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ISSUE NO. 10
AUGUST 1962

RECREATIONAL MATHEMATICS magazine

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— by J. Charles Clapham

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AUGUST 1962 ISSUE NUMBER 10

PUBLISHED AND EDITED BY JOSEPH S. MADACHY
ASSOCIATE EDITOR J. A. H. HUNTER
JUNIOR DEPARTMENT EDITOR HOWARD C. SAAR

ARTICLES

DEPARTMENTS

MISCELLANEOUS

ILLUSTRATIONS

RECREATIONAL MATHEMATICS MAGAZINE is published bimonthly by Joseph S. Madachy. Second-class postage paid at Idaho Falls, Idaho. Subscription rates (worldwide) $3.25 per year, 75 cents per copy. All correspondence concerning changes of address, subscriptions, reprints, and advertising should be sent to the Editor, RMM, Box 35, Kent, Ohio. See inside front cover for manuscript mailings. RECREATIONAL MATHEMATICS MAGAZINE is printed by The Falls Printing Co., Idaho Falls, Idaho.

From the Editor

We are late again with this August 1962 issue of RMM. The next, October 1962, issue will also be late - but not as late as the present issue. The December 1962 issue of RMM should be right on time - the first week of December - if all goes reasonably well!

* * * * *

The first *Junior Department* will be found on pages 43-48. As requested there, we want any comments - orchids or onions - you may feel about it.

* * * * *

The Editor has settled down, somewhat, in Ohio and things are not quite as hectic as before.

Your attention is directed to page 24. We wish constantly to improve RMM and this can only be done by founding out what RMM readers think. What we want are as many replies as possible - even if you have very little to say, say it!

* * * * *

Many readers have commented on the presence of material coming from all parts of the world and asked just where RMM goes. The Editor will compile a complete state-by-state (all 50) and country-by-country (over 35) table for the December 1962 issue which will give the circulation figures in those categories.

RMM is in the midst of a subscription campaign which should see the mailing out of about 70,000 letters by the end of the year. No doubt, some RMM subscribers will receive these letters (mailing lists unavoidably produce duplication and screening is much too costly), but we hope the letters will be passed on to some interested party.

The Editor is convinced that each RMM subscriber knows at least three other persons who would subscribe to RMM *if they knew about it.* Send in the names and addresses of those persons and we'll send a letter telling them about RMM. Better yet, be a good friend and give them a gift subscription!

J. S. M.

Regular Polygons from Knotted Strips

by Donald L. Vanderpool

Ever since Martin Gardner stated that a regular pentagon can be formed from a knotted strip of paper, I have been wondering if it would be possible to form other regular polygons in a similar manner - especially polygons with an odd number of sides. Mr. Gardner says "The equilateral triangle, square, hexagon and octagon are quite easy to fold, but the pentagon offers special difficulties."*

This report will show how regular polygons with an odd number of sides can be formed from a strip of flexible material by a process of knotting. Paper-folding methods to form equilateral triangles and even-sided polygons will also be shown.

If a strip of paper is simply knotted (Figure 1) and, then, carefully tightened, and flattened, the regular pentagon is formed (Figure 2). However, if you do not tighten this knot, you can form a knot that will produce a regular heptagon by bending the end *B* away from you, up and under the whole configuration, coming out near *A*, and then feeding the end *B* over all white portions, but under all shaded portions, obtaining the knot depicted in Figure 3.

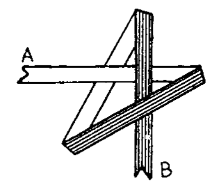


Figure 1

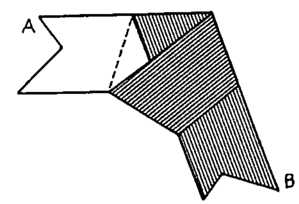


Figure 2

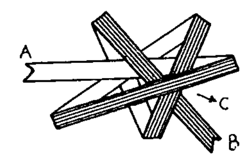


Figure 3

If this knot is carefully tightened and kept neatly flat, then the regular heptagon in Figure 4 will be formed.

Similarly, if you bend the end of the strip, *B*, away from you, bringing it up and under the whole configuration to a point near *A*, and then feeding the end, *B*, over all white portions, but under all shaded portions, coming out at *C* (above *B* in Figure 3), you will have a loose knot that will tighten (very carefully) into a regular nonagon (Figure 5).

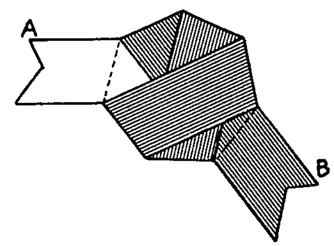


Figure 4

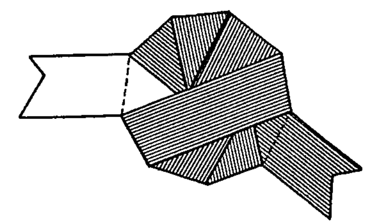


Figure 5

By repeating the procedure - bending *B* away from you, up and under the whole configuration to a point near *A*, then feeding *B* over

* MARTIN GARDNER, *Mathematical Games,* SCIENTIFIC AMERICAN, July 1959, page 138.

all white portions, but under all shaded portions, coming out above the previous position of B - it is possible to form knots that can be tightened into regular polygons of any odd number of sides.

The equilateral triangle and regular even-sided polygons cannot be derived from knotted strips for reasons a bit too long to explain here. Regular polygons with 10, 14, 18, etc., sides are possible by intertwining two duplicate knots of 5, 7, 9, etc., sides, respectively, but difficulties arise in the tightening and flattening of these knots.

An easier method of forming regular even-sided polygons is to fold them from regular polygons with half the number of sides. For example, a regular polygon of 10 sides can be derived from the regular pentagon by creasing the pentagon as shown by the heavy lines in Figure 6 and, then, by bisecting each angle at C by folding tthe line AC onto $A'C$, then $A'C$ onto $A''C$, etc., thus forming the dashed lines BC, $B'C$, etc., you obtain the points B, B', etc. Now if you fold the corners of the pentagon on the dotted lines BB′, etc., the result will be a regular polygon of 10 sides (a decagon).

This method is applicable to every polygon and so if we can form an equilateral triangle and square, we can fold a polygon of *any* number of sides (as we can form a polygon of any *odd* number of sides).

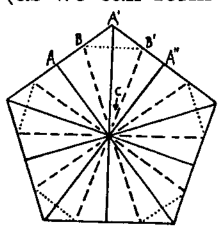


Figure 6

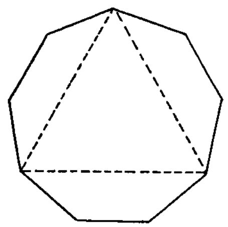


Figure 7

The equilateral triangle is easily derived from the regular nonagon as shown in Figure 7. However, it is easier to form as shown in Figure 8: (1) place the edge of the end A on some point B (marked by pencil or creasing) and (2) slide thi sedge upward, but always on point B, until a funnel-shaped form results with its peak at C (3). By carefully creasing this funnel along BC and turning the strip over you find the equilateral triangle BCD (4).

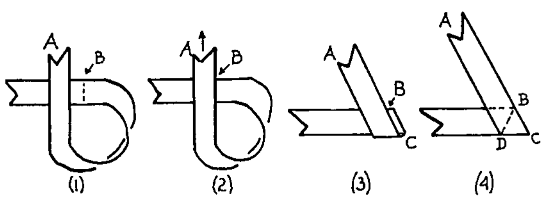


Figure 8

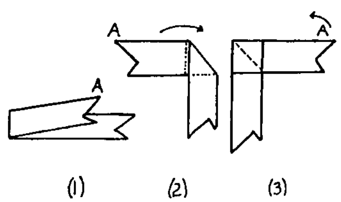


Figure 9

To form a square fold the strip on itself and make a crease [Figure 9, (1)]. Open the strip and fold one end down along this crease (2), and over (arrow) as shown. If you fold end A back away from you, you will have a sauare (3).

The hexagon and octagon are derived from the folding of an equilateral triangle and square as explained previously.

Fibonacci and Hero

by Walter W. Horner

The purpose of this note is to show how Fibonacci numbers can be used to generate Heronian integers. Such integers represent the sides of a triangle whose area is also integral. It is obvious that a Heronian triangle can be constructed from any two Pythagorean triangles having a leg of one equal to a leg of the other. The equal legs can be made to coincide to form the altitude and the sum or difference of the other two legs will form the base. Charles W. Raine has pointed out how to generate Pythagorean numbers from the Fibonacci series. (See *Scripta Mathematica*, Vol. XIV, page 164)*

The procedure is as follows. Take five terms of the series as

$$U_n,\ U_{n+1},\ U_{n+2},\ U_{n+3},\ U_{2n+3}$$

Note that the first four terms are consecutive and the fifth is as indicated by its subscript. Then from the four consecutive terms the product of the extreme terms and twice the product of the mean terms are the legs of a Pythagorean triangle. The fifth term is the hypotenuse, i.e.

$$(U_n \cdot U_{n+3})^2 + (2U_{n+1} \cdot U_{n+2})^2 = (U_{2n+3})^2$$

To simplify matters let

$$a = U_n \cdot U_{n+3} \qquad 2b = 2U_{n+1} \cdot U_{n+2} \qquad c = U_{2n+3}$$

Then $a^2 + (2b)^2 = c^2$.

From another set of five terms determine in the same manner another set of Pythagorean numbers as d, $2e$, f. Note that the sets of five terms from the series may or may not overlap.

Now we have

Set 1: a $2b$ c
Set 2: d $2e$ f

Multiply Set 1 first by d and then by $2e$. Multiply Set 2 first by a and then by $2b$. This gives the four sets of Pythagorean numbers below.

ad	$2bd$	cd
$2ae$	$4be$	$2ce$
ad	$2ae$	af
$2bd$	$4be$	$2bf$

Note that there are four common terms in the sets above. Combining pairs of triangles with common sides as before noted, we have the following Herodian triangles:

S_1	S_2	S_3	Area
af	cd	$2ae \pm 2bd$	$ad(ae \pm bd)$
$2bf$	$2ce$	$2ae \pm 2bd$	$4be(ae \pm bd)$
$2bf$	cd	$4be \pm ad$	$bd(4be \pm ad)$
af	$2ce$	$4be \pm ad$	$ae(4be \pm ad)$

Thus it is evident that any two sets of Fibonacci numbers chosen as described above will generate eight Heronian triangles with sides and areas as indicated above.

*For a method of generating Pythagorean triangles from four Fibonacci terms see RMM No. 4, August 1961, page 61.

Now for a numerical example. From the Fibonacci series let us choose the following sets:

Set 1: 1, 1, 2, 3, 5
Set 2: 1, 2, 3, 5, 13

Then from Set 1 we have $a = 3$, $b = 2$, $c = 5$. From Set 2 we have $d = 5$, $e = 6$, $f = 13$. Substituting these values in the expressions above we have:

S_1	S_2	S_3	*Area*
39	25	56	420
39	25	16	120
13	15	14	84*
13	15	4	24*
52	25	63	630
52	25	33	330
13	20	21	126*
13	20	11	66*

* Highest common factors removed.

* * *

Playful Mice

by J. Charles Clapham

There is an old puzzle involving four mice chasing each other at constant speed along the edges of a square room. At one instant, one of the mice decides to leave the wall edge and always head towards the mouse in front. At exactly the same instant, the other three mice decide to do the same thing. If the mice were originally equally spaced (say, in the corners of the room) you are asked to prove that when the mice meet in the middle of the room, each will have travelled a distance equal to the original distance between them. When it is discovered that the paths are logarithmic spirals, most puzzle-solvers forget about proof!

It occurred to me that the puzzle could be generalized to include mice in any room with a regular polygonal shape. The results show that the distance travelled is not in general equal to the distance between the mice. For an equilateral triangle they travel less than their original spacing, and in a 5-or-more sided room they travel further.

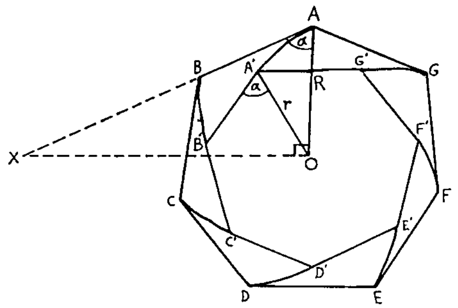


Figure 1

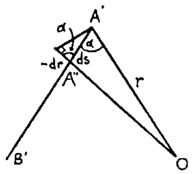


Figure 2

GIVEN: A mouse at each vertex of a regular polygon $ABCD\ldots$, each mouse running towards the next at speeds which at any instant are all equal.

PROVE: That when the mice meet at the center, the mouse at A will have run a distance equal to AX, where X is on AB and $\angle AOX = 90°$.

CONSTRUCTION: Figure 1 shows the situation at time t, when each mouse has run a distance s to $A'B'C'D'\ldots$, where

$$OA' = OB' = OC' = OD' \quad \ldots = r$$
$$OA = OB = OC = OD \quad \ldots = R$$

and α is the semi-angle of the polygon $ABCD\ldots$ and, by symmetry, of A'B'C'D'. . .

Figure 2 shows the situation at A' an infinitesimal time later, when the mouse at A' has advanced a distance ds to A'' in the direction of B'.

PROOF: From Fig. 2 $ds = -dr \cdot \sec\alpha$
Hence, integrating $s = (R-r)\cdot\sec\alpha$, since $r = R$ and $s = 0$.
When mice meet, $r = 0$, so $s = R\cdot\sec\alpha = AX$

I have not been able to get any result for an irregular figure, such as a general triangle. Will two of the mice meet first, and then travel in a straight line head-on to the third? Or will they spiral to meet simultaneously in a point? If so, where is that point?

Editorial Comment: If two mice are at the ends of a very narrow hall (a "two-sided" polygon) the formula indicates, quite correctly, that they will collide head-on halfway down the hall. If three mice are in the corners of an equilateral triangular room, they will travel ⅔ the length of a side of the room. When we consider that mice are rather prolific, in addition to being playful, we may imagine the frustration of an infinite number of mice in a large roundhouse (an "infinite-sided" polygon) trying to catch each other! The proof above indicates that they will forever run around the roundhouse, never getting to the center, and never catching each other - unless they run an infinite distance!

The slight use of calculus in the proof is unavoidable. Even though RMM is advertised as containing "no calculus or Ph.D-level mathematics", there are bound to be occasions when a touch of such material is required. Else we must avoid the publication of RMM-type material - as the *Playful Mice* certainly is. A trigonometery table is all that is required to solve the final equation given above.

On Setting Alphametics

by Alan Sutcliffe

It is always more interesting, and generally more difficult, to set puzzles than to solve them. In the case of alphametics the setter must first decide on a suitable form to try, and then test whether it has a unique solution. This can only be done by carrying through the normal method of solution. Sidney Kravitz has dealth fully with some systematic methods of solving alphametics[1]. These are bound to be fairly lengthy for any worthwhile puzzle, so it is certainly profitable to spend a little time beforehand to check whether there is *likely* to be just one solution. I shall set out here a method for estimating the expected number of solutions to any alphametic, based on the calculation of simple premutations. While no certainty can be attached to the result of the test, if it shows that a given puzzle is likely to have about 10 solutions for example, there is little use taking that particular puzzle further.

