1962-1963, Number 1, published in Antwerp, Belgium, devotes a full page to the project for the year 1962. Among the interesting results is a tabulation of *part* of the complete listing for the year 1962 formed as the sum of three 3-digit numbers using the digits from 1 to 9. A total of 576 distinct arrangements can be derived.

NINE-DIGIT DETERMINANTS FAREWELL SALUTE TO 1962

by Charles W. Trigg
Los Angeles City College

Note the relative positions of the digits 1, 9, 6, 2 in the determinants.

$$\begin{vmatrix}1&9&6\\2&7&5\\8&3&4\end{vmatrix}=1 \qquad \begin{vmatrix}1&5&4\\9&6&2\\8&7&3\end{vmatrix}=9 \qquad \begin{vmatrix}1&4&2\\3&9&6\\5&7&8\end{vmatrix}=6 \qquad \begin{vmatrix}1&2&3\\9&6&8\\7&4&5\end{vmatrix}=2$$

$$\begin{vmatrix}1&5&2\\4&9&6\\7&3&8\end{vmatrix}\cdot\begin{vmatrix}1&2&3\\9&6&8\\4&5&7\end{vmatrix}\cdot\begin{vmatrix}1&2&3\\8&9&4\\5&7&6\end{vmatrix}\cdot\begin{vmatrix}1&4&8\\9&3&2\\5&6&7\end{vmatrix}=2\cdot3\cdot3\cdot109=1962$$

$$\begin{vmatrix}1&9&6\\8&5&2\\3&7&4\end{vmatrix}\overset{R}{=}18 \qquad \begin{vmatrix}1&9&6\\4&2&5\\3&8&7\end{vmatrix}\overset{M}{=}13 \qquad \begin{vmatrix}1&3&7\\9&6&4\\8&5&2\end{vmatrix}\overset{M}{=}13$$

RECREATIONAL MATHEMATICS magazine
JUNIOR DEPARTMENT

All correspondence and material relating to the *Junior Department* should be sent to:

Howard C. Saar
1014 Lindell Avenue
Petoskey, Michigan

Wherever you might be reading this, it is the hope of the editor that there was less snow on the ground than there was here in Petoskey. Since the winter season is over and inasmuch as this is the third *Junior Department*, perhaps it is time that you the reader have an opportunity to give the editor your reactions to our initial efforts. I have received many pieces of correspondence from teachers, students, and others from throughout the country, many of you have attempted (and solved) the problems proposed. However, too few have submitted any constructive suggestions for inclusion in the *Junior Department*.

Consequently, I am asking that everyone give consideration to what you personally would like to see in the *JD*. If it's already there, fine, we'll try to keep it coming. If you have other ideas of items not included thus far, let's hear about them. The difficult ones we'll do in the next issue, the impossible ones might take a bit longer. Also, and this is particularly directed to the student readers, you don't have to be a mathematical genius to get your ideas into print. We at RMM are looking for your ideas and manuscripts, your original problems and solutions - anything that might be of some interest to *you* will be of interest to other readers of the *JD*. Our rules are very easy:

(1) All articles, problems, solutions, ideas, etc., should be submitted to the *JD* editor typewritten or neatly written in ink on 8½ x 11 paper (except those items obviously too large to fit those specifications).

(2) RMM reserves the right to edit all manuscripts accepted for publication to conform to reasonable literary standards. If we plan any major alterations you will be notified. All materials submitted to us for consideration become our property and will not normally be returned.

For those of you in the student category, really your writing is good - good enough that you should endeavor to do more. If you're one of those who just feel they cannot write, try something of specific interest to you. Perhaps you might even ask your English teacher to read it and criticize - you might be better than you think!

To be more specific! Last December the editor received a letter from Lloyd V. Rogers of O'Connor School in Menlo Park, California. Mr. Rogers made several excellent suggestions for potential *JD* material: elementary alphametics, mathematical games (perhaps some from outside the USA), short biographies of famous mathematicians written by students as the result of some simple research on their part, student-prepared book reviews, original geometric constructions, and original puzzles and problems. Why not give it a try?

