*Continued -*

Check your schools and libraries to make sure *they* get a reprint if necessary.

Remember: payment of 65 cents for each copy of RMM # 1 desired must be included in the request. Reprints, if made, will be sent only to those who have submitted payment. Thereafter, a separate paid request must be made for a reprint.

It is understood that errors in RMM # 1 will be corrected before reprinting. This will protect the value of the *original* RMM # 1 as a collectors' item.

Other past issues of RMM are still available at 65 cents each. There are limited quantities of the April, June and August issues. Orders for these issues need not be accompanied by payment and this can be forwarded upon receipt of the requested copies.

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Effective March 1, 1962 *All* subscriptions to RMM will be $3.25 per year. NOTE: there is to be no more $3.50 general public rate or $3.00 teacher-student-library-school rate.

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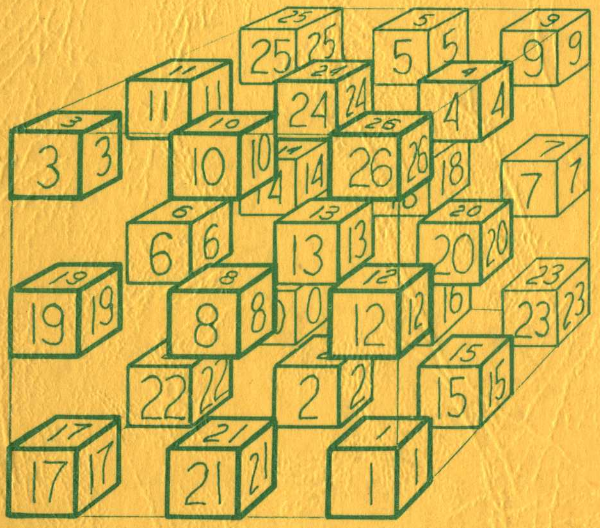
Some subscriptions are due to terminate with the next (December 1961 - DEC61) issue. Renewal must be made, in this case, by January 15, 1962 in order to receive the February 1962 issue of RMM.

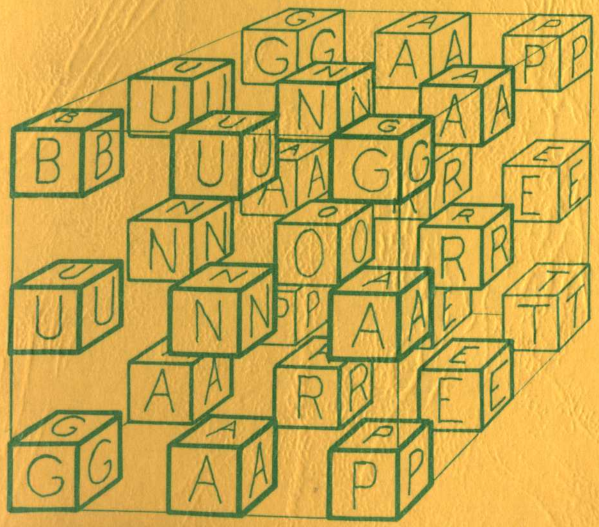
# RECREATIONAL MATHEMATICS magazine

ISSUE NO. 5
OCTOBER 1961

65¢

## Magic Cubes of NUMBERS





## and WORDS

# RECREATIONAL MATHEMATICS magazine

BOX 1876 or 150 FIRST STREET
IDAHO FALLS, IDAHO

OCTOBER 1961 ISSUE NUMBER 5

PUBLISHED AND EDITED BY JOSEPH S. MADACHY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RECREATIONAL MATHEMATICS MAGAZINE is published bimonthly by Joseph S. Madachy at The Falls Printing Co., Idaho Falls, Idaho. Second-class postage paid at Idaho Falls, Idaho. Subscription rates (worldwide): $3.00 per year until March 1, 1962 - thereafter all subscriptions will be $3.25 per year. Reprints of any material, at 5c per page, must be requested within the month of publication. All correspondence concerning changes of address, subscriptions, reprints, advertising and manuscripts should be sent to the Editor, Recreational Mathematics Magazine, 150 First Street, Idaho Falls, Idaho.

## From the Editor

Those interested in a reprint of the February 1961 RMM (RMM # 1) should see the inside back cover for details. The back cover gives some vital information about new subscription rates for RMM effective March 1, 1962.

* * *

Many of you may notice the increase in the number of puzzle-solvers listed in the answer sections of this issue. The editor apologizes for omissions and errors. Keep sending in your answers, though.

* * *

The collection of references and proofs to Lehmus' Theorem (see the *Letters to the Editor* on page 59 of the June 1961 RMM) is growing and next year in the April or June issue the editor will compile the bibliography and add reader comments and discoveries about Lehmus' Theorem. In the meantime, RMM readers should submit bibliographic references (especially recent ones) and their own proofs.