I shall first illustrate the method on a simple multiplicative alphametic: DUNKED·A = FILLING. We have to look at the two sides of the equation separately, the left hand side (l.h.s) where we are free to substitute any digits for the letters, and the right hand side (r.h.s.) where the values of the letters are determined once we have set values on the other side.

There are 6 different letters on the l.h.s. - E, D, K, N, U, A. Leaving aside, for the moment, even the most obvious restrictions there are on these, such as $D \neq 0$ and $A \neq 0$ or 1, the K may have any of ten values from 0 to 9, D may have any of the remaining nine values, E may have any of eight, and so on to give a total number of possibilities equal to $10{\cdot}9{\cdot}8{\cdot}7{\cdot}6{\cdot}5 = 151{,}200$. Now each of these sets of values of the six letters determines a number equal to DUNKED·A, which we wish to take the form FILLING.

There are five different letters on the r.h.s. F, G, I, L, N. N has already been specified, while L may take the value of any of the four digits not used on the l.h.s., F any of the remaining three, and so on, giving 24 possibilities in all. This is out of a total of 9,000,000 numbers with 7 digits. Thus for each of the 151,200 possible values of the l.h.s., there is a chance of 24/9,000,000 of the r.h.s being of the form FILLING. Therefore we may say that the expected number of solutions is $151{,}200{\cdot}24/9{,}000{,}000 = 0.432$.

Clearly this is only an approximate indication, firstly because of the essentially statistical nature of the test, and secondly because we have ignored obvious restrictions on values of letters. I have found in practice that values between 0.3 and 3 are certainly worth following up, but with values much outside this range, there is little likelihood of a unique solution. The problem of the restrictions on the values of letters, which we ignored above, is one of knowing where to stop: the test can be applied at any stage of a solution, though the permutations may become fairly hard to calculate in some cases. In the example

[1] SIDNEY KRAVITZ, *The Art of Solving Multiplication Type Alphametics*, RMM No. 2, April 1961, pages 9-14.

given above, if we had taken into consideration $D \neq 0$, and $A \neq 0$ or 1 on the l.h.s and $F \neq 0$ or 1 on the r.h.s., then the expected number of solutions becomes $(8{\cdot}8{\cdot}8{\cdot}7{\cdot}6{\cdot}5) \cdot (3{\cdot}3{\cdot}2)/8{,}000{,}000 = 0.32$ approximately. The expression $(3{\cdot}3{\cdot}2)$ is not quite correct, but is based on the fact that usually just one of the values prohibited for F, that is 0 or 1, will have been used on the l.h.s., so that F will be able to take any of three out of the four digits not used on the l.h.s. Of course, sometimes both 0 and 1 will already have been used, and sometimes neither will have been used: this is just another example of the sort of thing that is just not worth calculating any more accurately because of the essentially approximate nature of the answer. In this case our more careful calculation has made little difference to the result, but we shall see in the next example that such refinements can be important. Incidentally, solution by the normal method shows that this puzzle has just one solution - to be given in the December 1962 issue of RMM.

We are now in a position to state the method for the general type of puzzle which can be written simply as an equation, that is where no subsidiary working, as in long division, is shown.

I. Calculate the total of numbers which satisfy the form of the l.h.s.: let this total be a.

II. Calculate the total of numbers which satisfy the form of the r.h.s. allowing for the digits already used on the other side, but regardless of the arithmetic equality of the two sides: let this total be b.

III. Calculate the total of numbers with the same number of digits as the r.h.s.: let this total be c. (Clearly, for a number with n digits, $c = 9 \times 10^{n-1}$, where the only restriction is that the first digit is not zero.)

IV. The "*expected number*" of solutions is ab/c.

Of course, the expressions for the l.h.s. and the r.h.s. won't be strictly correct for many puzzles that are laid out in the way that the sum would be for calculation.

I shall now take an example of an addition alphametic to illustrate some further points about the method. This puzzle was actually composed for the society whose name appears as the total and whose members must pass an I.Q. test before admission.

$$\begin{array}{r} \text{MEN} \\ \text{OF} \\ \text{SOUND} \\ \text{SENSE} \\ \hline \text{MENSA} \end{array}$$

On the l.h.s. (or above the the line) we have eight letters D, E, F, M, N, O, S and U, while on the r.h.s. (below the line) only one further letter appears, namely A. Thus for a first calculation, ignoring any restrictions, we have

$$a = 10{\cdot}9{\cdot}8{\cdot}7{\cdot}6{\cdot}5{\cdot}4{\cdot}3 \qquad b = 2 \qquad c = 90{,}000.$$

From these we get the expected number of solutions as 40.32. However, there are some important restrictions that can be placed on some of the letters without giving much study to the problem. Clearly, M, O or $S \neq 0$, and further, since $M > S$, $M \neq 1$. Looking at the thousands column we see that O must be either 8 or 9, depending on whether a 1 or a 2 is carried from the hundreds column. As a result of these restrictions, we make a drastic reduction in the value of a: taking the letters in the order O, M, S followed by the others, we have $a = 2 \cdot 7 \cdot 7 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 246{,}960$. This gives a new value for the expected number of solutions as 5.49 approximately. The lesson here is that while the reduction for a particular letter of one or two in the number of values it can take may safely be ignored, the reduction *to* one or two values may not.

Another point to be noticed about this puzzle is that if one solution exists, then it is not unique, for there are certain values that can be interchanged in any given solution. Since D and F occur only in the units column, they are clearly interchangeable. Not quite so obvious is the interchangeability of E and N: they occur once each in the units and tens columns, and their appearance in the hundreds and thousands columns is such that in fact any free digit could be substituted for either of them. Combining these two interchanges gives us a total of four solutions from any single solution. In fact there are two distinct solutions, giving eight solutions in all, which is in fair agreement with our revised estimate of over five. These solutions will be given in the December 1962 issue of RMM. Clearly it is important to check any prospective puzzle for this type of ambiguity before even beginning to estimate the expected number of solutions, and this example shows that care may be needed in spotting such ambiguities.

In the case of alphametics which show some intermediate working, as well as two sides of an equation, such as long division and long multiplication, the method of estimating the likely number of solutions is not quite so easy to apply. Such puzzles are inherently likely to have a solution at all, since there are more conditions to satisfy - all those relating to the intermediate working. The chance that there will be more than one solution, unless there are many ambiguities, is very small.

However, if it is wished to test an alphametic of this sort, it is necessary to split it up into its component parts: that is, for multiplication, into a number of multiplications by one digit followed by an addition. These can then be dealt with in succession, and each one will give a further reduction of the number of solutions expected. If the puzzle is one where some of the intermediate working is shown, by letters, the remainder merely being shows as so many places, then it is not possible to give fixed method of procedure, but the method described above can easily be adapted to give a rough estimate of the expected number of solutions.

Alphametics

H. E. Dudeney, one of the greatest puzzilists of all time, seems to have been the originator of the classic alphametic SEND MORE MONEY. J. R. Whalley of Sussex, England sends in the information that in *Strand Magazine* for July 1924, Dudeney, under "Perplexities", composed four alphametics. They include the classic SEND MORE MONEY and the (TWO) (TWO) = THREE previously quoted in the February 1962 issue of RMM on page 13.

Here are Dudeney's alphametics which he entitled "Verbal Arithmetic":

ADDITION is an imposition

$$\begin{array}{r} \text{S E N D} \\ \text{M O R E} \\ \hline \text{M O N E Y} \end{array}$$

SUBTRACTION is as bad

$$\begin{array}{r} \text{E I G H T} \\ \text{F I V E} \\ \hline \text{F O U R} \end{array}$$

MULTIPLICATION is vexation

$$\begin{array}{r} \text{T W O} \\ \cdot\ \text{T W O} \\ \hline \text{T H R E E} \end{array}$$

DIVISION drives me mad

$$\begin{array}{l} \text{T W O) S E V E N (T W O} \\ \qquad\qquad\ \text{B O B} \\ \qquad\qquad\ \overline{\ \ \text{J O E}} \\ \qquad\qquad\ \ \ \text{O V V} \\ \qquad\qquad\ \ \ \overline{\text{V E S N}} \\ \qquad\qquad\ \ \ \text{V E S N} \\ \qquad\qquad\ \ \ \overline{\qquad\quad} \end{array}$$

The answers to Dudeney's alphametics will be given in the December 1962 issue of RMM. We cannot guarantee that all the above have unique solutions.

Now, the only alphametic published in RMM whose authorship remains unknown is:

$$\frac{\text{EVE}}{\text{DID}} = .\text{TALKTALKTALK} \ldots$$

which has two solutions.

* * * * *

To complete this issue's page of alphametics, we present two regular addition problems.

The sign seems to be set high at the bus stop. What do you make of that STOP?

(*A. G. Bradbury;* North Bay, Ontario)

$$\begin{array}{r} \text{P O S T} \\ \text{T O P S} \\ \hline \text{S T O P} \end{array}$$

* * * * *

Teamwork is the modern battle cry. You may require help in solving this timely alphametic for it has TWIN solutions.

(*J. A. H. Hunter;* Toronto, Ontario and *Joseph S. Madachy;* Kent, Ohio)

$$\begin{array}{r} \text{N O W} \\ \text{T W O} \\ \text{I N} \\ \text{T W I N} \\ \hline \text{O R B I T} \end{array}$$

* * * * *

We refer the reader to Alan Sutcliffe's article *On Setting Alphametics* to be found on page 8 of this issue.

Some Studies of the Moebius Tape and the Moebius Rings

by Roger Hayward

The question was asked in the June 1962 issue of RMM (page 18):

> *What is the intersection which a Moebius strip would make with itself if it were infinitely wide?*

Figure 1 below shows the mathematics and visual aid required to arrive at the answer. The intersection is a straight line at an angle of 45° to the plane of the generating circle.

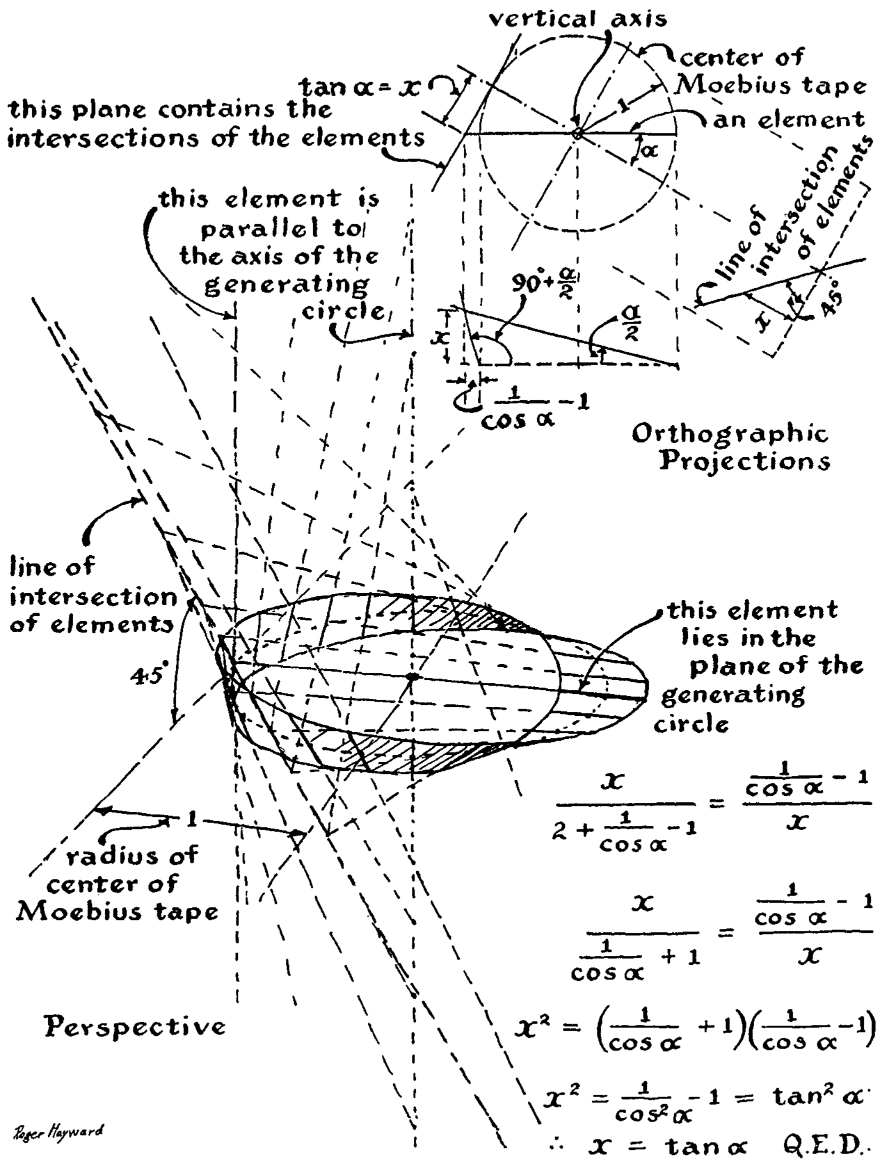


Figure 1

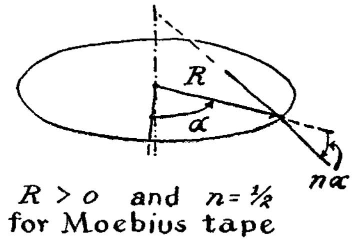


Figure 2

Figure 2 shows the process of generating the original Moebius tape. A line segment is centered on a circle of radius R. As the center of the circle moves around the circle through an angle α the line segment rotates about the moving center in a plane which contains the axis of the circle and through an angle $n\alpha$. Thus, when the line segment reaches its starting point it will exactly meet the original position if n is the inverse of an odd number. If n is the inverse of an even number then the line segment meets itself exactly reversed and the common Moebius tape is the case where $n = \frac{1}{2}$.

Figure 3 shows a stereo perspective* of the usual Moebius tape and the two parallel elements have been extended and are parallel to the line of intersection which the tape would make with itself if it were infinitely wide.

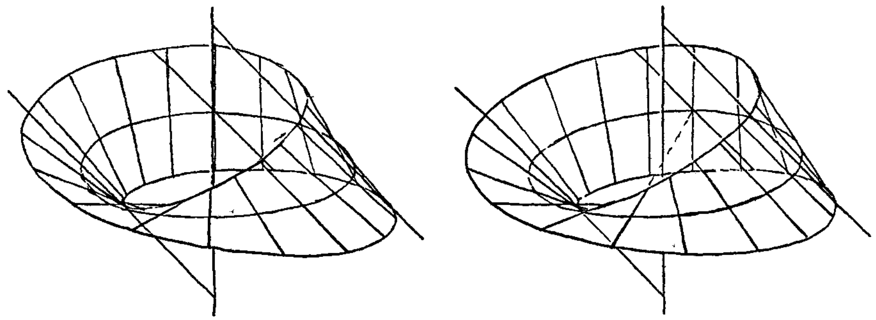


Figure 3

* In most perspectives which the writer has made there have been groups of lines which appear to radiate from a vanishing point. In cases of this kind a stereo pair can be constructed by first finding the vanishing points for the systems of parallel lines in the object. The stereo mate to one drawing is made by merely starting another drawing displaced an appropriate amount laterally, using the same vanishing points. As drawn these drawings are immediately useful by the crossed-eye method of viewing (i.e. viewing the two views and crossing the eyes until the two views fuse together — with practice, most persons will then see a definite illusion of a three-dimensional object). In the cases here presented there are many lines with no or very few parallels to be represented. It is easier to find the perspective location of two points on a line than to find one point and a vanishing point for the line. Therefore I simply constructed a plan and elevation. With two station points for the plan I ticked off on a strip of paper the horizontal perspective position for each of the points. I kept track of them by numbering. (One can easily get lost.) Then I used one vanishing point at the same distance from the elevation to find the vertical perspective locations for the same points. The stereo pair so produced have to be viewed by the wall-eyed method which can be assisted with a stereo viewer. In these examples I have used the convention that all vertical lines are parallel in spite of the fact

If the value of n is zero then the tape degenerates into either a cylinder or a plane depending whether the generating line is started at an angle with R or coinciding with it. If n is even then the generating line meets itself all right but is not reversed with the result that the tape has two sides and two edges and there is no way to get from one side to the other without crawling over an edge. As a help in understanding some of the properties of the normal Moebius tape it seemed an interesting idea to let $R = 0$. The tape then has some of the properties of a cone. All the elements (the generating lines) pass through a point. This form can indeed be developed in a sheet material just as a cone can be unwrapped although it does intersect itself and, as one would expect, at the same place where the opposite elements are parallel. Figure 4 is a stereo perspective of this conical version.