FERMAT'S LAST THEOREM

Probably the most famous of all unsolved problems of mathematics is that known as *Fermat's Last Theorem.* The historical background of this problem has added very much to its appeal. The story begins with Diophantos of Alexandria (250 A.D.?), one of the great mathematicians of the golden period of Greek mathematics. It is known that Diophantos wrote a treatise in thirteen books called the *Arithmetics**, but unfortunately only six or seven of the books have been preserved. These books deal particularly with a type of problem in which it is required to find integers or fractions which satisfy certain conditions. The following examples are typical of over 130 such problems discussed by Diophantos.

Book II, *Problem* 29: Find two square numbers such that when one forms their product and adds either of the numbers to it, the result is a square.

Book III, *Problem* 7: Find three numbers such that their sum is a square and the sum of any two of them is a square.

Book IV, *Problem* 11: Find two numbers such that their sum is is equal to the sum of their cubes.

Book II, *Problem* 8: Decompose a given square number into the sum of two squares.

This last example is of particular interest in our story. It may be restated: Given a^2, find integers or fractions x and y such that

$$a^2 = x^2 + y^2.$$

The solution is not difficult. Let m denote an arbitrarily chosen number. Set $y = mx - a$, and solve for x:

$$x = \frac{2am}{m^2+1}.$$

For example, suppose $a = 4$. If $m = 3$, then $x = {}^{12}\!/_5$ and $y = {}^{16}\!/_5$. As a check $({}^{12}\!/_5)^2 + ({}^{16}\!/_5)^2 = {}^{144}\!/_{25} + {}^{256}\!/_{25} = {}^{400}\!/_{25} = 16 = 4^2$. Again, suppose $a = 5$ and $m = 2$. Then $x = 4$ and $y = 3$. The familiar relationship $4^2 + 3^2 = 5^2$ was known to the ancient Egyptians, and similar triplets were known to the Hindus as early as the fourth century B.C. Pythagoras stated a rule for finding any desired number of such triplets, and Euclid also discussed them. Diophantos' contribution to the theory of these triplets was a generalization and extension of Pythagoras' rule.

A copy of the *Arithmetics* of Diophantos became a favorite text of Fermat (1601?-1665), a celebrated French mathematician of the seventeenth century. It was Fermat's practice to write notes in the margins of his books, and opposite Problem 8 of Book II he wrote, "However, it is impossible to write a cube as the sum of two cubes, a fourth power as the sum of two fourth powers, and in general any power beyond the second as the sum of two similar powers. For this I have discovered a truly wonderful proof, but the margin is too small to contain it." In algebraic terms, Fermat asserted that *it is impossible to find integral solutions of the equation* $x^n + y^n = z^n$ *when* n *is greater than* 2.

A few years after the death of Fermat, his son published a new edition of the *Arithmetics* of Diophantos, including therewith all of Fermat's marginal notes. These notes furnished mathematicians with

* The Greeks of this period used the term *arithmetics* to designate what we now call the theory of numbers. They referred to ordinary computations as *logistics*.

material for many years of study and investigation. Every theorem that Fermat noted has been suitably dealt with except the theorem just quoted. It is therefore commonly referred to as Fermat's "last" theorem.

Many eminent mathematicians have labored long and hard on the problem. Euler proved the theorem for all cases when n is a multiple of 3 or 4, and later Legendre (1752-1833) proved the theorem for all multiples of 5. Kummer (1810-1893) thought he had devised a proof of the general theorem but he based his reasoning on a fallacious assumption which a critic immediately pointed out to him. Undaunted, Kummer tried to revise his proof and in so doing he created the theory of algebraic numbers, a theory which has proven of tremendous importance in modern mathematics. Although Kummer failed to find a rigorous proof of the general theorem, he was able to apply his algebraic numbers to show that the theorem was true for many special cases of n, in particular for all multiples of primes which do not exceed 100.

Interest in the theorem was intensified in 1907 when it was announced that a prize of 100,000 marks (about $25,000) had been established for anyone who could offer a complete solution. The prize was created by the will of a German mathematician Wolfskehl and was to remain open until the year 2007. As might be expected, a host of would-be solvers with little or no mathematical training or ability hastened to print their "solutions." They not only wasted their own time but that of many mathematicians who conscientiously examined and pointed out the errors in hundreds of proposed proofs. Fortunately for all concerned, the German inflation of 1918 wiped out the prize and most of the dilettanti left the field to the serious mathematicians.