* * *

The December RMM will carry the conclusion of Solomon W. Golomb's "General Theory of Polyominoes" plus some interesting observations by RMM readers; a collection of dot and line games will be presented; *Deployment*, a new strategy game by William H. McGrail will be offered to RMM readers; another piece of cryptographic fiction by Norvin Pallas; and more, of course. An author-title-subject index will appear in the coming December issue and in all December issues thereafter.

The next issue will carry a notice about some future plans the editor has for RMM - mostly suggestions and ideas from you, the readers of RMM. If you have some positive commendations or condemnations about RMM, please write.

7 October 1961 J. S. M.



# The General Theory of Polyominoes

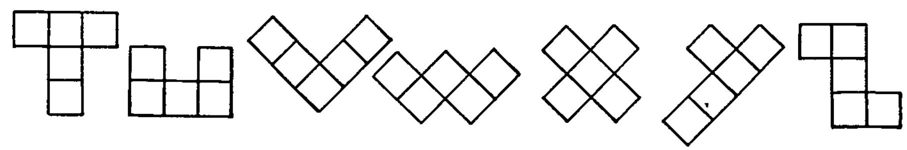
by Solomon W. Golomb

## Part 2 - Patterns and Polyominoes

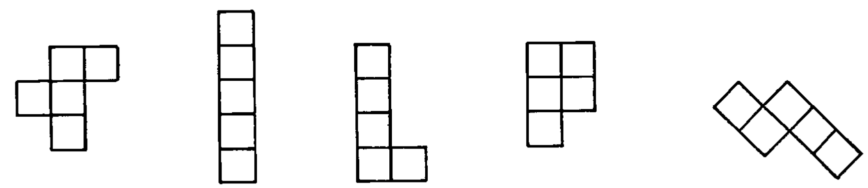
Readers of this magazine will remember polyominoes, the intriguing shapes which cover connected squares on a checker board and which were discussed in the June issue of RMM. Since that article appeared, a number of interesting facts and configurations have come to light.

### *Pentominoes*

The shapes which cover *five* connected squares are called *pentominoes*. There are twelve of these, and the following christening is recommended for remembering them:



T U V W X Y Z



F I L P N

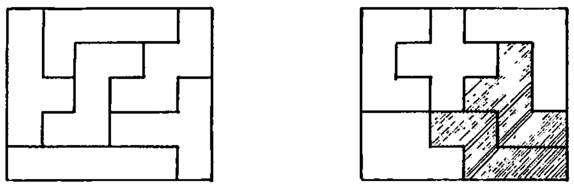
As a mnemonic device, one has only to remember the end of the alphabet (TUVWXYZ), and the word FILiPiNo.

In the previous article, it was shown that the twelve pentominoes, which cover a total of sixty squares, can form such patterns as a 3 x 20 rectangle, a 4 x 15 rectangle, a 5 x 12 rectangle, and a 6 x 10 rectangle. They can all be fitted onto the 8 x 8 checker board, with the four excess squares occupying a 2 x 2 rectangle at any specified location on the board. Given any pentomino, nine of the others can be used to triplicate it—that is, to form a scale model of it, three times as long and three times as high.

A new class of pentomino problems which will be discussed here, are *superposition* problems. Several examples will now be considered.

1. Arrange the twelve pentominoes into *two* 5 x 6 rectangles.

Solution:

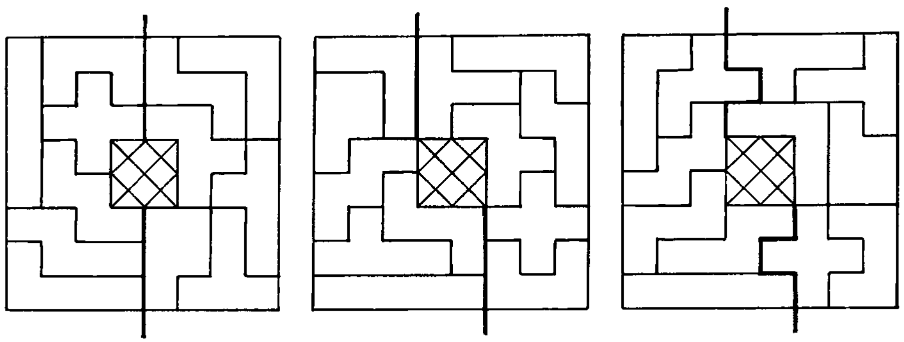


This solution is the only one known, except for the fact that the F and N pentominoes (shaded) can be fitted together in another way and still occupy the same region.