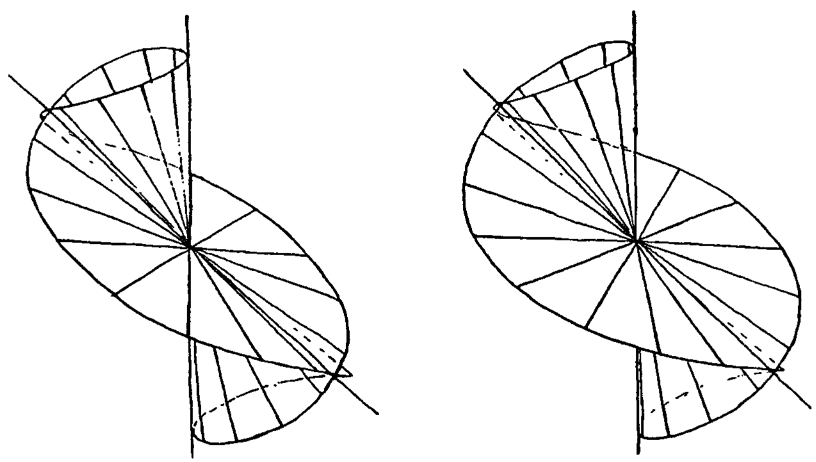


Figure 4

Figure 5 is the case of the Moebius tape with $R = 0$ and $n = 1/3$. By following this through it will be seen that the elements meet facing the same way. There are two edges and two faces. If R is larger than zero then such a complex tape will intersect itself if sufficiently wide. The lines of intersection will in general be probably hyperbolic in character but not lying in a plane and they will be asymtotic to the directions where the surfaces in Figure 4 intersect or kiss. The vertical axis is an example of the latter. Even mathematical forms have fun.

that they will be properly viewed with the eye considerably below the proper station-point. The usual analogue is the photograph which has been taken with a view camera with a rising front. People are so familiar with viewing perspectives from station-points which are in fact incorrect that the errors go unnoticed and do not interfere with proper interpretation of the image.

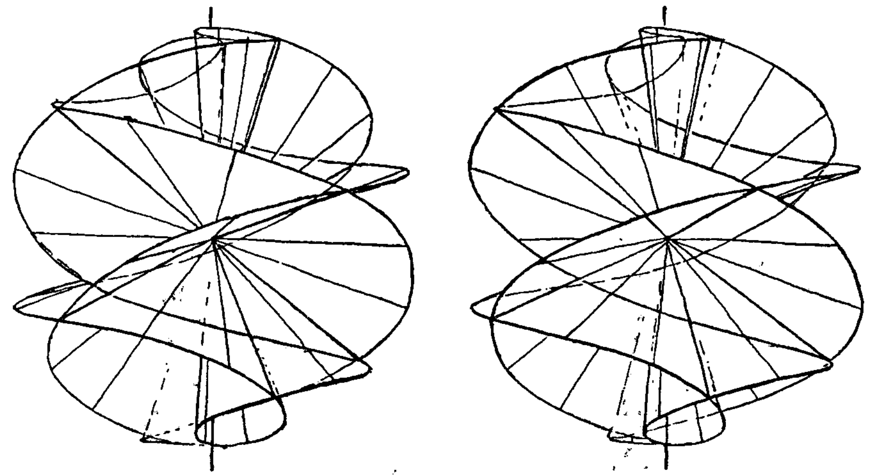


Figure 5

Figure 6 shows a more general form of the Moebius tape. The point which moves around the axis at radius R is the center of a circle lying in a plane containing R and the axis. This latter circle has a radius r at the end of which a segment of line normal to it rotates through an angle $n\alpha$. The case where $n = 1/2$ is not illustrated but is shown in figure 28, page 64 of Martin Gardner's *Scientific American Book of Mathematical Puzzles and Diversions* (Simon & Schuster, 1959). Incidentally in his figure there is an unnecessary joint. Only one joint is needed. When $n = 1/2$ and r is smaller than R then space is not enclosed. There are two faces and two edges and one may insert a Moebius tape between the two layers.

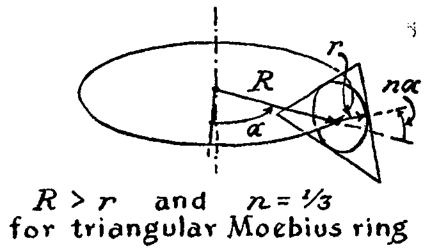


Figure 6

When r is smaller than R and $n = 1/3$ or some other inverse integer then space is enclosed and a ring of solid material may be constructed. Figure 7 shows the case where $n = 1/3$ and one can see a phantom representation of a stereo perspective of a ring with a triangular cross-section. This figure rotates one face per revolution about the axis.

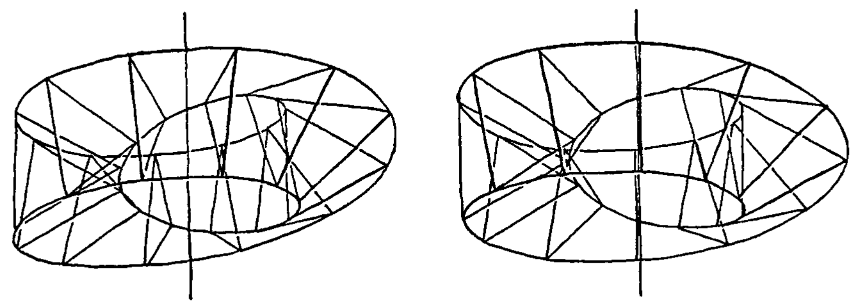


Figure 7

In Figure 8 the same case is illustrated but with $R = r$ and it appears that in many of the sections the opposite sides of the ring intersect. The intersections of opposite sections haven't been specifically indicated. One would expect the intersections of opposite parts of the ring to become asymtotic to the intersection directions in Figure 4 if the elements are extended to infinity.

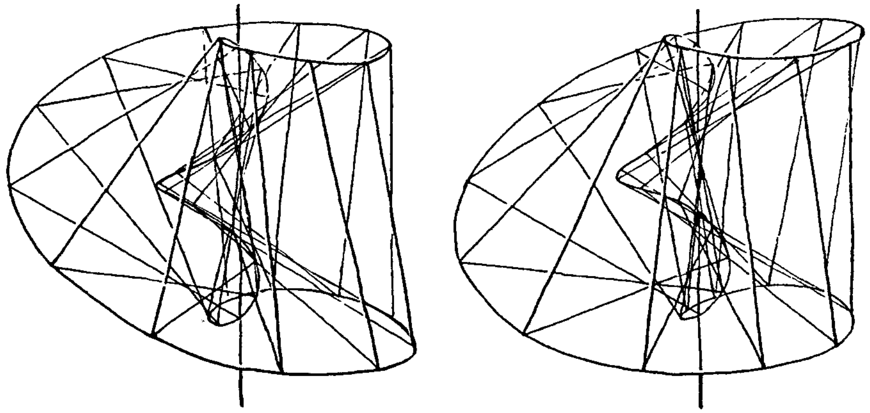


Figure 8

* * *

Mathematical Word Rebuses

by Charles W. Trigg

Complete the following equations with a mathematical term. Example: Word used with either + noise + devoured = *or din ate*

1. A color + slang word for gentleman =
2. ½ (Hawaiian lava) + daintily attractive =
3. Brazilian estuary + 39.37 inches =
4. $\sqrt{-1}$ + call for help + legs =
5. Netherlands meter + upper and lower mouth + logarithmic base =
6. Unscramble: Boron, Copper, Carbon, Iodine =
7. Gas from a muffler + gas in a light tube =
8. To arrange the hair + nominative me + Mother's Day flower without the automobile =
9. To musically utter + type of boat + Malaysian gibbon =
10. Sticky − Kentucky + knot =
11. Two pints + the result thereof =
12. Old witch + one + departed − epsilon =
13. Phonetic: Where the body lies + a flat plate + leg joint + you and me =
14. Mongrel dog + velvet − frown =
15. Company + 5 cent piece + abbreviated heat unit =
16. Into + golfing aid without excellence + marvelous =
17. First two letters + Ed Wynnie + musical instrument =
18. At home + ungrammatical galloping dominoes =

19. Egyptian sun god expires − plural suffix + the Golden State =
20. To squeal + two vowels =
21. Chinese weight + close =
22. Advance + electrified particle =
23. Phonetic reverse of not a single feline =
24. Most magnetic element + abbreviated advertisement =

(*Answers will be found on page* 19)

* * *

The Lost Chord

by W. H. Cozens

A chord which is allowed to go wandering round its circle on the loose can produce some fascinating effects on its travels.

Draw a circle and divide its circumference into any number of equal parts. The diagrams shown here, except Figure 6, are marked out with 10° spacing, making 36 parts. If you are the thorough type you may prefer 9° or 6° or even 5°, giving 40, 60 or 72 divisions. If on the other hand you believe that work is only a necessary evil, you may get away with 12°, 15° or even 18°, giving 30, 24 or 20 divisions.

First Excursion

Let a chord amble round the circumference - *the front end going twice as fast as the back end.* The arithmetically minded may like to number their divisions, as shown in Figure 1, and recite their two-times table as they join up 1-2, 2-4, 3-6, 4-8, etc. The geometrically inclined will be content to remember that every time the back end of the straightedge moves one space the front end moves two spaces.

A moment's thought now shows by the time the fast end has completed the circle, the slow end has travelled just half-way round (Figure 2). Meanwhile, although every line has been drawn with a straightedge, something uncommonly like a curve is emerging.

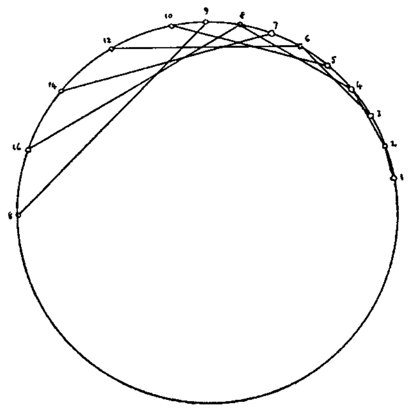


Figure 1

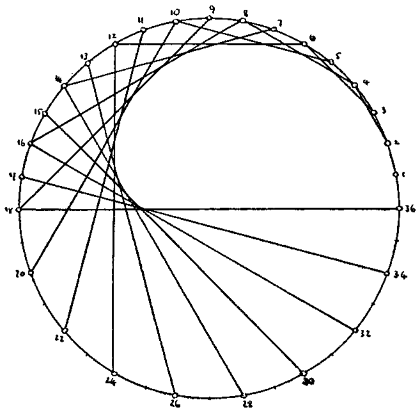


Figure 2

What now? If you work by arithmetic you will have to extend your numbering. If you had 36 divisions, mark number 1 becomes 37, number 2 becomes 38 and so on. But this is not essential: continue *exactly* as before, the front end - now on the right - going twice as fast as the back end. Then by the time the slow end has traversed its full circle the fast end will have completed its second circuit. They will in fact arrive at the tape together, the fast end having lapped its partner, and the result is shown in Figure 3 The elegant curve which our eyes insist is present was constructed - by entirely different methods - in the 17th century, and has been known since the 18th century as the cardiod, or heart curve.

To anyone with a spark of geometrical curiosity a large field of investigation now opens up. What if the leading end of the chord had travelled *three* times as fast as the other? Try it.

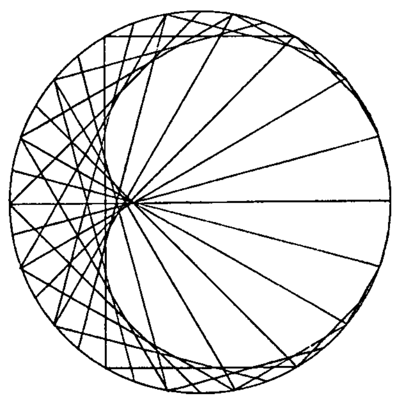


Figure 3

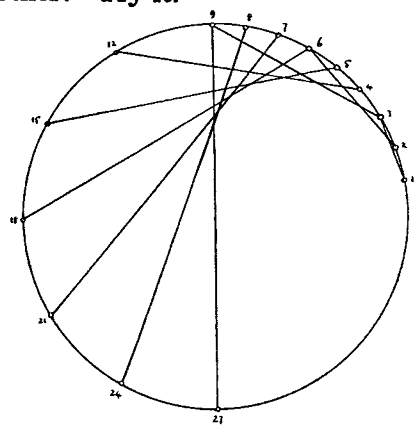


Figure 4

Second Excursion

When the slow end has covered one right-angle the fast end will have covered three right-angles, so that the chord will have become a diameter (just as in Figure 2) and as the chord must now begin to shorten again this must mark the position of a cusp (Figure 4). By the time the slow end has completed its circle the fast end will have done three laps, and a symmetrical two-cusped curve is visible. This one also was known in the 17th century; it is called the nephroid, or kidney curve (Figure 5).

These two results suggest that when the front end moves n times as fast as the back, a figure with $n-1$ cusps will be produced, and this is in fact true, as can easily be verified by experiment. We give one more example.

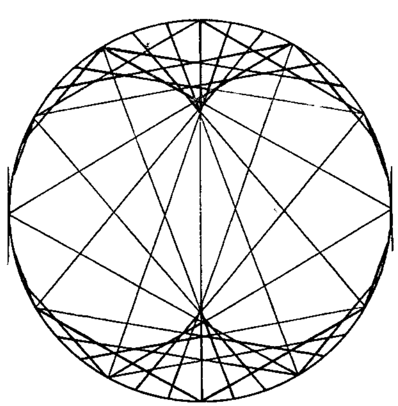


Figure 5

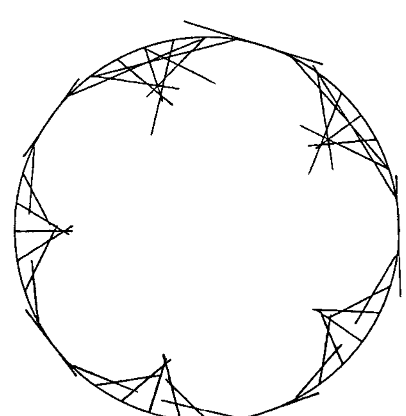


Figure 6

Third Excursion

Let the fast end move *six* times as fast as its trailing partner. As we expect five cusps the 36 divisions we have been using will not be very suitable. As 36 is not a multiple of 5 the cusps will not be precisely defined. We therefore switch to 9° spacing, which will give 40 divisions. For clarity we draw only the ends of the chords, omitting what would otherwise be a fearsome spider's web in the middle, as Figure 5 already showed. The result, shown in Figure 6, is a five-petalled curve which has no common name. It looks like a buttercup, so what about ranunculoid?