Investigations on Fermat's Last Theorem continue with new results reported regularly. At the present time the theorem has been proven true for all values of n less than 617 as well as for many particular values greater than 617. But the general theorem itself still remains to be proved.

* * * * * *

IBN BANNAA

by Ali R. Amir-Moéz
University of Florida

"Ahmad! Why are you wasting your time and all that paper?" asked Mohammad, Ibn Bannaa's father. "You are supposed to study your lesson and write it on these papers."

"This is a beautiful trick of arithmetic," said Ahmad, "I am trying to find some more of them."

Abol-Abbas Ahmad Ben Mohammad Ben Othman Ibn Bannaa Marrakeshi was that day amusing himself with this attractive rule of arithmetic. He wrote down

$$9 \times 9 = 81.$$

Then $$99 \times 99 = 9801.$$

He was surprised when he saw $999 \times 999 = 998001$.

"I should try more examples." he thought, and went on to

$$9999 \times 9999 = 99980001.$$

"However, I can never be sure until I find a rule."

To multiply a number whose digits are all 9's by itself we first write 8. If the number of digits in this number is n, we write $n-1$ nines on the left and $n-1$ zeros on the right of 8. Then we put down a 1 on the very end. As we see, examples of this rule have been given above. We will leave the proof to the reader.

Then Ahmad Ibn Bannaa went ahead and tried a few more tricks:

$9 \times 1 = 9$	$9 \times 2 = 18$	$9 \times 4 = 36$
$99 \times 11 = 1089$	$99 \times 22 = 2178$	$99 \times 44 = 4356$
$999 \times 111 = 110889$	$999 \times 222 = 221778$	$999 \times 444 = 443556$

Thus another rule was developed - which we leave to the reader to discover.

Finally, we give another arithmetic trick of Ibn Bannaa's:

$$\begin{aligned}
1 \times 1 &= 1 \\
11 \times 11 &= 121 \\
111 \times 111 &= 12321 \\
1111 \times 1111 &= 1234321 \\
11111 \times 11111 &= 123454321 \\
111111 \times 111111 &= 12345654321 \\
1111111 \times 1111111 &= 1234567654321 \\
11111111 \times 11111111 &= 123456787654321 \\
111111111 \times 111111111 &= 12345678987654321
\end{aligned}$$

He stopped here because he was doing arithmetic in base-10. If we do arithmetic in base-15, for example, then we can continue this rule up to 15. But the rule can be modified to continue in base-10. For example:

$$\begin{array}{r}
111111111111 \times 111111111111 = 123456789 \\
10 \\
11 \\
12 \\
11 \\
10 \\
987654321 \\
\hline
= 123456790123320987654321
\end{array}$$

Ibn Bannaa was born in Morocco in 1258 and lived until 1339. The most important work of Ibn Bannaa is *Shortcuts in the Operations of Arithmetic*. This book was translated into French by Aristide Marre and in 1865 was published in Rome.

PROBLEM CORNER

1. *1963 Puzzle*

```
OOO×O = ____
 OO×O =  ___
 OO×O =  ___
        1963
```

Arrange the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in the circles shown at the left so that the sum of the indicated products is 1963.

(*Paul M. Nemecek*; Riverside, Illinois)

2. *A Train and Tunnel Problem*

A train travelling 60 miles per hour approaches a tunnel and takes 3 seconds to enter completely inside and 30 seconds to pass completely through the tunnel. What is the length of the train and how long is the tunnel?

3. *A Number Puzzle*

A certain four-digit number is divisible by one less than the sum of its digits, the resulting quotient being 81. If the sum of its digits is subtracted from this number a new number is obtained which can be written as $1a59$, where a stands for a digit that must be identified.

What must this digit be, and what must be the original number?