Note that the solution to this superposition problem simultaneously solves the 5 x 12 rectangle and the 6 x 10 rectangle problems, simply by putting the two 5 x 6 rectangles together in two different ways.

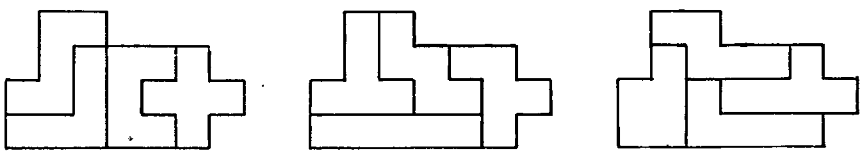
2. Find solutions to the 8 x 8 pattern with the hole in the middle which will separate into two congruent pieces, each piece using six of the pentominoes.

Typical solutions:



3. Divide the twelve pentominoes into three groups of four each. Find a twenty-square region which *each* of the three groups will cover.

Solution:



No other solutions to this problem have yet been discovered.

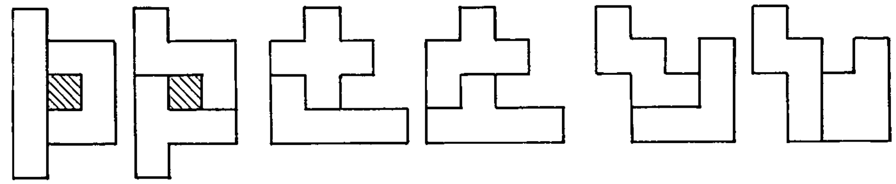
4. Divide the twelve pentominoes into three groups of four each. Subdivide each group into two pairs of pentominoes. For each group,





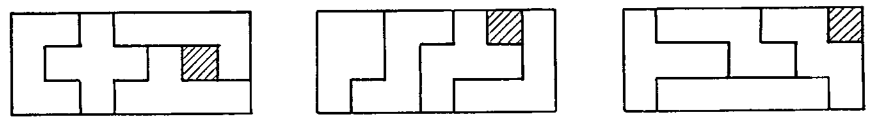
find a ten-square region which each of the two pairs will cover.

Solution:



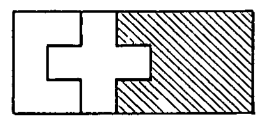
It would be interesting to find other solutions, especially one which eliminates "holes" in all three regions.

5. Divide the twelve pentominoes into three groups of four each. To each group add a monomino, and form a 3 x 7 rectangle.



This solution is known to be unique, except that in the first rectangle the monomino and the Y-pentomino can be re-arranged and still occupy the same region.

The uniqueness proof follows a suggestion of Dr. C. S. Lorens. To begin with, the X-pentomino can be used only in conjunction with the U-pentomino, in the pattern:



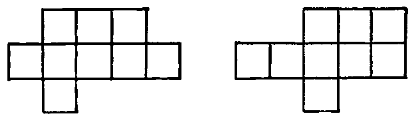
Next, neither the F nor the W-pentomino can be used to complete this rectangle. Also, with the U-pentomino needed to support the X, it is impossible to use F and W in the same 3 x 7 rectangle. Hence, of the three 3 x 7 rectangles, one will contain XU, another will contain W (but not U), and the third will contain F (but not U). When all possible completions of these three rectangles are listed and compared (a very time-consuming enterprise) it is found that the solution given above is the only possible one.

6. Divide the twelve pentominoes into four groups of three each. Find a fifteen-square region which *each* of the four groups will cover.

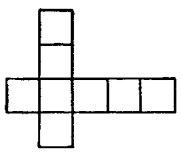
No solution to this problem is known. On the other hand, the problem has not been proven impossible.

7. Find the smallest region on the checker board onto which each of the twelve pentominoes will fit, one at a time.

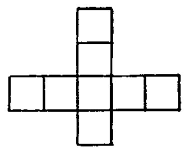
The minimum area for such a region is nine squares. The two examples of such minimal regions are:



The adequacy of these regions is proved by observing that each pentomino in turn will fit. The impossibility of *fewer* than nine squares is proved as follows: If it were possible to use a region with fewer than nine squares, then in particular the I, X, and V pentominoes would fit on a region of no more than eight squares. The I and X pentominoes will then have three squares in common. (Otherwise either nine squares are needed, or else the longest straight line has *six* squares, a needless extravagance.) This can happen in two distinct way, and fixes also the position of the V pentomino.