We hope to pursue other wanderings of the lost chord in a future article.

* * *

ANSWERS TO *MATHEMATICAL WORD REBUSES* ON PAGE 16

1. tan gent
2. a cute
3. Para meter
4. i sos celes
5. el lips e
6. Cu B I C
7. CO Ne
8. comb i (car)nation
9. sing u lar
10. tac(Ky) node
11. quart ic
12. hex a gon(e)
13. ex tra ne ous
14. cur ve(lour)
15. Co Ni cal
16. in te(e) grand
17. ab so lute
18. in dices
19. Ra di(es) Cal
20. rat io
21. li near
22. progress ion
23. (nary a cat) = cat e nary
24. Dy ad

A Committee Problem

by Brother U. Alfred, F.S.C.

The following is a popular type of committee problem of the puzzle variety: If there are six people and four committees and each person is to belong to two committees, while each committee is to have three members, but no two committees are to have more than one person in common, how are the six people to be distributed among the four committees? The brilliant and intuitive solution that is often held up for admiration is based on a simple geometric construction. Draw four lines forming a complete quadrilateral. If the lines represent committees and the points of intersection people, the problem has been solved.

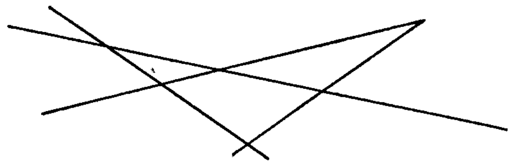


Figure 1. Solution of Simple Committee Problem

Unfortunately, it is difficult to conceive how this approach could lead to the discovery or solution of similar problems. For example, try to imagine the complex of lines and points that would solve the committee problem when there are ten committees and ten people.

While this particular geometric solution leads nowhere, it is interesting to note that a related structure in space opens up new avenues into the problem. The four lines and six points in the plane may be conceived as the cross-section of a space configuration derived from four non-coplanar points. These determine four planes and six lines, each plane containing three lines and two planes going through each line. This evidently satisfies the conditions of the committee problem.

Proceeding to five points in space, there would be ten planes and ten lines determined with each plane containing three lines and three planes passing through each line. The cross-section by a transversal plane would consist of ten lines and ten points of intersection, three points being on each line and three lines passing through each point. Moreover, each line cannot have more than one point in common with any other line without being identical with it. Thus, the corresponding committee problem would read: Given ten committees and ten people, arrange the committee membership so that each committee will have three members, each person will belong to three committees and no two committees will have more than one person in common.

The space arrangement provides us with a convenient method of solving this problem. Let the points in space be Q_1, Q_2, Q_3, Q_4,

Q_5. They will determine planes α_i as follows:

$\alpha_1(Q_1, Q_2, Q_3)$	$\alpha_6(Q_1, Q_4, Q_5)$
$\alpha_2(Q_1, Q_2, Q_4)$	$\alpha_7(Q_2, Q_3, Q_4)$
$\alpha_3(Q_1, Q_2, Q_5)$	$\alpha_8(Q_2, Q_3, Q_5)$
$\alpha_4(Q_1, Q_3, Q_4)$	$\alpha_9(Q_2, Q_4, Q_5)$
$\alpha_5(Q_1, Q_3, Q_5)$	$\alpha_{10}(Q_3, Q_4, Q_5)$

The lines formed will be:

$\mathrm{P}_1(Q_1, Q_2)$	$\mathrm{P}_6(Q_2, Q_4)$
$\mathrm{P}_2(Q_1, Q_3)$	$\mathrm{P}_7(Q_2, Q_5)$
$\mathrm{P}_3(Q_1, Q_4)$	$\mathrm{P}_8(Q_3, Q_4)$
$\mathrm{P}_4(Q_1, Q_5)$	$\mathrm{P}_9(Q_3, Q_5)$
$\mathrm{P}_5(Q_2, Q_3)$	$\mathrm{P}_{10}(Q_4, Q_5)$

Let p_i be the lines corresponding to the planes α_i when the configuration is cut by a transversal plane and P_i the points corresponding to the lines in space P_i.

This committee problem can be solved by noting which lines pass through which points and which points lie on which lines. These point and line relationships are:

$P_1(p_1, p_2, p_3)$	$p_1(P_1, P_2, P_5)$
$P_2(p_1, p_4, p_5)$	$p_2(P_1, P_3, P_6)$
$P_3(p_2, p_4, p_6)$	$p_3(P_1, P_4, P_7)$
$P_4(p_3, p_5, p_6)$	$p_4(P_2, P_3, P_8)$
$P_5(p_1, p_7, p_8)$	$p_5(P_2, P_4, P_9)$
$P_6(p_2, p_7, p_9)$	$p_6(P_3, P_4, P_{10})$
$P_7(p_3, p_8, p_9)$	$p_7(P_5, P_6, P_8)$
$P_8(p_4, p_7, p_{10})$	$p_8(P_5, P_7, P_9)$
$P_9(p_5, p_8, p_{10})$	$p_9(P_6, P_7, P_{10})$
$P_{10}, p_6, p_9, p_{10})$	$p_{10}(P_8, P_9, P_{10})$

If the lines represent committees and the points people, the above table shows which individuals belong to a given committee and the committees of which each person is a member. It is clear too that no two committees have more than one person in common.

If six points in space are employed, there are:

$$(6{\cdot}5{\cdot}4)/(1{\cdot}2{\cdot}3) = 20 \text{ planes} \quad \text{and} \quad (6{\cdot}5)/(1{\cdot}2) = 15 \text{ lines.}$$

The related committee problem is this: Distribute fifteen people among twenty committees so that each committee has three members, each person belongs to four committees and no two committees have more than one person in common.

It is interesting to note that a dual problem is possible if planes are taken to be people and lines committees. The problem then reads: Distribute twenty people among fifteen committees so that each committee has four members, each person belongs to three committees and no two committees have more than one person in common. This latter condition still maintains since if two planes (committee members) are to go through two lines (committees), they would have to be coincident.

This space derivation of committee problems has one evident limitation. If we consider the planes to be committees, then each committee will always have a limit of three committee members; if the planes are taken to be people, then each person will belong to but three committees.

If we hold to planes as committees and lines as people, the following table summarizes the type of committee problem that may be created by taking n points in space. In each case, there would be three members to a committee.

n	*Number of Committees*	*Number of People*	*Number of Committees per Person*
4	4	6	2
5	10	10	3
6	20	15	4
7	35	21	5
8	56	28	6
9	84	36	7
10	120	45	8
11	165	55	9

As mentioned previously, the committees and people can be interchanged, so that in the last instance, we might have 165 people to be allocated among 55 committees, nine people to a committee and each person belonging to three committees.

How to enlarge the scope of this problem? We might proceed to four-space in which case we would be able to have four people on each committee. At this point, it would be more in order to ascertain what it is that enables these space analogies to provide a frame on which the committee problem may be hung. The answer is quite simple. Essentially, we have n objects which are taken r at a time in the first instance and then $r-1$ at a time in the second. In other words, starting with n objects and forming the ${}_nC_r$ combinations of these objects, we have a set of quantities which correspond to committees; then the ${}_nC_{r-1}$ combinations of these objects will correspond to committee members. Obviously, there will be symmetry in the arrangement, each committee having the same number of members and each member belonging to the same number of committees. But the main point is that no committee will have more than one committee member in common with any other committee.

This may be seen as follows. Take one combination of r objects. If there are two combinations of $r-1$ objects whose elements are found in this combination of r objects, they must differ from each other in at least one element. But if they have $r-2$ elements in common and their other two elements different, this makes up precisely the r objects in the given combinations of r objects. If these two sets of $r-1$ objects were to belong as well to another set of r objects, then the two sets of r objects would have to be identical.

Committee problems of the type we have been considering, therefore, can be set up at will. If there are ${}_nC_r$ committees and ${}_nC_{r-1}$ committee members, how many people will belong to each committee and of how many committees will each person be a member? This question is answered by the following combinatorial relation:

$$k\,{}_nC_k = (n-k+1)\ {}_nC_{k-1}$$

Stated in words: Each committee has k members and each person belongs to $n-k+1$ committees; or since the holes may be interchanged, there might be $n-k+1$ members on each of ${}_nC_{k-1}$ committees and each of the ${}_nC_k$ people might belong to k committees.

The situation for several different arrangements is tabulated below.

FOUR PEOPLE TO A COMMITTEE

n	*Number of Committees*	*Number of People*	*Number of Committees per Person*
4	1	4	1
5	5	10	2
6	15	20	3
7	35	35	4
8	70	56	5
9	126	84	6
10	210	120	7
11	330	165	8
12	495	220	9

FIVE PEOPLE TO A COMMITTEE

n	*Number of Committees*	*Number of People*	*Number of Committees per Person*
5	1	5	1
6	6	15	2
7	21	35	3
8	56	70	4
9	126	126	5
10	252	210	6
11	462	330	7
12	792	495	8
13	1287	715	9

SIX PEOPLE TO A COMMITTEE

n	*Number of Committees*	*Number of People*	*Number of Committees per Person*
6	1	6	1
7	7	21	2
8	28	56	3
9	84	126	4
10	210	252	5
11	462	462	6
12	924	792	7
13	1716	1287	8

Opinion Please

There comes a time when one should take stock. This goes for magazines, too. The Editors of RMM would like to know exactly what RMM readers think of RMM. Here are some questions - send us the answers:

........What do you want more of? less of? Have some topics been too heavily dealt with? Have others been sadly neglected?

........What do you want that has not yet appeared? What do you *not* want that *has* appeared (and threatens to appear again)?

........The omission of mailing envelopes for U. S. subscribers (all foreign subscribers will continue to receive RMM in envelopes) has been taken as an economy measure and also as a time-saving device. What are your feelings about this?

........The lack of advertising has not been to the Editors' liking. Some subscribers are happy that RMM is not loaded down with ads - *but* advertising helps pay the bills! Subscriber payments cannot meet all the bills including the Editor's own bills (the Editor-Publisher devotes his full time to RMM). RMM is *intended* as a profit-making business. It is not subsidized by any Government or Organizational grants.

........We would like to have at least 500 readers send in their opinions and answers to the above questions. You do not need to include your name and address if you do not wish. But we do need a good sampling of opinion. If you have comments that are not covered by the above, don't hesitate to tell us. We cannot guarantee acknowledging all the letters, but we will *use* the comments in trying to produce a better RMM.

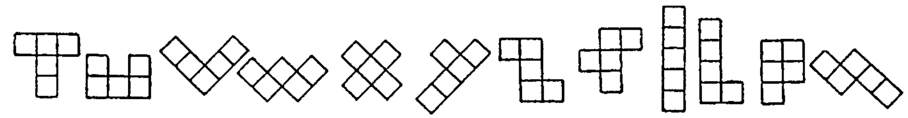
Please send all comments to the *KENT, OHIO* Editorial offices (see Table of Contents page, or the inside front cover for the full address).

So many RMM subscribers move with the school year that we wish to emphasize the importance of notifying us of your change of address *as soon as possible* before you actually move. Copies of RMM are generally *not* returned to us by the Post Office if a subscriber has moved. We only get a change of address notice from the Post Office and we must then mail another copy - a delay which can be irritating. When you send us a change of address, *please* send us your *old* address as well as your *new* address. If possible, send an old mailing label or at least the RMM IBM code number on each label.

Polyominoes - The "Twenty" Problem and Others

The June 1962 issue of RMM (pages 25-30) gave some conditions necessary for solutions to the "Twenty" problem:

Divide the twelve pentominoes into three groups of four each. Find a twenty-square region which each of the three groups will cover.


T U V W X Y Z F I L P N

The Twelve Pentominoes and Their Names

That article, by Jean H. Anderson, shows that there are two necessary, though not sufficient, conditions to be met for a solution to the "Twenty" problem to exist.

Condition 1: To be a solution to the "Twenty" problem, a region must hold each of tthe twelve pentominoes separately in such a way that the number of squares isolated by each piece is a multiple of five.

Condition 2: To be a solution to the "Twenty" problem, a region must cover at least eight but no more than twelve black squares of the checkerboard.

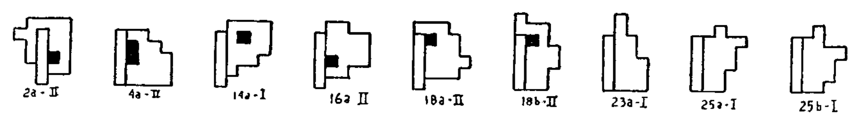
In connection with Condition 2, we find there are three groups of solutions:

Group I: Regions covering a maximum of 10 black squares and 10 white squares.

Group II: Regions covering a maximum of 11 black squares and 9 white squares.

Group III: Regions covering a maximum of 12 black squares and 8 white squares.

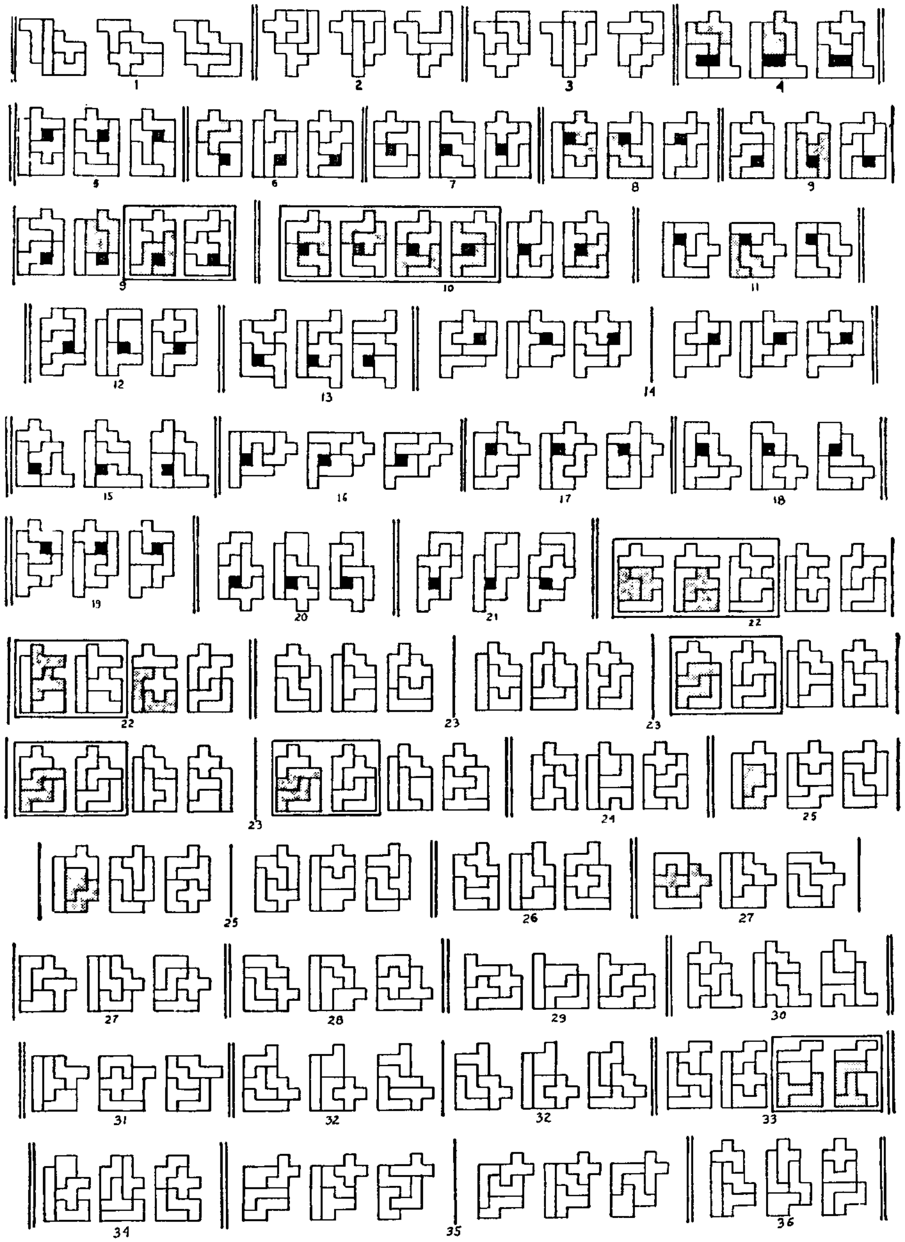
The June issue showed 54 regions, or shapes, which the solutions take. To those 54 regions must now be added nine more which have been found by Mrs. Anderson and Maurice Povah of England. They are shown in Figures 1 to 9 below. Figure 1 is labelled as Region 2a, Figure 2 as Region 4a, and the others so that the figures are placed in a logical position among the 54 regions given in the June issue of RMM (page 30). The proper grouping is also indicated.