(*Dale Kozniuk*; Delburne, Alberta)

4. *Sums of Primes*

The smallest prime number that is the sum of three different prime numbers is 11, i.e. $11 = 1 + 3 + 7$. What is the smallest prime number that is the sum of two different groups of three prime numbers? Of three different groups of three prime numbers? Of four different groups of three prime numbers? Of five different groups of three prime numbers? (*Clifford R. Dickinson*; Camas, Washington)

5. *A Designing Stunt Section*

The card stunt section for a small high school consists of twenty members. Each member has a card that is blue on one side and gold on the other side. How many different designs could they make, such that ten blue and ten gold cards are showing each time?

(*Clifford R. Dickinson*; Camas, Washington)

* * * * *

ANSWERS TO PREVIOUS JUNIOR DEPARTMENT PROBLEMS
(August 1962 and October 1962)

AUGUST 1962 RMM (Pages 43-44)

1. 3750 peanuts were shelled daily.
2. The radius of a circle whose area is numerically the same as its circumference is 2 (Area = Circumference = 4π).
3. 28 pennies, 127 nickels, 56 dimes, 28 quarters ($19.23).
4. Nine (IX) can be changed to six by writing an S in front: SIX.
5. Making the even digits equal the odd digits by using arithmetical operations brought a number of answers, e.g.:

$$3 + 5 - 7 + (9 \div 1) = (2)(4) + 8 - 6$$
$$3 + 5 + 7 - (9 \div 1) = (8 \div 2) - 4 + 6$$

6. Both triangles (with sides 5, 5, 6 and 5, 5, 8) are of equal area. Both can be formed by rearrangement of two pair of 3, 4, 5 triangles.

OCTOBER 1962 RMM (Page 46)

1. It must be noted here that this problem was quoted from H. E. Dudeney's *Amusements in Mathematics* (problem number 3) published by Dover Publications. Jakes owned 7 animals, Hodge owned 11 animals, and Durrant owned 21 animals.

2. There are quite a number of ways of using the digits from 1 to 9 in order and with mathematical operations to equal 100. The reader is referred to RMM No. 1, February 1961, pages 39-42; and to *Scientific American*, January 1963, page 10. Here are just a few:

$1+2+3-4+5+6+78+9=100$ $\qquad 1^{2345}+6(7+8)+9=100$

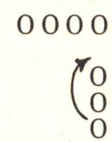
$123-45-67+89=100$ $\qquad -1+2\sqrt{-3-4+56}+78+9=100$

3. The missing digit is 4.

4. Boy, girl, widow in the ratio 8:3:13.

5. Each boy gets one apple - but one boy gets the box with his apple in it.

6. Move one penny to transform 7 pennies in 2 rows (3 pennies in one row, 4 in the other) to make 2 rows with 4 pennies in each row.



Junior Department Problem Solvers:

The August and October lists are combined here - the solvers of the August problems are listed without the issue date, the October solvers are indicated by 'Oct' preceding the problem numbers.

Don Adams, Midland, Texas (1,2,4,5,6); *John Alvord*, Exeter, N. H. (2,3,5,6); *Randy Blaisdell*, Arlington Heights, Ill. (2,5); *Louis Brill*, Westtown, Pa. (6; Oct-2,5); *Matt Clarkson*, Urbana, Ill. (5); *John Crabbe*, Sacramento, Calif. (2,3,4,5,6); *John Craig, Jr.*, Yorktown, Texas (2); *Laura Crowe*, Atlanta, Ga. (5); *William Davis*, Danville, Ill. (1,2,3,5); *Arthur Garcy*, Atlanta, Gal (5); *Charles Jones*, Orlando, Fla. (Oct-2,3,4); *Thad Jones*, Orlando, Fla. (1,2,5,6); *Jonathan Khuner*, Berkeley, Calif. (1,2,5); *Dale Kozniuk*, Delburne, Alberta (Oct-1,2,3, 4,5,6); *Greg Langston*, Atlanta, Ga. (5); *Charles Nash*, Atlanta, Ga. (1,5); *Ann Radley*, Bainbridge Island, Wash. (1,3,4); *Judy Raucher*, Brooklyn, N. Y. (4); *Shirley Ann Respondik*, Yorktown, Texas (4,5); *James Robinson*, Somerset, N. J. (2,5); *Roy Rogers*, Atlanta, Ga. (5); *Tom Schmitt*, Atlanta, Ga. (5); *Jill Scholten*, Grandville, Mich. (1,3,4); *Patsy Sills*, Atlanta, Ga. (5); *Ronald Snipet*, Atlanta, Ga. (5); *John Stark*, Gridley, Minn. (1,2,3,5,6); *Ellen Verdel*, Atlanta, Ga. (4).