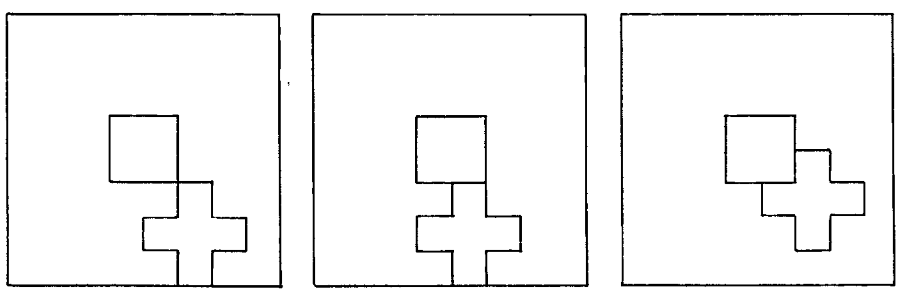


or



In either case, however, the fitting of the U pentomino would require a ninth square. Thus eight squares is not enough, whereas nine squares has been shown by example to be sufficient.

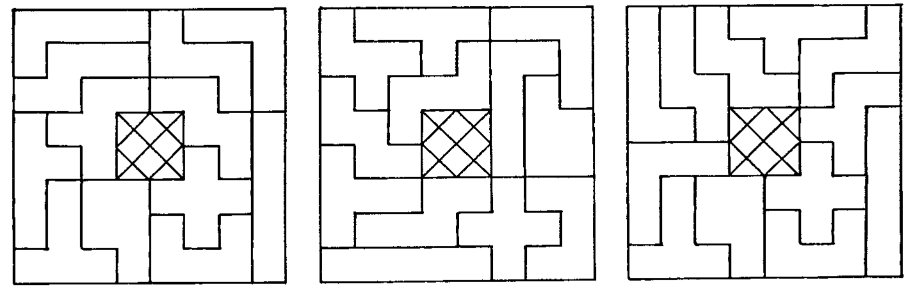
Recently the resources of modern electronic computing have been turned loose on various pentomino problems A technical report (*Programming a Combinatorial Puzzle*, 10 June 1958, Department of Electrical Engineering, Princeton University) by Dana S. Scott, describes two problems which were programmed for and solved by the MANIAC computer. The first problem was the fitting of the twelve pentominoes onto a 3 x 20 rectangle. It was verified that the two solutions already known are indeed the only possible solutions. The second problem was to find all ways of fitting the twelve pentominoes onto the 8 x 8 checker board, leaving a 2 x 2 hole in the center. It was discovered that there are sixty-five basically different solutions (in the sense that two solutions differing only by rotation or reflection of the checkerboard are not regarded as distinct). The program began with the astute observation that there are only three basically different locations for the X pentomino, as follows:



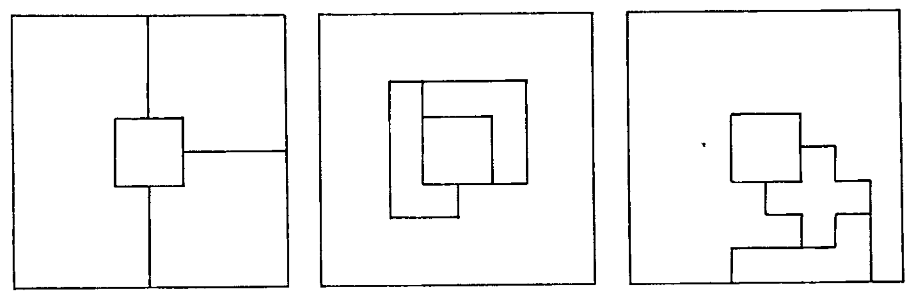




The number of ways to finish covering the board in each of these cases is 20, 19, 26, respectively. Some of the more interesting solutions include:



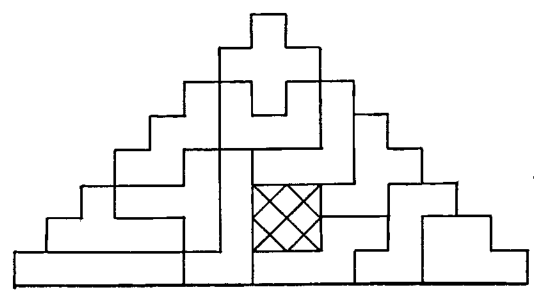
A number of plausible problems turn out to be impossible, since they are absent from Scott's listing. These include:



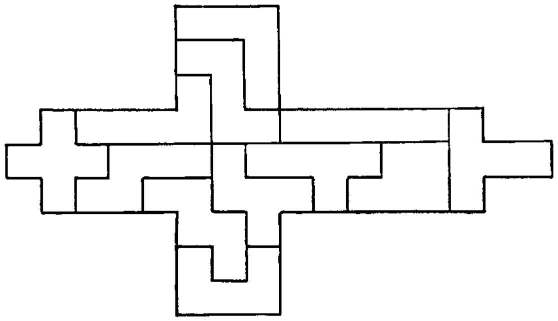
Professor C. B. Haselgrove recently programmed a computer at Manchester University, in England, to find all ways of placing the twelve pentominoes into a 6 x 10 rectangle. Excluding rotations and reflections, he found 2,339 basically distinct solutions! He also verified the results of Dana Scott's program for the checker board and the 3 x 20 rectangle.