Figures 1 *to* 9 *- New Regions for the "Twenty" Problem*

As explained in the June 1962 issue of RMM (page 25), all solutions are drawn in a standard orientation placing the I pentomino to the lower left as much as possible.

A total of 140 solutions are now known to the "Twenty" problem. These are shown in Figure 10 in the same order as the 54 Regions in the June issue. The nine new regions (shown in Figures 1 to 9) have a total of 16 solutions and two more solutions to Region 8 are added. Each set of solutions in a given region is separated by double lines. Different solutions within the same region are separated by single lines. Different arrangements of the same pieces in a given solution are "framed" in a rectangle. A different arrangement resulting from a simple rearrangement of two pieces is shown by shading the two pieces.



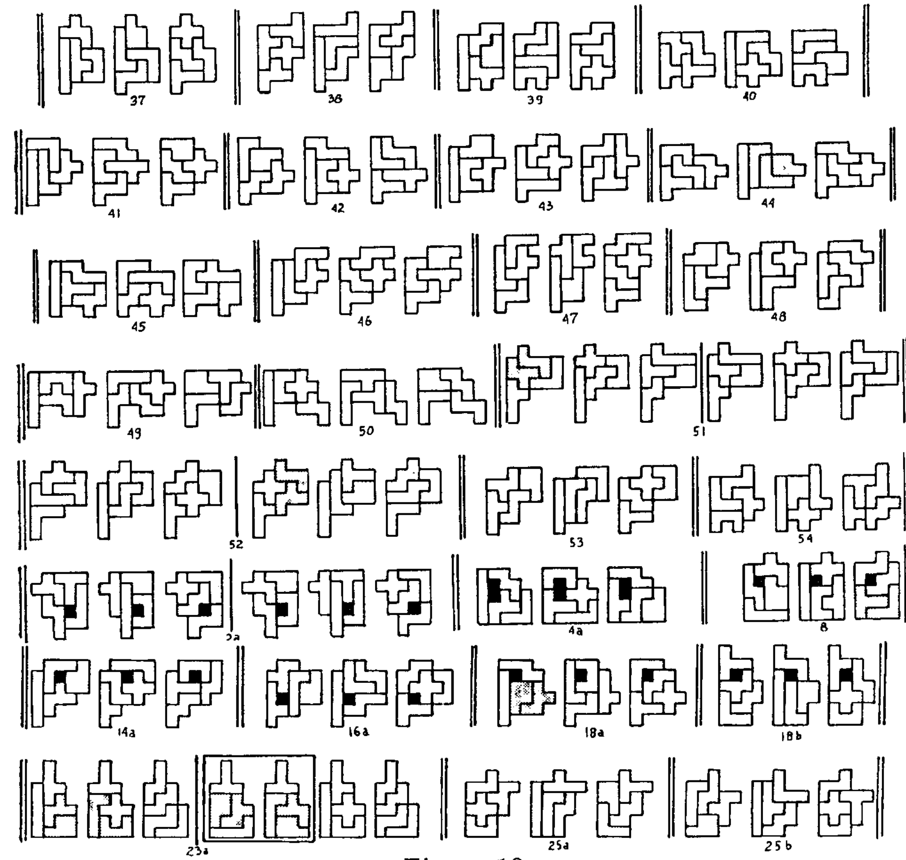


Figure 10

The Solutions to the "Twenty" Problem - 63 Regions

Considering each individual arrangement along with other members of each trio, the total of 140 solutions is reached. For example, Region 10 includes four arrangements in a "frame". With the other two members of the trio, this constitutes four solutions. However, the shaded portion in all of the "framed" arrangements and in one of the other members adds an additional ten solutions - a total of fourteen solutions for Region 10 alone.

Many of these solutions were submitted simultaneously by different readers. Here is a list of solutions (by Region numbers) submitted by readers. The Benjamin and Stead solutions had appeared in a 1954, or earlier, issue of the *Fairy Chess Review.*

Jean H. Anderson - Minneapolis, Minnesota: 1, 2, 3, 4, 4a, 5, 7, 10 (1 solution), 11 (1 solution), 12, 14 (1 solution), 15, 16, 17, 18a, 19, 22 (1 solution), 23 (3 solutions), 24, 25 (2 solutions), 26, 27 (3 solutions), 28, 29, 30, 31, 41, 42, 43, 44 (1 solution), 45, 46, 52 (2 solutions).

H. D. Benjamin (In Fairy Chess Review): 27 (2 solutions), 32.

Solomon W. Golomb - La Canada, California: 36.

Jack H. Halliburton - Los Angeles, California: 25 (1 solution)

Joseph S. Madachy - Kent, Ohio: 4 (2 solutions), 10 (2 solutions), 44 (1 solution), 52 (1 solution)

Maurice Povah - Lancashire, England: 2a, 6, 7, 8 (2 solutions), 9, 10, 11, 13, 14, 14a, 16a, 18b, 20, 21, 22, 23 (10 solutions), 23a, 24, 25 (1 solution), 25a, 25b, 27 (1 solution), 31, 32, 33, 34, 35, 37, 38, 41, 49, 50, 51, 53, 54.

Diane N. Schneider - Minneapolis, Minnesota: 18, 32, 35, 39, 40, 47, 48.

W. Stead (In *Fairy Chess Review*): 8, 25 (2 solutions).

* * *

James H. Schmerl of Piedmont, California submits a pentomino problem giving his six solutions known at this time. Can RMM readers find others? We shall call it the "Double Duplication" Problem:

Arrange each of two pairs of pentominoes in the same shape. With the remaining eight pieces construct a scale model, twice as long and twice as wide, of this shape.

Here are the solutions known at the present time.

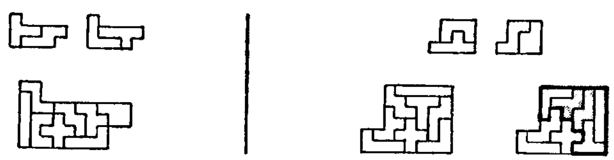


Figure 11

The heavy line in the third large set encloses a symmetrical section which can be rotated to give an additional solution. At the same time, the shaded section indicates that the two pieces can be rearranged within the shaded area - a total of four solutions with this set.

* * *

In later issues we shall show the 17 possible solutions to the "Ten" problem (see RMM No. 5, October 1961, pages 4-5: *Divide the twelve pentominoes into three groups of four each. Sub-divide each group into two pairs of pentominoes and for each group find a ten-square region which each of the two pairs will cover.*); a full set of solutions to the "Triplication" problem (see RMM No. 4, August 1961, page 9: *Given a pentomino, use nine of the others to construct a scale model three times as wide and three times as high as the given pentomino.*); and some miscellaneous new configurations to several other pentomino problems.

? ? ?

by Robert Abbott

Would anyone be interested in experimenting with a new game?

We are developing a game that so far has shown great promise of rivalling chess in interest. The game has yielded complex and intriguing strategies; and we feel these strategies are only part of the surface, that further play will reveal deeper and more intricate strategies.

However, since it has been difficult to recruit players of the right ability and interest to follow the game through any extensive play, the game in its present form has not received an adequate amount of testing. Thus there is the possibility that further play will yield not deeper strategy but instead will reveal a perfect strategy which would always obtain a win or draw for the player that follows it. This of course would make the game trivial, unless it were possible to change certain of the moves or captures.

For this reason we thought some of the readers of RMM would like to try out the game. It is the sort of game you would probably find interesting and, if any of you find yourselves playing it extensively, perhaps you might want to write us your experiences with it or write any suggestions you would have for changes in rules. You probably have noticed that one thing this game lacks is a name; so we are especially interested in suggestions for this.*

Our game uses the regular chess board and pieces, but it shouldn't be thought of as strictly a fairy chess, as it is quite far from chess. Nor is it like any of the "simplified" chess games you may have seen on the market. These games may have rules that are simpler than chess, but their play often seems arbitrary and confused. Although the rules for our game are complex, after it is played once or twice it is possible to see quite clearly what is going on and thus to develop strategies that involve several moves ahead of time. There is, unfortunately, some opportunity for sneak attacks and blunders, but you should find these reduced as experience is gained — just as in chess.

The pieces are set up exactly the same way as for chess except each player should place a piece of tape on the rook on his left or in some way mark it to distinguish it from the other rook. This marked rook is called the "immobilizer" in the game. The knights are called "long-leapers," the bishops are called "chameleons," the queen is called the "withdrawer," the king is still called "king" and the rook on the right, the one that is not marked, is called the "coordinator". The pawns are still called "pawns" for want of a new name.

Our game is to a large degree eclectic, and its basic idea came about through readings in the history of games. It can be noted that there are very many different forms of captures used in the various games throughout history. However, each particular game uses only one form of capture. Chess, for instance, has different moves for its pieces, but each piece captures in the same fashion, by moving onto the same space as the piece it is capturing. Checkers has one form of the

**Editor's Footnote*: Two FREE years of RMM (new subscription or extension of present subscription) to the one who suggests the name eventually chosen by Mr. Abbott. In case of duplicate entries, the first entry received will be considered.

capture, the short leap, even though the men and the kings have different powers of movement.

The basic concept behind our game was to construct a game that used several different forms of capture. Thus each of the seven different pieces uses a different form of capture. There are, however, only three different powers of movement in the game.

With this in mind I can proceed to an explanation of the movements and means of capture of each piece.

KING: The king moves and captures in exactly the same fashion as the chess king. The object of the game is to capture this king. The same rules for declaring check apply as in chess.

PAWNS: The pawns can move any number of unoccupied squares orthogonally (horizontally or vertically, but not diagonally). Thus their power of movement is the same as the rook in chess. The pawns use a form of the "interception" capture. This is the oldest form of capture found in war games, predating the replacement capture of chess games and the leap of alquerque and checkers. Interception was the form of capture used in ancient games such as the Saxon Hnefatafl and the Roman game Latrunculi.

If a pawn moves onto a square that is *orthogonally* next to an enemy piece, and if there is a friendly piece (*any* friendly piece, not necessarily another pawn) on the other side of that enemy piece, then the enemy piece is captured and removed from the board.

As an example, if the pawn in Figure 1 moves up to the head of the arrow, it captures the enemy withdrawer. (In these figures the friendly pieces are represented by letters and the enemy pieces by letters enclosed in circles. The pieces are represented by the first letter of their name, except for the chameleon which is arbitrarily designated by "S" to distinguish it from the coordinator, "C".)

To use this form of capture, however, the piece that does the moving must be the pawn. If the long-leaper in Figure 2 moves to the head of the arrow, it would not capture the immobilizer, even though the immobilizer is now between the pawn and the long-leaper. The long-leaper uses a different form of capture, and in this game, it is the piece that does the moving that determines what form of capture may be used.

A pawn may capture more than one piece in a move. If the pawn in Figure 3 moves to the head of the arrow, it captures three enemy pieces, the withdrawer, the coordinator and the chameleon. It does not capture the enemy long-leaper, since it has not moved to the square orthogonally next to it.

A piece may move to the square between two enemy pawns without fear of being captured by them on the enemy's next turn.

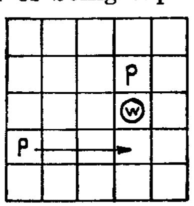


Figure 1

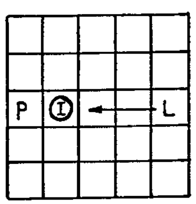


Figure 2

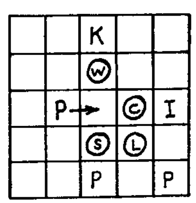


Figure 3

LONG-LEAPER: This piece is named after its capture, a variation of the long leap which is found in Polish and Spanish checkers. The long-leaper may move any number of unoccupied squares orthogonally or diagonally (as the queen's move in chess). In addition, if it can approach an enemy piece by a legal move, and if the next square beyond the enemy is vacant, it can leap over the enemy piece to that vacant square. The piece lept over is then captured.

The long-leaper in Figure 4 could capture the enemy coordinator by leaping over it to space 1; it could capture the immobilizer by leaping over it to space 2; or it could capture the withdrawer by leaping over it to space 3. This last move might be called a short leap, but this is also a capture the piece can perform. The long-leaper in the figure cannot capture any of the enemy pawns, since there are no vacant spaces on the other sides of them.

A long-leaper can capture only one enemy piece in a turn and it cannot leap over friendly pieces.

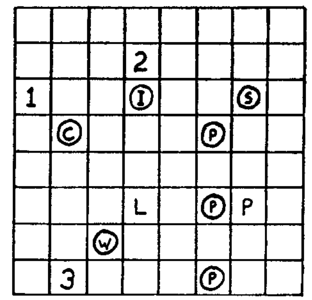


Figure 4

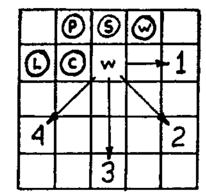


Figure 5

WITHDRAWER: The withdrawer can move any number of unoccupied squares in an orthogonal or diagonal direction. It can capture a piece next to it by moving any number of unoccupied squares directly away from that piece.

Thus the withdrawer in Figure 5 could capture the coordinator by moving any number of squares along the arrow marked 1, or it could capture the pawn by moving along the arrow marked 2, or it could capture the chameleon if it moved along the arrow marked 3, or it could capture the enemy withdrawer by moving along arrow 4.

This withdrawal capture was found in the description of a Madagascan game called Fanorona. I have not seen it in any other game.