PUZZLE SOLVERS: We have listed the solvers of the various puzzles in the August 1962 *and* October 1962 issues of RMM (the solvers list was omitted from the October 1962 issue). The first set of answers are the August 1962 list of solved problems; the October 1962 list follows 'Oct'. See pages 22-23 in this issue for the October 1962 puzzle answers.

ALPHAMETICS

Merrill Barnebey, Grand Forks, N.D. (1,2,3,4,5,6,7; Oct-1,2,3a,4,5,6); *Richard L. Breisch*, Royersford, Pa. (3,4,6); *John J. Coble*, Orlando, Florida (Cover,2,3,4,6,7; Oct-1,3a); *John Crabble*, Sacramento, Calif. (Cover,1,6); *Clifford R. Dickinson*, Camas, Wash. (Cover,1,2, 3,4,6,7; Oct-1,2,3a,4,5); *John Eckelman*, Hudson, N. Y. (6,7; Oct-1, 2,4,6); *Ronald L. Enyeart*, Los Angeles, Calif. (1,2,3,4,6,7); *Fr. Victor Feser, OSB*, Richardton, N. D. (1,2, 3,4,6,7); *Michael Garvey*, San Anselmo, Calif. (6); *Harry M. Gehmen*, Buffalo, N. Y. (1,2, 3a,4,5,6); *Alan Gold*, Downsview, Ontario (Oct-1,2,3a,6); *Edward Harlan*, Midland, Texas (3,4,6); *Charles Jones*, Orlando, Fla. (Cover,1,2,3,4,6,7; Oct-1,3); *Edward Joris*, Antwerp, Belgium (Oct-1,2,3a,4,5,6); *Prof. Edgar Karst*, Springfield, Mass. (Oct-1,2,3a,3b,4,5,6,7); *Felix Kestenholz-Seiler*, Liestal, Switzerland (Cover); *Jonathan Khuner*, Berkeley, Calif. (Cover,1,2,3,4,5,6,7); *Dale Kozniuk*, Delburne, Alberta (1,2,3,4,6,7; Oct-1,2,3a,4,6); *Richard McCreless*, Midland, Texas (1,3,6,7); *Bertha McDaniel*, Stayton, Ore. (Cover,1,2, 3,4,5,6,7; Oct-1,2,3a,4,5,6) *Thomas J. Morris*, San Juan, Puerto Rico (1,3,6,7); *Derrick Murdoch*, Toronto, Ontario (2); *Harry L. Nelson*, Livermore, Calif. (Cover,1,2,3,4,6,7; Oct-1,2,3a,4,5,6); *Paul Nemecek*, Riverside, Ill. (Cover,6); *Wade E. Philpott*, LIma, Ohio (Oct-1); *Robert Prall*, Urbana, Ill. (Oct-1,2,3a,4,6); *George Propper*, Bronx, N. Y. (Oct-1,2, 3a,4,5); *Rusell Rew*, Aurora, Colo. (Oct-1); *Osga Richardson*, Otwell, Ind. (Oct-1,2,4, 5,6); *Lloyd V. Rogers*, Menlo Park, Calif. (6,7); *R. Robinson Rowe*, Sacramento, Calif. (Cover,1,2,3,4,5,6,7; Oct-1,2,3a,4,6) *Robert Ruderman*, Columbia Univ., N. Y. (1,2,3,4, 6,7); *Martin Sobin*, Brooklyn, N. Y. (2,4,); *Donval R. Simpson*, College, Alaska (Cover,6,7 Oct-1,2,3a,4,6); Arlene Steinberg, Lincolnwood, Ill. (3a,3b); *C. W. Sweitzer*, San Diego, Calif. (1,3,4,6,7); *C. W. Sweitzer, II*, San Diego, Calif. (Oct-1,2,3b,4,5); *John H. Sweitzer* Princeton, N. J. (Cover,1,2,3,4,5,6,7); *Alfred Vasko*, Swanton, Ohio (1,2,3,4,6,7; Oct-1,2, 3b4,5,6); *Robert Winer*, Lincolnwood, Ill. (Oct-1); *Larry Zeman*, Lincolnwood, Ill. (Oct-3b); *Anneliese Zimmermann*, Bad Godesberg, West Germany (2,4,6,7; Oct-1,2,3a,4,6,7).