Before leaving the subject of pentomino patterns, several special configurations seem worthy of mention:

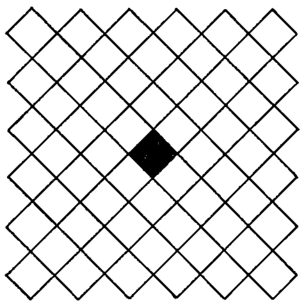
1. The sixty-four square triangle, filled with the twelve pentominoes and the square tetromino. (Other tetrominoes may also be specified as the thirteenth piece.)



2. Another difficult configuration is the elongated cross:

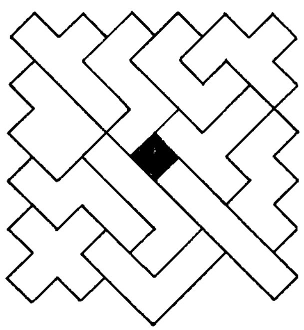


3. Until quite recently, the following configuration was unsolved (neither constructed nor proved impossible).



The ingenious and painstaking solution found by Spencer Earnshaw will be described in the article to appear in the December issue of RMM.

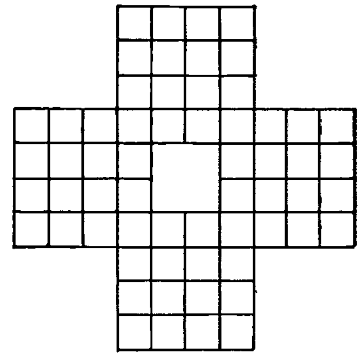
Even if the monomino is moved to another location, no solution to placing the twelve pentominoes on the remaining sixty squares has been found. The closest approximation yet known to this pattern is:



4. Also long believed impossible is Scott Taylor's configuration:

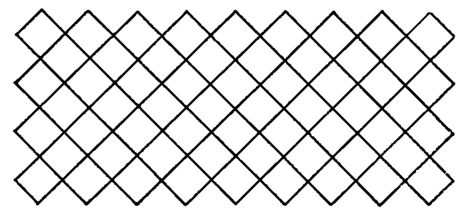






Again, this case has recently been painstakingly settled, and the solution will be given in December.

5. Fortunately, not all such problems are so hard to resolve. The following pattern, composed of sixty squares, was proved by Professor R. M. Robinson, of the University of California at Berkeley, to be incapable of holding the twelve pentominoes:



In particular, there are twenty-two edge squares (including corners) in this pattern. If the pentominoes are examined separately, and the maximum number of edge-squares each could contribute is listed, the total is only 21, as follows:

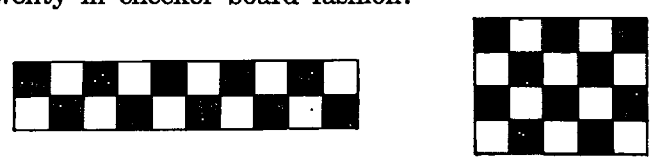
| | | | | |
|---|---|---|---|---|
| T = 1 | W = 3 | Z = 1 | L = 1 | |
| U = 1 | X = 3 | F = 3 | P = 2 | Total = 21 |
| V = 1 | Y = 2 | I = 1 | N = 2 | |

This type of reasoning is used in solving jig-saw puzzles, where it is common practice to separate the edge pieces from the interior pieces.

* * *

*Tetrominoes*

The polyominoes which cover *four* squares of the checker board are called *tetrominoes*. Unlike the pentominoes, the five distinct tetrominoes will not form a rectangle. To prove this, color the rectangles of area twenty in checker board fashion:

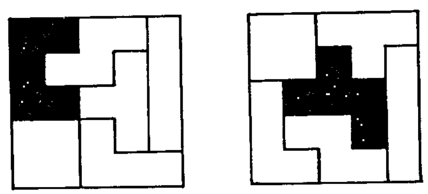


Four of the five tetrominoes will always cover equally many dark and light squares. These are

However, the remaining tetromino always covers three squares of one color and one square of the other:

Hence, altogether, the five tetrominoes will cover an *odd* number of dark squares and an *odd* number of light squares. However, the rectangles in question have *ten* squares of each color, and ten is an even number.