COORDINATOR: The coordinator uses an original capture not found in other known games. The coordinator can move any number of unoccupied squares in an orthogonal or diagonal direction. When it finishes its move, it captures any enemy piece that is on an intersection of the orthogonal lines that pass through the coordinator and through the friendly king.

In Figure 6, if the coordinator moves to the head of the arrow, it captures the enemy pawn; for this pawn is on the intersection of the vertical dotted line, which passes through the king, and the horizontal dotted line, which passes through the point where the coordinator finished its move.

A coordinator can also capture two pieces in a single move. In Figure 7, if the coordinator moves up one space, it captures the enemy chameleon and long-leaper. The orthogonal lines that run through the king and through the coordinator at the end of its move are drawn in the diagram.

Moving the king, instead of the coordinator, does not effect a capture by the coordinator even if enemy pieces would then be on the intersection of coordinate lines. The coordinator must be the piece that moves if you wish to effect the coordinate capture.

It might appear difficult to anticipate an attack from the coordinator. However, if a player watches what pieces he has on a line with the enemy king, he will be able to see which are vulnerable to the coordinator.

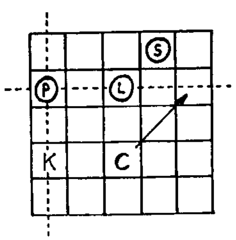


Figure 6

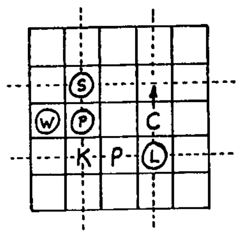


Figure 7

IMMOBILIZER: The dread immobilizer is another original sort of piece. The immobilizer does not capture its victims, for it does not remove them from the board, but instead paralyzes any piece it is next to in an orthogonal or diagonal direction. Any enemy piece the immobilizer moves next to, or any enemy piece that moves next to the immobilizer, loses its powers of movement. The powers of movement, however, are restored if the immobilizer moves away or is captured. The immobilizer can move any number of unoccupied squares in an orthogonal or diagonal direction.

In Figure 8 the immobilizer moves to the head of the arrow and paralyzes the enemy king, withdrawer, coordinator and three pawns. However, by this same move it frees the enemy long-leaper which had previously been immobilized. It is not necessary to announce check before immobilizing a king.

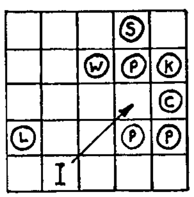


Figure 8

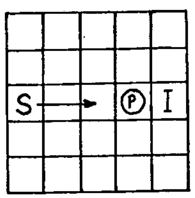


Figure 9

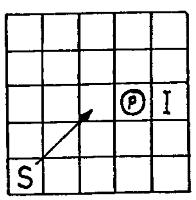


Figure 10

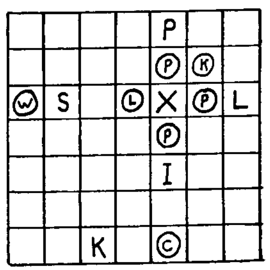


Figure 11

A piece may move past an enemy immobilizer without being paralyzed, but if it finishes its move on a square next to the immobilizer it loses its power of movement. If the two immobilizers come together, they immobilize each other as well as any enemy pieces in contact. Neither immobilizer thus can move unless the other is captured.

CHAMELEON: The purpose of the chameleon can be simply stated. It does to pieces what they do to other pieces. It captures an enemy piece in the manner that the enemy piece captures.

When it is not capturing, it may move any number of squares in an orthogonal or diagonal line. However, when it captures it can move only in a way that the piece it is capturing could move. Thus the chameleon in Figure 9 could move to the head of the arrow and capture the pawn by interception, the pawns' method of capture. However, the chameleon in Figure 10 could move to the head of the arrow, but it could not capture the pawn since it has moved diagonally, a move that a pawn can't make.

A chameleon can do many things in a single move. In Figure 11 the chameleon leaps over the long-leaper to the spot marked "x." It thus captures the long-leaper; it captures the withdrawer (by withdrawal); it captures the coordinator by placing it on the intersection of the orthogonal lines through the chameleon and the friendly king; it captures the three pawns by interception; and it gives check to the king. A chameleon can give check to a king by moving onto the space next to the king, for on the next move it could act as a king and capture the king.

A chameleon can immobilize an immobilizer by moving next to it, or an immobilizer becomes paralyzed if it moves next to a chameleon. In either case both pieces become paralyzed and neither can move unless the other is captured. The immobilizer in this case continues to paralyze any other enemy piece next to it, although the chameleon of course lacks this ability.

A chameleon cannot capture another chameleon.

There is one special move in this game, that of suicide. A player may use a turn to remove from the board one of his own pieces (except his king) that is immobilized. This is sometimes a valuable move since it may clear the way for an attack on the immobilizer. However, a player cannot return a piece to the board which he has removed in the suicide move.

This completes the description of the game. If any points are not clear, the best procedure would be to follow whatever interpretation of the rule you think works best in the situation. We would appreciate any report of situations where the above rules do not make it clear as to what should be done.

You may recall that the April 1962 issue of RMM, page 42, reprinted the card game Babel from a book I privately published called *Four New Card Games* (which incidentally is no longer available). The four games, plus about three additional card games, will be included in a book *Abbott's Card Games* to be published next Spring by Stein and

Day Publishers. Also included will be the game described here, even though it is not a card game. We would therefore welcome any suggestions for naming the game, improvement of the game and points of strategy. We are especially interested in discovering the minimum number of pieces needed for checkmate.

Please address all correspondence - especially game-naming efforts to obtain the free two years of RMM - to the writer:

Robert Abbott
Box 1861 - General Post Office
New York 1, New York

* * *

Puzzles and Problems

1. *More Number Curiosities*

H. E. Dudeney, in his *Amusements in Mathematics,* states that the lowest square number containing the nine digits (excluding 0) is 139,854,276 ($11{,}826^2$) and the highest is 923,187,456 ($30{,}384^2$).

What is the lowest and highest square numbers containing the nine digits (excluding 0) exactly twice? Exactly three times? Exactly four times? Etc.?

(*Donald L. Vanderpool;* Towanda, Pa.)

2. *Another Balance Scale Problem - Eight Coins*

The balance scale problem was analyzed fully by J. A. H. Hunter in RMM No. 3, June 1961, page 44. Here is a more difficult variation on the theme.

We have eight coins identical in appearance, but each is slightly different in weight from any other. A balance scale, without weights, is available and we may mark the coins as desired.

How can we determine the order of weights of the coins, from heaviest to lightest, in the least number of weighings? For those of us with extreme patience, we might try to determine the least number of weighings required to determine the order of weight of N coins, each differing in weight.

(*Kobon Fujimura;* Osaka, Japan)

3. *Publisher's Dilemma*

A magazine publisher has sold 151 subscribtions to his magazine. Four types of subscriptions have been sold: one-year at \$2.25, two-year at \$4.00, three-year at \$6.00 and trial subscriptions at \$0.95. The publisher knows that there are more trial subscriptions sold than one-year subscriptions, more one-year than two-year subscriptions, more two-year than three-year subscriptions, and that there are less than 10 of the three-year subscriptions. The total income from the sales is \$308.95.

What is the number of each type of subscription, assuming the largest possible number of trial subscriptions, owing to the fact that it was a special sales price?

(*Paul Nemecek;* Riverside, Illinois)

4. *Trouble at the Trestle*

An express train is travelling toward a trestle at 45 miles per hour. Mr. Smith, our accountant friend, out for a walk, is walking across this same trestle and is exactly two-thirds of the way across when he sees the train coming toward him. Always the mathematician - regardless of circumstances - he calculates that if he runs as fast as he can the remaining one-third of the way he'll just make it. But he also figures that if he runs back the long way, he'll also just make it.

How fast can Mr. Smith run?

(*Harvey Hahn;* Valparaiso, Indiana)

5. *Problem of the Generations*

Two grandfathers were talking and one observed that "You have a large family I understand. I only have two children myself, and now they each have two children also."

The other grandfather thought about it and replied "Yes, but each of my children has one less child in his or her family than were in my family."

The first grandfather then did some mental calculation and commented, "You know, if the trend continues in your family for the number of children per family to be one less in each succeeding generation, then in just three generations of your family you will have five descendants more than I shall have if my descendants each have two children for as many generations as you have children."

Of how many children was the first grandfather the father?

(*F. Gross;* London, England)

6. *Toy Bricks*

A professor of mathematics was visiting the home of some friends, when their five-year-old son, Tim, came into the room.

"Professor," said Tim, "would you like to see my bricks?"

The professor assented and followed Tim into the nursery. The bricks were scattered all over the floor.

"You *have* got a lot," the professor remarked, "Do you know exactly how many?"

"Yes," replied Tim, "but I'm not saying how many, though I've over two thousand. You may also like to know that I've enough to make two unequal piles of bricks which are perfect cubes."

"Using all the bricks to make the piles?" inquired the professor.

"Yes," said the boy. "And do you know? By using all the bricks again, I can make two more unequal piles which are also perfect cubes, but I can't make a third pair of unequal piles which are perfect cubes."

"Very interesting," commented the professor. "You certainly have less than ten thousand bricks and they are all perfect cubes. I can tell you how many bricks you have, you know."

The professor was correct, so how many bricks did Tim have?

(*N. A. Longmore;* Jerez de la Frontera, Spain)

7. *The Farmer's Fields*

A farmer owned three small fields each of which was in the form of a right triangle. Asked what their areas were, he replied that each field was the same number of square yards, but added that all nine sides were of different lengths, though all in whole yards. He also vouchsafed the interesting information that had the triangles been of smaller area it would not have been possible to have *three* such fields.

What were the areas of the three fields?

(*N. A. Longmore;* Jerez de la Frontera, Spain)

8. *Simplified Multiplication*

Johnny's teacher was showing the class the peculiarities of certain numbers.

"Take 1,012,658,227,848 for example," he said. "To multiply it by eight there is no need to go through the normal process of multiplying each digit. You simply transfer the digit on the extreme right to the extreme left."

The boys were most intrigued and wished that all numbers were as easy to multiply.

"Now boys," the teacher went on, "there is another thirteen digit number which has similar properties, that is, to multiply by eight you simply move the digit on the extreme right to the extreme left. Your assignment is to find that other number."

Johnny thought that this was more a problem for his father who, for some strange reason, liked mathematical puzzles. After much head-scratching, not to mention numerous cups of coffee, Johnny's father found the answer.

He then told Johnny to ask his teacher to find other numbers which are as easily multiplied by some number - that is, in which multiplication can be performed by moving digits from one end to the other. He also told Johnny to tell his teacher that there was a 28-digit number which can be multiplied by 3 by merely moving the last digit to the front.

(*N. A. Longmore;* and other RMM readers)

Numbers, Numbers, Numbers

J. A. H. Hunter, our Associate Editor, gave some examples of integers equal to the cubes of the sums of their digits (RMM No. 5, October 1961, page 53):

$$4913 = (4+9+1+3)^3 = 17^3$$
$$5832 = (5+8+3+2)^3 = 18^3$$

and, similarly $17576 = 26^3$ and $19683 = 27^3$

Several readers pointed out that Mr. Hunter had omitted the only other case involving cubes: $512 = (5+1+2)^3 = 8^3$.

Malcolm H. Tallman extended the idea using higher powers (RMM No. 6, December 1961, page 44):

$$2401 = (2+4+0+1)^4 = 7^4$$
$$234256 = (2+3+4+2+5+6)^4 = 22^4$$

and others.

The results from various RMM readers have been gathered together and here is presented a table of such numbers compiled by J. A. H. Hunter of Toronto, Ontario, Andrzej Makowski of Warsaw, Poland, Alan Sutcliffe of Yorkshire, England, Malcolm H. Tallman of Brooklyn, New York, H. E. Tester of Middlesex, England, and Donald Vanderpool of Towanda, Pennsylvania.

The general form of these number relationships is

$$abcde \ldots = (a+b+c+d+e+ \ldots)^n = P^n$$

where $abcde \ldots$ is an integer equal to P^n and $(a+b+c+d+e+ \ldots)^n$ is thé sum of the digits of $abcde \ldots$ raised to the nth power. We omit the $(a+b+c+d+e+ \ldots)^n$ portion to conserve space.

$$512 = 8^3 \qquad 2401 = 7^4 \qquad 17210368 = 28^5$$
$$4913 = 17^3 \qquad 234256 = 22^4 \qquad 52521875 = 35^5$$
$$5832 = 18^3 \qquad 390625 = 25^4 \qquad 60466176 = 36^5$$
$$17576 = 26^3 \qquad 614656 = 28^4 \qquad 205962976 = 46^5$$
$$19683 = 27^3 \qquad 1679616 = 36^4$$

$$34012224 = 18^6 \qquad 271818611107 = 43^7$$
$$8303765625 = 45^6 \qquad 1174711139837 = 53^7$$
$$24794911296 = 54^6 \qquad 2207984167552 = 58^7$$
$$68719476736 = 64^6 \qquad 6722988818432 = 68^7$$

$$612220032 = 18^7$$
$$10460353203 = 27^7 \qquad 20047612231936 = 46^8$$
$$27512614111 = 31^7 \qquad 72301961339136 = 54^8$$
$$52523350144 = 34^7 \qquad 248155480267521 = 63^8$$

$$81\ 92000\ 00000\ 00000 = 20^{13}$$
$$12157\ 66545\ 90569\ 28801\ 00000\ 00000\ 00000\ 00000 = 90^{20}$$
$$20\ 86444\ 84729\ 75628\ 94722\ 60059\ 81267\ 19444\ 70425\ 84001 = 207^{20}$$

Alan Sutcliffe also notes that, in the binary scale:

$101001 = (1+0+1+0+0+1)^{100} = 11^{100}$ [equivalent to $81 = (8+1)^2 = 9^2$]

Mr. Sutcliffe also adds that a more difficult variation is to find integers equal to the sums of the nth powers of each of their digits:

$$153 = 1^3+5^3+3^3$$
$$370 = 3^3+7^3+0^3$$
$$371 = 3^3+7^3+1^3$$
$$407 = 4^3+0^3+7^3$$

The higher powers, we leave to RMM readers.

For number relationships similar to those given we refer to the answer to problem 8 in the June 1962 issue of RMM (see page 42 of this issue).

* * * * *

PRINTER'S "ERRORS"

Donald L. Vanderpool of Towanda, Pa., brings our attention to H. E. Dudeney's *Amusements in Mathematics* (Dover Publications). For problem 115 Dudeney shows a printer's error which turns out to be correct: $2^5 \cdot 9^2 = 2592$. Mr. Vanderpool worked out a method and submits the following printer's "errors":

$$2^5 \cdot {}^{25}/_{31} = 25^{25}/_{31} \qquad 13^2 \cdot 7^6/_7 = 1327^6/_7$$
$$2^6 \cdot {}^{26}/_{63} = 26^{26}/_{63} \qquad 16^2 \cdot 6^6/_{17} = 1626^6/_{17}$$
$$12^2 \cdot {}^{122}/_{143} = 122^{122}/_{143} \qquad 19^2 \cdot 5^1/_3 = 1925^1/_3$$
$$100^2 \cdot {}^{334}/_{3333} = 1002^{334}/_{3333} \qquad 21^2 \cdot 4^9/_{11} = 2124^9/_{11}$$
$$11^2 \cdot 9^1/_3 = 1129^1/_3 \qquad 29^2 \cdot 3^{10}/_{21} = 2923^{10}/_{21}$$
$$2^6 \cdot 412698^{26}/_{63} = 26412698^{26}/_{63}$$

Some of these have periodic forms which can generate an infinite number of printer's "errors":

$$13^2 \cdot 7^6/_7 = 1327^6/_7$$
$$13^2 \cdot 7857142^6/_7 = 1327857142^6/_7$$
$$13^2 \cdot 7857142857142^6/_7 = 1327857142857142^6/_7$$

etc.

$$11^2 \cdot 9^1/_3 = 1129^1/_3$$
$$11^2 \cdot 93^1/_3 = 11293^1/_3$$
$$11^2 \cdot 933^1/_3 = 112933^1/_3$$

etc.