PUZZLES AND PROBLEMS

Merrill Barnebey, Grand Forks, N.D. (4,6; Oct-2,3,6,7,8); *Richard L. Breisch*, Royersford Pa. (4,5,8); *Louis Brill*, Westtown, Pa. (Oct-2,4,8); *John J. Coble*, Hattiesburg, Miss. (Oct - 2,6); *Richard Corry*, Manhattan, Kansas (4); *Clifford R. Dickinson*, Camas, Wash. (4,5,6,8; Oct-2,4,6); *John Eckelman*, Hudson, N. Y. (Oct-2,3.6); *Ronald L. Enyeart*, Los Angeles, Calif. (3,4,5,6; Oct-1,2,6,7,8); *Nick Farnum*, La Mirada, Calif. (4); *Fr. Victor Feser, OSB*, Richardton, N. D. (4,5,6,8); *Louise Glatz*, Pittsburgh, Pa. (3); *Alan Gold*, Downsview, Ontario (Oct-2,3,6,7,8); *Lance J. Hoffman*, Pittsburgh, Pa. (4,6); *Stanley S. Isaacs*, Washington, D.C. (Oct-2,3); *John Ivan*, Pittsburgh, Pa. (Oct-2); *R. S. Johnson*, Town of Mt. Royal, Quebec (3,4,5,6,8; Oct-1,2,3,6,7,8); *Charles Jones*, Orlando, Fla. (Oct-2,6); *Edward Joris*, Antwerp, Belgium (Oct-1,2,3,4,6,7,8); *Jonathan Khuner*, Berkeley Calif. (3,4,6,8); *Dale Kozniuk*, Delburne, Alberta (4,8; Oct-2,6,8); *Richard McCreless*, Midland, Texas (5); *Bertha McDaniel*, Stayton, Ore. (3,4,5,6; Oct-2,3,7,8); *Thomas Morris*, San Juan, Puerto Rico (4); *Harry L. Nelson*, Livermore, Calif. *(1,2,4,5,6,8;Oct*-1,2,3,4,6, 7,8); *Wade E. Philpott*, Lima, Ohio (3,4,7; Oct-2,3,6); *Luther J. Porter*, Wilmington, Del. (3,4,5,6,8); *Russell Rew*, Aurora, Colo. (Oct-2,3,4,6,8); *R. Robinson Rowe*, Sacramento, Calif. (2,4,5,6,7,8; Oct-2,3,4,6,7,8); *Jamie Scott*, Arcadia, Calif. (Oct-1,2); *Donval R. Simpson*, College, Alaska (4,6); *C. R. J. Singleton*, Sheffield, England (Oct-1,2,4,5,6,7,8) *Martin Sobin*, Brooklyn, N. Y. (Oct-2); *C. W. Sweitzer*, San Diego, Calif. (4,5,6); *C. W. Sweitzer, II*, San Diego, Calif. (Oct-2,3,6,7,8); *John H. Sweitzer*, Princeton, N. J. (3,4, 5,6); *Alfred Vasko*, Swanton, Ohio (4,6; Oct-2,3,4,6,7,8); *James Williams*, Springfield, Mo. (3,4,6,7); *Anneliese Zimmermann*, Bad Godesberg, West Germany (3,4,5,6,7; Oct-2,3,4,7,8).

ON SETTING ALPHAMETICS (Article by Alan Sutcliffe in August 1962 RMM)

Michael Garvey, San Anselmo, Calif. (MENSA); *Jonathan Khuner*, Berkeley, Calif. (MENSA, FILLING); *Derrick Murdoch*, Toronto, Ontario (FILLING); *Harry L. Nelson*, Livermore, Calif. (MENSA); *R. Robinson Rowe*, Sacramento, Calif. (MENSA); *Donval R. Simpson*, College, Alaska (MENSA).