On the other hand, any of several different pentominoes can be combined with the five tetrominoes to form a 5 x 5 square. Two examples are

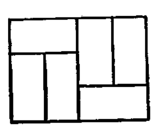


The reader is invited to investigate how many *different* pentominoes can be used in this manner.

* * *

*Masonry Problems*

Robert I. Jewett, a graduate student at the University of Oregon, has proposed the following problem: Is it possible to cover a rectangle with dominoes in such a way that every grid line (both horizontal and vertical) of the rectangle intersects at least one domino? For example, in the pattern



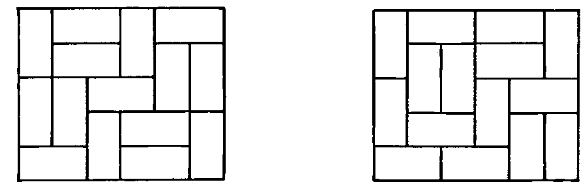
the middle vertical grid-line does not cut any dominoes. Thinking of dominoes as bricks, such a grid line represents a structural weakness. Jewett's problem is thus to find masonry patterns without "fault lines".

Many people who try this problem soon give up, convinced that there are no solutions. Actually there are infinitely many, but the

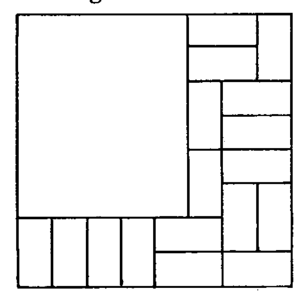




*smallest* of them requires fifteen dominoes, arranged in a 5 x 6 rectangle. In fact, there are two basically different ways to form such a rectangle:



It is not difficult to show that the minimum width for "fault-free" rectangles must exceed 4. (The cases of width 2, 3, and 4 are best treated separately.) Hence, since 5 x 5 is an odd number of squares, while dominoes always cover an even number of squares, the 5 x 6 rectangle is the smallest solution.



A 5 x 6 rectangle can be "extended" to an 8 x 8 checkerboard and still satisfy the fault-free condition.

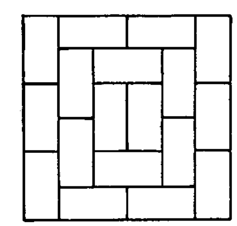
Surprisingly, there are no fault-free 6 x 6 rectangles. For this fact there is a truly remarkable proof:

Imagine any 6 x 6 rectangle covered entirely with dominoes. Such a figure contains 18 dominoes and ten grid lines (five horizontal and five vertical). For example:

A 6 x 6 RECTANGLE

COVERED WITH DOMINOES.

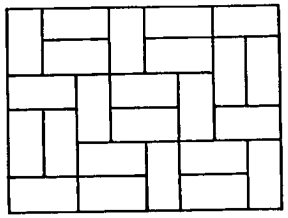
(NOT FAULT-FREE)



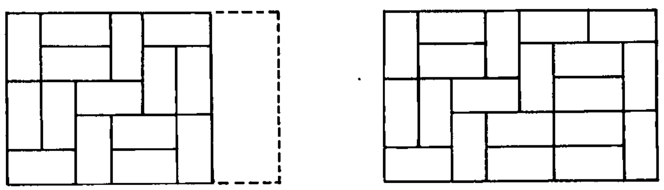
Such a figure is *fault-free* if each grid line intersects at least one domino. (Note that each domino is cut by *exactly* one grid line.) We shall show that each grid line intersects an *even number* of dominoes. Hence, in the fault-free case, each grid line must intersect at least *two* dominoes. With ten grid lines, at least twenty dominoes would be intersected; but there are only eighteen dominoes altogether!

It remains only to prove the assertion that each grid line cuts an even number of dominoes. Consider, for example, a vertical grid line. The area to the left of it is *even* (either 6, or 12, or 18, etc.). *Entire* dominoes to the left of it cover an even area, since each domino covers two squares. The dominoes *cut* by the grid line must also occupy an even area to the left of it, because this area is the difference between two even numbers (the total area to the left, and the area of the uncut dominoes to the left). Since each cut domino occupies *one* square to the left of the grid line, there must be an *even number* of dominoes cut by the grid line. Thus the proof is complete.

Similar reasoning shows that for a fault-free 6 x 8 rectangle to exist, every grid line must intersect *exactly* two dominoes. For example:



The most general result is the following: If a rectangle has even area, and both its length and width exceed 4, it is possible to find a fault-free covering of the rectangle with dominoes, *except* in the case of the 6 x 6 rectangle. Actually, coverings for all larger rectangles can be gotten from the 5 x 6 rectangle and the 6 x 8 rectangle, using a method of enlarging either the length or the width by 2. This method is easiest to explain by example. To go from a 5 x 6 rectangle to a 5 x 8 rectangle:



To extend horizontally by two, a horizontal domino is placed next to each horizontal domino at the old boundary, while vertical dominoes are shifted from the old boundary to the new, with the intervening space filled by two horizontal dominoes.