However, the more interesting printer's "errors" are those involving integers rather than fractions. Mr. Vanderpool gives us the following:

$$2^5 \cdot 9^2 = 2592 \text{ (Dudeney's)}$$
$$3^4 \cdot 425 = 34{,}425$$
$$31^2 \cdot 325 = 312{,}325$$

These last two generate an infinite number of "errors":

$$3^4 \cdot 4250 = 344250$$
$$3^4 \cdot 42500 = 3442500$$

etc.

and

$$31^2 \cdot 3250 = 3123250$$
$$31^2 \cdot 32500 = 31232500$$

etc.

Letters to the Editor

Dear Sir:

In RMM No. 9, June 1962, page 24, Amos Voil presented a set of *Square Reversals* - numbers such that the squares of their reverses equal the reverses of their squares. (e.g. $12^2 = 144$ and $441 = 21^2$)

The key to such numbers is that there is no "carrying": no result greater than 10, in either the digit by digit multiplications or in the addition of the partial products column by column. The multiplication condition implies that no digit greater than 4 can appear in the "number". But, for a more effective criterion, we must consider the addition condition.

In adding the partial products we find, in general, that the middle column contains the most digits. That the sum of this middle column be less than 10 is a necessary condition, but I conjecture that it is also a sufficient condition. This sum can be found without carrying out the complete squaring, by multiplying each digit by the corresponding digit equidistant from the other end of the number, and summing the products.

This criterion indicates, as supported by experience, that there can be no solution containing both 2's and 3's, and none containing more than one 3. More important, it is seen that from any solution we can obtain others by inserting any number of 0's anywhere within the number (or at either end, if numbers such as 0012 are accepted). (e.g. $102^2 = 10404$ and $40401 = 201^2$)

A palindromic number is not necessarily a solution: its square must also be a palindrome. My list of basic solutions without the insertion of 0's is: 1, 11, . . . 111,111,111; 2, 22; 3; 122, 212; 1212; 13, 113, 1113, 11,113; and any permutations of one 2 and from one to five 1's. (e.g. $1121^2 = 1256641$ and $1466521 = 1211^2$; $121111^2 = 14667874321$ and $12347876641 = 111121^2$)

I would be glad to hear from anyone who has found other solutions, or who has further information on the problem.

Assumption Abbey
Richardton, North Dakota

Fr. Victor Feser, O.S.B.

* * * * *

Dear Mr. Madachy:

On page 45 of the December 1961 issue of RMM you have conjectured that the equation

$$1+x+x^2+x^3+\ .\ .\ .\ +x^n = y^2$$

has no solution with $n < 10$ except the two given on the same page ($1+3+9+27+81 = 11^2$ and $1+7+49+343 = 20^2$). In fact the stronger theorem is true. W. Ljunggren proved in *Norsk Matematisk Tidsskrift*, 25(1943), p. 17-20, that the only solutions of the equation in integers x, n, y, all >1, are $x = 3$, $n = 4$, $y = 11$; $x = 7$, $n = 3$, $y = 20$. I think this reference may be of interest to RMM readers.

Warsaw, Poland

Andrzej Makowski

ANSWERS FOR THE JUNE 1962 *ISSUE OF RECREATIONAL MATHEMATICS MAGAZINE*

The list of puzzle solvers will be found on Cover III.

MAGIC CROSS-NUMBER PUZZLE (Page 11 - June 1962 RMM)

The answers are shown at the right.

^{1}2	^{2}2	3	■	^{3}3	■	^{4}4	^{5}7	^{6}7
^{7}3	3	■	^{8}9	8	1	■	^{9}2	2
^{10}7	3	^{11}3	■	3	■	^{12}3	5	4
9	■	^{13}8	^{14}3	■	^{15}4	1	■	3
■	^{16}4	4	7	1	9	2	^{17}1	■
^{18}2	2	4	1	■	^{19}6	2	5	^{20}6
^{21}3	3	4	■	^{22}5	■	^{23}7	3	3
^{24}1	5	■	^{25}8	4	8	■	^{26}1	1
^{27}1	6	2	■	4	■	^{28}9	4	2

ALPHAMETICS (Page 19 - June 1962 RMM)

(1) ALAS LASS NO MORE CASH = 5157 1577 38 2804 9576*

(2) AIR AIR WARMER = 491 491 241081**

(3) HIGH DIVING ACT = 2102 917150 436

(4) FIRS FOR THE BIRDS = 9063 976 584 10623

* CASH was to have been as large as possible.

** Many RMM readers turned in 496 496 246016 as an answer to this alphametic. We refer them to the *actual* set-up shown in the June 1962 issue which will point out their error.

PUZZLES AND PROBLEMS (Pages 48-49 - June 1962 RMM)

1. *Let's Play the Numbers Games* Many, many answers were turned in. We cannot give even one expression for the integers from 1 to 1000, but here are two of the integers listed as having escaped solution:

$$914 = 26 \cdot 35 + 4 \qquad\qquad 987 = 5^4 + 362$$

2. *A Bag of Coins*: 17 coins were lost.

3. *Fallout Shelter*: The maximum number of sides, all triangular in shape, appears to be 37.

4. *A Problem in Confusion*: (Correctly stated!) Bert was 72, Ben was 80 and Bill was 85.

5. *The* 15-*Coin Puzzle*: We are to determine the fake coin among 14 other genuine coins and determine whether it is heavier or lighter, using only a weighing machine with no weights supplied.

Mark the 15 coins A, B, C, D, E, F, G, H, I, J, K, L, M, N, O.

1st Weighing: $A+B+C+D+E+F+G+H = w_1$ say
2nd Weighing: $E+F+G+H+I+J+K+L = w_2$ say

Case I: $w_1 \neq w_2$

3rd Weighing: $A+B+I+J = w_3$ leading to:

(1) $w_1 = 2w_3$ Fake is K or L.
4th Weighing: $K = w_4$
If $w_1 \neq 8w_4$ Fake is *K*, weight is w_4
If $w_1 = 8w_4$ Fake is L, weight is $(w_2 - 7w_4)$

(2) $w_2 = 2w_3$ Fake is C or D.
4th Weighing: Repeat procedure in (1)

(3) $w_1 = 2(w_2 - w_3)$ Fake is I or J. 4th Weighing, repeat (1).

(4) $w_2 = 2(w_1 - w_3)$ Fake is A or B. 4th Weighing, repeat (1).

Case II: $w_1 = w_2$

3rd Weighing: $A+E+F+M+N = w_3$
4th Weighing: $E+G+M+O = w_4$

Now use the following weight relationships:

$$q = w_1/8 \qquad r = w_3/5 \qquad s = w_4/4 \qquad t = (w_1 - w_3)/3$$
$$u = (w_1 - w_4)/4 \qquad v = w_3 - w_4$$

Then, if:

$t = u = v$, the fake is E, weight is $w_3 + w_4 - w_1$
$s = t$, the fake is F, weight is $w_3 - w_4$
$r = u$, the fake is G, weight is $w_1 - (7/5)(w_3)$
$r = s = v$, the fake is H, weight is $w_1 + 2w_4 - 3w_3$
$q = v$, the fake is M, weight is $4w_4 - 3w_3$
$q = s = u$, the fake is N, weight is $w_3 - w_4$
$q = r = t$, the fake is O, weight is $w_3 + w_4 - w_1$

6. *All Boys*: The solution hinges on three clues: (1) Andy knew the house number, but still could not be sure about its 4-factor breakdown. Hence, it could be expressed as a 4-factor product in more than one way within limitations on the sum of the factors. (2) At least one 4-factor breakdown must have included unity as a factor, from Andy's question. (3) Ted's reply, whether "Yes" or "No" removed all doubt.

Now, $3+4+5+6 = 18$, so the smallest factor in any acceptable breakdown must have been 1 or 2. A little thought will show that at least one acceptable breakdown must have had 2 as its minimum factor. $2 \cdot 4 \cdot 5 \cdot 6 = 240$, so the house number $\leqq 240$.

Tabulate the 7 possible groupings that include 2 as a factor, and for each corresponding house number seek groupings that include 1 as a factor.

We derive house numbers 240, 210, 180, 192, 168, 144 and 120 having 2 as a factor: of these, only 120 has acceptable factors that include 1, i.e. $1 \cdot 3 \cdot 5 \cdot 8$ and $1 \cdot 4 \cdot 5 \cdot 6$.

The house number, then, must have been 120. Ted must have replied "No," and the families must have comprised 2, 3, 4, and 5 sons.

7. *Another Cube Cutting Problem*: There was too little time to prepare the drawings of some of the interesting solutions to this problem. We will gather them together for the December 1962 issue of RMM.

8. *Curious Number Relationships*: Many thanks to the readers who supplied various forms. We refer readers to the *Numbers, Numbers, Numbers* department in this issue. Here are a few forms supplied by RMM readers: (We want more!)

$1!+4!+5! = 145$ (H. V. Gosling - as supplied in the June issue)

$1^1+3^2+5^3 = 135$	(Dale Kozniuk)
$5^1+9^2+8^3 = 598$	(Dale Kozniuk)
$12^2+33^2 = 1233$	(Jerry Roger)
$1^4+6^4+3^4+4^4 = 1634$	(Alan Bostrum)
$4^5+1^5+5^5+0^5 = 4150$	(Alan Bostrum)
$4^5+1^5+5^5+1^5 = 4151$	(Alan Bostrum)
$8^4+2^4+0^4+8^4 = 8208$	(Alan Bostrum)
$9^4+4^4+7^4+4^4 = 9474$	(Alan Bostrum)
3^3(base-8) $= 33$(base-8)	(Thaddeus C. Jones)
2^4(base-6) $= 24$(base-6)	(Thaddeus C. Jones)

ERRATA AND ADDENDA FOR THE JUNE 1962 *ISSUE OF RMM*

Page 3: The *Carus Mathematical Monographs* are now published by the Mathematical Association of America.

Page 11: Clue 1 Down *should* read (18 down) + 68

Page 18: Second line should start "*two* faces twice. . . "

Page 28: Third paragraph: Three more Group II regions cannot be covered by the five tetrominoes besides regions 5, 6 and 17. They are regions 20 and 21.

Fourth paragraph: Problems 4 and 7 (not 6) have been solved.

Page 29: Fifth paragraph, second line: XYVW *should* read XTVW.

Page 46: Leon Bankoff's note concerning 870^3 should have been referred to as a *CUBE OF AN INTEGER EXPRESSIBLE AS THE SUM OF THREE CUBES.*

Page 50: The answer to the alphametic (3) *should* read TWO = 846.

Page 51: Answer to problem 5 *should* read $22+2 = 24$.

Page 52: The Editor really confused the issue! The correction concerning the Robinson prime was an error! The *correct* form for the Robinson prime on page 28 of the APRIL 1962 RMM is ${}_{53}R_{485}$ (as it is actually printed in the APRIL issue). The APRIL cover is correct. It is the subscript of this prime on page 25 of the APRIL 1962 issue which is incorrect. It *should* read ${}_{53}R_{485}$.

RECREATIONAL MATHEMATICS magazine

JUNIOR DEPARTMENT

All correspondence and material relating to the *Junior Department* should be sent to:

Howard C. Saar
1014 Lindell Avenue
Petoskey, Michigan

* * * * *

Welcome to the *Junior Department* and we sincerely hope that you will be with us each and every issue! This is your department and we encourage students of all ages (through secondary school) to submit anything they feel would be appropriate for the *Junior Department.* Also, we trust that you will submit problems for consideration by other readers, solutions to the proposed problems, and any suggestions you might have for articles for your age group. Most especially, for this first *Junior Department,* we want your comments about the material - is it what you want? Is it too difficult, or too elementary? Is it too long, or too short?

We want this department to be 100% yours.

* * * * *

PROBLEM CORNER

Students of all age who have not yet graduated from secondary school or its equivalent are invited to send in solutions to any or all the problems here. Solutions should be written out neatly on paper 8½ x 11 inches, contain the name and school of the student submitting them, and should be sent to the *Junior Department* Editor. Correct solutions will be acknowledged in the next issue of RMM. Original creative problems are also requested from students and the same manuscript and mailing instructions apply as have been previously stated.

1. 33⅓% of a peanut-sheller's daily number of peanuts shelled is 50% of another sheller's output. The slower sheller shells 1,500 peanuts daily. What is the daily number of peanuts shelled? (*Vicki Brown;* Reprinted from *The Explorer* published by the 7th GRADE MATH SOCIETY of the Weston, Connecticut Junior High School.)

2. Amos Voil of Azusa, California asks what is the radius of a circle whose area is numerically the same as its circumference?

3. During a school play, $19.23 was collected from the patrons in pennies, nickels, dimes and quarters. After the money was sorted, it was found out that there were as many quarters as there were pennies, and that there were twice as many dimes as there were quarters. But one mathematically-minded student also noticed that the number of nickles was equal to one more than half as much again as the sum of the number of pennies and dimes. How many coins were collected, and how many of each denomination? (*Harvey Hahn;* Valparaiso, Indiana)

4. A baker sent a boy to deliver an order for nine donuts. The baker placed the donuts in a box and wrote IX on the cover to indicate

the number of donuts inside. On the way, the boy ate three of the donuts, and then decided it might be a good idea to change the number on the box. His pencil had no eraser, so how did he change the number?

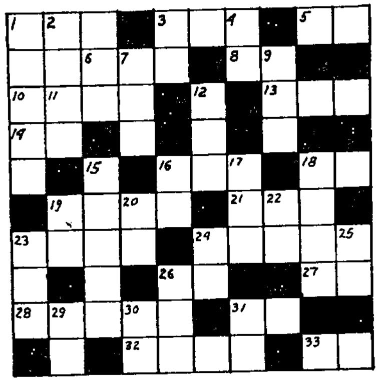
5. The Editor offers something to keep students busy: The odd digits 1, 3, 5, 7, and 9 total 25, while the even digits 2, 4, 6, and 8 total 20. Can you arrange the digits using arithmetical signs and operations so that the odd digits add up to the same amount as the even digits?

6. Which has the larger area, a triangle with sides 5, 5 and 6 or one with sides 5, 5 and 8? (*Hint*: You do not even have to calculate areas!)

* * * * *

CROSS-NUMBER PUZZLE

A cross-number puzzle is worked the same way as a crossword puzzle, but we use *numbers* instead of letters or words.



Across

1. $48.72 + 39.41 + 34.87$
4. 564.55 rounded off to the nearest integer
5. The area of a trapezoid with bases 6 and 8, and height of 5
6. An arithmetic sequence
8. Inches in $2\frac{2}{3}$ yards
10. $40581 \div 27$
13. 45% of 980
14. Pints in $2\frac{1}{4}$ gallons
16. 25^2
18. An unlucky number for some
19. Annual salary at \$423 a month
21. How far can you go on a 20-gallon tank of gas if you get 22 m.p.g.?
23. $15 \cdot 241$
24. The first and last digits are twice the middle digit.
26. Nickels needed for \$1.40
27. $84 \div 2\frac{5}{8}$
28. Two months wages at \$377.72 a month
31. Perimeter of a 12 x 11 rectangle
32. 2.8×10^3
33. Nine degrees greater than a right angle.

Down

1. Eleven in Base-2
2. Number of dimes needed for \$34.00
3. Average of 77, 48, 31
4. Born 1898 - died 1957. How old?
6. 50 decreased by 20%
7. 67 is what percent of 201?
9. Area of a circle with radius 8
10. $40\frac{2}{3} + 14\ 5/6 + 55\frac{1}{2}$
11. 29 is $\frac{1}{2}$ of what?
12. Volume of an 8-inch cube
15. Premium of an insurance policy for \$5000 at \$40.25 per \$1000
16. Days from August 15 to October 20
17. Interest on \$3000 at 6% for 3 years
18. $10^3 + 3$
19. Perimeter of a regular octagon 7 inches on a side
20. Gum drops at 3 for a penny How many for 25 cents?
22. Inches in 1 yard, 2 feet
23. Days in Leap Year
24. Acute angle by 2°
25. 50° warmer than freezing
26. Double each digit to get the next
29. How many years from now is 2034?
30. 201—159
31. Twenty in Base-5

Aboo-Bakkre Mohammad Al-Karkhi

by Ali R. Amir-Moéz

Around the year 1000, in Persia (Iran), the schools were not the same as what we have nowadays in the United States. The Mosque, the house of worship, was also the school and the teacher, who lived there, was called Master. Pupils and Master sat on the floor and it was considered quite impolite for a pupil to say anything or ask questions.

But one day Mohammad was so uncomfortable that he couldn't sit still - for which, of course, he risked punishment.

"What is the matter, Mohammad?" asked the teacher, "Are you ill?"

"No, Master," answered Mohammad with a low voice, "I think I can do arithmetic with pictures."

The teacher became curious and asked him to explain what he meant by that.

Mohammad had really found a trick of arithmetic by using simple pictures and explained it to his teacher.

"We draw a square and call it *one*. We put two of them together and call it *two*. Continuing this way, we can draw *three*, *four*, etc. (Figure 1)

"Now if we want to find

$$1+2+3+4$$

we just put these pictures together as in Figure 2 and make another picture just like it, upside down, and put this over the first figure as in Figure 3. We see that the two pieces fit together (Figure 4). It is easy to count the squares of this picture because we have four strips and each strip has five squares. This way we get

$$4 \times 5 = 20.$$

This means that $1+2+3+4$ is one-half of 20 or $1+2+3+4 = 10$.

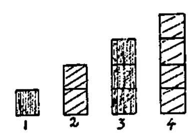


Fig. 1

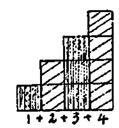


Fig. 2

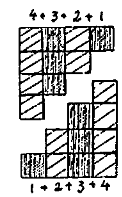


Fig. 3

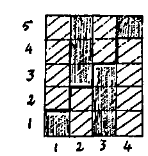


Fig. 4

"I am sure that this will work for more than four numbers. For example, if we want to add through fifty, we don't even have to write it or draw pictures. We say

$$1+50 = 51.$$

This is one of the strips in our picture and, since we have fifty strips, we can write

$$50 \times 51 = 2550.$$

But this is twice the sum we want, so one-half of 2550 is 1275. This means that if we add the numbers from one through fifty we get 1275."

"This is very good," said the teacher, "Tomorrow I'll let you study the books of Euclid. The thirteen books of Euclid are the first books written on geometry."

Mohammad grew up to be a famous mathematician and astronomer. We find his name in most mathematical histories. Unfortunately, we don't know his birthday since the Persians did not celebrate such days. We only know that he lived about the year 1000.

Later Karhi mastered his geometrical methods for proving propositions of arithmetic. We shall give same examples of his work.

(1) *The sum of the first* n *odd numbers is* n^2, i.e.

$$1+3+5+\ldots+(2n-1) = n^2$$

First, Karkhi did the problem for some special case:

$$1+3+5+7 = 4^2 = 16$$

He then chose 1 to be the square of unit area, 3 to be three of these squares, 5 to be five of them, and 7 to be seven of them put together as in Figure 5. All these areas fit together to make 4^2 as in Figure 6. Karhi generalized this idea, but we shall not go into that.

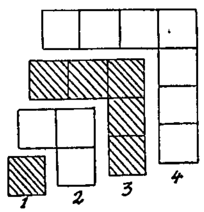


Fig. 5

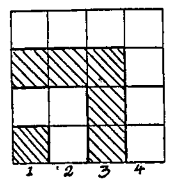


Fig. 6

(2) *Prove that* $(1\times2)+(2\times3)+(3\times4)+\ldots+[n\times(n+1)] = \frac{1}{3}[(n+1)^3-(n+1)]$.

We only give the geometric demonstration of the formula for $n = 3$.

Take the three bricks with edges 1, 1, 2 and put them together as shown in Figure 7. It is clear that the volume of this solid is $2^3-2 = 3(1\times2)$. Now let us put together three bricks with edges 1, 3, 3 (Figure 8). The volume of this solid is $3(2\times3)$. Finally we put together three bricks with edges 1, 3, 4 (Figure 9). The volume of this solid is $3(3\times4)$. Now we observe that these three figures fit together as shown in Figure 10. But the volume of the solid in Figure 10 is 4^3-4 because it is a cube of edge 4, from which four unit cubes along the diagonal have been removed. Thus

$$4^3-4 = 3\ [(1\times2)-(2\times3)-(3\times4)]$$
$$(1\times2)+(2\times3)+(3\times4) = \tfrac{1}{3}(4^3-4)$$

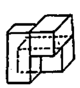


Fig 7

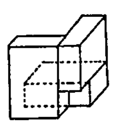


Fig. 8

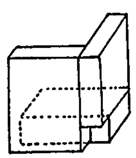


Fig. 9

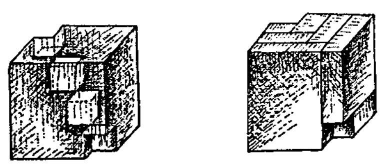


Fig 10

We leave the cases of larger numbers and the general case to the reader.

Prime Twins and Goldbach's Conjecture

The most difficult problems in mathematics are often those which are rather easy to state. A number of problems of this nature are quite famous. Though they have been the objects of extensive study for many years, they have withstood the attacks of both amateur and professional mathematicians.

One such problem is concerned with prime numbers. Some of the younger readers of RMM are no doubt acquainted with the prime numbers which have appeared in several issues. A whole number greater than one is said to be prime if it has no integral (whole number) divisors other than itself and 1. Thus, 13 is a prime number, the only divisors being 1 and 13. 15 is not prime since it has the divisors 3 and 5 in addition to itself and 1, and as such, it is called a composite number.

Prime numbers have attracted the attention of mathematicians from the earliest days. The study of primes suggested many intriguing questions which required the utmost skill and ingenuity to answer. Perhaps the first question of any importance to be considered was "How many primes are there?". This question was answered brilliantly by Euclid (300 B.C.) who proved that the number of primes is infinite - there being no prime which could be called the largest *possible* prime (there is a largest *known* prime, but there are larger primes which, so far, defy discovery). Euclid's proof is a masterful example of the beauty that is found in serious mathematics. Euclid showed that the assumption that there was a last, or largest, prime would lead to a contradiction. This method of mathematical proof is called *reductio ad absurdum* of which G. H. Hardy once remarked "... a chess player may offfer the sacrifice of a pawn or even a piece, but the mathematician offers the game."

Here is a list of the first fifty prime numbers: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227 and 229 . An examination of this list reveals the presence of several prime pairs, that is, consecutive primes which differ by two. In fact, sixteen such twins are to be found among the first fifty primes. This observation leads to the question "Are there an infinity of prime twins?"

Empirical evidence points to the conclusion that there are indeed an infinity of prime twins. It has been shown, for example, that there are fifteen pairs of twins between 999,999,990,000 and 1,000,000,000,000 (the largest pair being 999,999,999,959 and 999,999,999,961) and there are twenty pairs between 1,000,000,000,000 and 1,000,000,010,000. However, no one has yet suceeded in solving the proposed problem. In fact, no progress whatever seems to have been made towards a solution.

The study of prime numbers suggested another famous unsolved problem of mathematics. Observe that any even number which might be selected can be expressed as the sum of two primes. For example, $18 = 7+11$, $32 = 13+19$, $100 = 29+71$, etc., (You can verify this for other even numbers.) This fact was noted by Goldbach (1690-1764), a Russian mathematician of no other interest in the history of mathematics. Goldbach conjectured that the proposition was true in general, that is, every even number can be expressed as the sum of two prime numbers. Note that the sum of any two primes (except the prime 2, which is the only even prime) is always even - but this is not the same as saying that any given even number is the sum of two primes.

Being unable to prove his conjecture, Goldbach requested assistance from the peerless Swiss mathematician, Leonard Euler (1707-1783). After much fruitless effort he acknowledged he was unable either to prove or disapprove the conjecture. It should be noted Goldbach's proposition is proven false if but a single even number can be found which cannot be expressed as the sum of two primes.

There is considerable evidence to support Goldbach's conjecture. Thus, the proposition has been verified for all even numbers up to 100,000. But not the slightest progress was made towards a proof of the proposition until 1931 when Schnirelmann (1905-1938), a young unknown Russian mathematician, succeeded in proving that every even number can be represented as the sum of not more than 300,000 primes.

Now, Schnirelmann's theorem is not as ludicrous as it might appear on comparison with Goldbach's conjecture. That it was indeed a real advance is evidenced by the fact that in 1937, another Russian mathematician named Vinogradoff, adapted Schnirelmann's method to prove that every even number beyond a certain point is the sum of at most four primes. Unfortunately, he was not able to indicate just what limit is implied by "beyond a certain point".

Even today as you read this, mathematicians everywhere are seeking an answer to this question. Though some progress has been made towards a solution to the problem, Goldbach's conjecture remains a tantalizing problem still unsolved.

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Howard C. Saar, our **Junior Department** Editor received his B.S. and M.S. in mathematics at Illinois State (Normal, Illinois); was with State Farm Automobile Insurance Co. for four years; taught high school prior to joining the mathematics staff at Albion College. Very recently he was appointed Head of the Department of Mathematics and Physics at North Central Michigan College. When he is not in the classroom or working on the **Junior Department,** he loves to fish.

PUZZLE SOLVERS: The solvers of the various puzzles are listed below. See pages 40, 41, 42 for the answers to the problems posed in the June 1962 RMM.

ALPHAMETICS

Merrill Barnebey, Grand Forks, N. D. (1,2,4); Corine Bickley, Warson Woods, Mo. (1,2,3,4); Richard Breisch, Royersford, Pa. (2,4); Sam Cannella, Atlanta, Ga. (2); John Coble, Orlando, Fla. (2); Will Cochran, State College, Pa. (2); Ronald L. Enyeart, Los Angeles, Calif. (1,2,3,4); Jack Falk, Idaho Falls, Idaho (1,2,3,4); Wendell F. Fogg, Lexington, Mass. (1,2,3,4); Harry M. Gehman, Buffalo, N.Y. (1,2,3,4); Harvey Hahn, Valparaiso, Ind. (1,2,4); Thaddeus C. Jones, Orlando, Fla. (2); Edward Joris, Antwerp, Belgium (1,2,3,4); Dale Kozniuk, Delbourne, Alberta (1,2,3,4); Gerald Kirk, Toppenish, Wash. (1,2,3,4); Albert Koehler, Milwaukee, Wis. (4); P. H. Mabey, East Molesey, England (1,3,4); Andrzej Makowski, Warsaw, Poland (2); Richard McCreless, Odessa, Texas (2); Bertha McDaniel, Stayton, Ore. (1,2,3,4); Thomas J. Morris, San Juan, P.R. (1,2,3,4); Harry L. Nelson, Livermore, Calif. (2,3,4); Wade E. Philpott, Lima, Ohio (2); Luther J. Porter, Wilmington, Del. (2); R. Robinson Rowe, Naubinway, Mich. (2,3,4); Donval R. Simpson, College, Alaska (1,2,3,4); John H. Sweitzer, Princeton, N.J. (4); W. J. Sweitzer, San Diego, Calif. (2,3,4); Vesper Taylor, Decatur, Illinois (2,3,4); Jack Travis, Corning, N.Y. (2,3,4); Alfred Vasko, Swanton, Ohio (1,2,3,4).

CROSS-NUMBER PUZZLE

Merrill Barnebey, Grand Forks, N.D.; Jeanette Bickley, Warson Woods, Mo.; Dennis Bosch, Whittier, Calif.; R. S. Johnson, Town of Mt. Royal, Quebec; Thaddeus C. Jones, Orlando, Fla.; Edward Joris, Antwerp, Belgium; Harry L. Nelson, Livermore, Calif.; Luther J. Porter, Wilmington, Del.; Jerry Roger, Tulsa, Okla.; John H. Sweitzer, Princeton, N.J.; Frederic Weiner, Bethlehem, Pa.

PUZZLES AND PROBLEMS

Merrill Barnebey, Grand Forks, N.D. (1,2); Richard H. Beck, Mt. Vernon, N.Y. (2); Corine Bickley, Warson Woods, Mo. (2); Jeanette Bickley, Warson Woods, Mo. (4,8); Bob Bortolussi, Toronto, Ontario (2,5); Dennis Bosch, Whittier, Calif. (2); Alan Bostrum, Bakersfield, Calif. (8); Richard Breisch, Royersford, Pa. (2); David E. Broadhead, Hartford, Conn. (2,4,6,8); Sam Cannella, Atlanta, Ga. (2); Lewis Carver, McAllen, Texas (1); John Coble, Orlando, Fla. (1,2); S. N. Collings, Surrey, England (1); Dan Gribbin, Orlando, Fla. (1); Harvey Hahn, Valparaiso, Ind. (2); R. S. Johnson, Town of Mt. Royal, Quebec (2,3,4,5,6,8); Thaddeus C. Jones, Orlando, Fla. (1,2,8); Edward Joris, Antwerp, Belgium (2,4,5,6,7); Dale Kozniuk, Delbourne, Alberta (8); Bertha McDaniel, Stayton, Ore. (2,4); Thomas J. Morris, San Juan, P.R. (1,2); Moe Moss, Montreal, Quebec (1); Harry L. Nelson, Livermore, Calif. (2,56); Paul Nemecek, Riverside, Ill. (8); E. A. R. Newson, Willowdale, Ontario (2,8); Wade E. Philpott, Lima, Ohio (2,6); Luther J. Porter, Wilmington, Del. (2,3,4,6); Jerry Roger, Tulsa, Okla. (2,8); R. Robinson Rowe, Naubinway, Mich. (2,4,5,6,7,8); John H. Sweitzer, Princeton, N.J. (2); W.J. Sweitzer, San Diego, Calif. (1,2,4); Donval R. Simpson, College, Alaska (2); Jack Travis, Corning, N.Y. (2); Alfred Vasko, Swanton, Ohio (2,4,5); Frederic Weiner, Bethlehem, Pa. (2).