The reader may find it interesting to study *trominoes* as bricks. In particular, what is the smallest rectangle which can be covered by two or more "straight trominoes" (1 x 3 rectangles) without any fault lines?





## NOTES FROM THE EDITOR

Several items of interest were brought to the attention of the editor. Perhaps other readers can add more comments about Mr. Golomb's series of articles.

Lewis Carver of McAllen, Texas writes that "rubber or vinyl floor tile is an excellent material for making pentominoes. It is available in a wide range of colors. One tile, 9 x 9 inches, ruled in 1⅛-inch squares and cut according to any of the various patterns already given by Mr. Golomb [pages 7 and 8, August RMM; pages 4 and 7 in this issue] will produce the 12 pentominoes plus a 2¼-inch square.

"A sharp, thin-bladed knife is best for cutting the tile. The pieces can be fitted by filing and the flat surfaces and edges finished with fine garnet paper. Since vinyl tile is affected by temperature, I have found that it is easiest to cut at 80° F or over, while it takes a finer sanded finish at lower temperatures."

* * * * *

In the August issue Mr. Golomb described a game which can be played with pentominoes. Don Long of the University of California reports that a variation on the game has been put on the market.

"It is commercially manufactured," writes Mr. Long, "under the name of *Pan-Kai*. An additional rule is introduced in the game which makes the game more interesting, in my opinion. No piece may be played in such a manner as to close off an area of less than five squares. *Pan-Kai* also uses a 10 x 10 rather than an 8 x 8 board. Further, each player has his own complete set of pentominoes. Hence, a principle of strategy is to attempt to reserve a space of five or more spaces for your opponent which can only be occupied by a piece he has *already* played."

The questions that come to Mr. Long's mind is how many essentially different first moves are possible in *Pan-Kai*? And which one is the best?

*Pan-Kai* is manufactured by Phillips Publishers, Inc. of Newton, Massachusetts. The two sets of pentominoes are supplied in tan and dark-brown colors and the squares are ¾-inch in size.

* * * * *

Martin Gardner in his *first* SCIENTIFIC AMERICAN BOOK OF MATHEMATICAL PUZZLES & DIVERSIONS reports that "a box of colored plastic pentominoes was marketed in 1957 by Tryne Products, Inc., 233 Broadway, New York, New York, under the trade name of *Hexed*."

In Mr. Gardner's *second* SCIENTIFIC AMERICAN BOOK OF MATHEMATICAL PUZZLES & DIVERSIONS (just published by Simon & Schuster, who published the first volume) is a little note that if the twelve pentominoes are made from cubes, adding a third dimension, the twelve pieces can be put together to form a 3 x 4 x 5 solid. Other solids of 2 x 5 x 6 and 2 x 3 x 10 can be formed.

* * * * *

Related to the solid polyominoes just described are the well-known SOMA cubes. A set consists of six ways in which four cubes can be joined face-to-face, plus a three-cube L arrangement (there are more than six ways of arranging four cubes so that faces are joined - I leave it to readers to discover them). The Gem Color Co., Inc. of Paterson, New Jersey manufactures a regular SOMA game with seven pieces and also a Double SOMA which is merely two sets of seven pieces. The object is to form various shaped solids by putting the 7 (or 14) pieces together. (Mr. Gardner devotes a chapter to SOMA cubes in his *second* SCIENTIFIC AMERICAN BOOK, etc. . . .)

# A Classic Reprint

## A Contribution to the Mathematical Theory of Big Game Hunting

*by H. Pétard*

This little known mathematical discipline has not, of recent years, received in the literature the attention which, in our opinion, it deserves. In the present paper we present some algorithms which, it is hoped, may be of interest to other workers in the field. Neglecting the more obviously trivial methods, we shall confine our attention to those which involve significant applications of ideas familiar to mathematicians and physicists.

The present time is particularly fitting for the preparation of an account of the subject, since recent advances both in pure mathematics and in theoretical physics have made available powerful tools whose

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very existence was unsuspected by earlier investigators. At the same time, some of the more elegant classical methods acquire new significance in the light of modern discoveries. Like many other branches of knowledge to which mathematical techniques have been applied in recent years, the Mathematical Theory of Big Game Hunting has a singularly happy unifying effect on the most diverse branches of the exact sciences.

For the sake of simplicity of statement, we shall confine our attention to Lions (*Felis leo*) whose habitat is the Sahara Desert. The methods which we shall enumerate will easily be seen to be applicable with obvious formal modifications, to other carnivores and to other portions of the globe. The paper is divided into three parts, which draw their material respectively from mathematics, theoretical physics, and experimental physics.

The author desires to acknowledge his indebtedness to the Trivial Club of St. John's College, Cambridge, England; to the M.I.T. chapter of the Society for Useless Research; to the F. o. P., of Princeton University; and to numerous individual contributors, known and unknown, conscious and unconscious.

### 1. MATHEMATICAL METHODS

1. THE HILBERT, OR AXIOMATIC, METHOD. We place a locked cage at a given point of the desert. We then introduce the following logical system.

AXIOM I. *The class of lions in the Sahara Desert is non-void.*

AXIOM II. *If there is a lion in the Sahara Desert, there is a lion in the cage.*

RULE OF PROCEDURE. *If p is a theorem, and "p implies q" is a theorem, then q is a theorem.*

THEOREM I. *There is a lion in the cage.*

2. THE METHOD OF INVERSIVE GEOMETRY. We place a *spherical* cage in the desert, enter it and lock it. We perform an inversion with respect to the cage. The lion is then in the interior of the cage, and we are outside.

3. THE METHOD OF PROJECTIVE GEOMETRY. Without loss of generality, we may regard the Sahara Desert as a plane. Project the plane into a line, and then project the line into an interior point of the cage. The lion is projected into the same point.

4. THE BOLZANO-WEIERSTRASS METHOD. Bisect the desert by a line running N-S. The lion is either in the E portion or in the W portion; let us suppose him to be in the W portion. Bisect this portion by a line running E-W. The lion is either in the N portion or in the S portion; let us suppose him to be in the N portion. We continue this process indefinitely, constructing a sufficiently strong fence about the chosen portion at each step. The diameter of the chosen portions

approaches zero, so that the lion is ultimately surrounded by a fence of arbitrarily small perimeter.

5. THE "MENGENTHEORETISCH" METHOD. We observe that the desert is a separable space. It therefore contains an enumerable dense set of points, from which can be extracted a sequence having the lion as a limit. We then approach the lion stealthily along this sequence, bearing with us suitable equipment.

6. THE PEANO METHOD. Construct, by standard methods, a continuous curve passing through every point of the desert. It has been remarked[1] that it is possible to traverse such a curve in an arbitrarily short time. Armed with a spear, we traverse the curve in a time shorter than that in which a lion can move his own length.

7. A TOPOLOGICAL METHOD. We observe that a lion has at least the the connectivity of the torus. We transport the desert into four-space. It is then possible[2] to carry out such a deformation that the lion can be returned to three-space in a knotted condition. He is then helpless.

8. THE CAUCHY OR FUNCTIONTHEORETICAL METHOD. We consider an analytic lion-valued function $f(z)$. Let $\lambda$ be the cage. Consider the integral

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-\lambda}\,dz,$$

where $C$ is the boundary of the desert; its value is $f(\lambda)$, i.e., a lion is in the cage[3].

9. THE WIENER TAUBERIAN METHOD. We procure a tame lion, $L_0$, of class $L(-\infty, \infty)$, whose Fourier transform nowhere vanishes, and release it in the desert. $L_0$ then converges to our cage. By Wiener's General Tauberian Theorem[4], any other lion, $L$ (say), will then converge to the same cage. Alternatively, we can approximate arbitrarily closely to $L$ by translating $L_0$ about the desert[5].

[1] By Hilbert. See E. W. Hobson, The Theory of Functions of a Real Variable and the Theory of Fourier's Series, 1927, vol. 1, pp. 456-457.

[2] H. Seifert and W. Threlfall, Lehrbuch der Topologie, 1934, pp. 2-3.

[3] *N. B.* By Picard's Theorem (W. F. Osgood, Lehrbuch der Funktionentheorie, vol. 1, 1928, p. 748), we can catch every lion with at most one exception.

[4] N. Wiener, The Fourier Integral and Certain of it Applications, 1933, pp. 73-74.

[5] N. Wiener, *l. c.* p. 89.





## 2. METHODS FROM THEORETICAL PHYSICS

10. THE DIRAC METHOD. We observe that wild lions are, *ipso facto*, not observable in the Sahara Desert. Consequently, if there are any lions in the Sahara, they are tame. The capture of a tame lion may be left as an exercise for the reader.

11. THE SCHROEDINGER METHOD. At any given moment there is a positive probability that there is a lion in the cage. Sit down and wait.

12. THE METHOD OF NUCLEAR PHYSICS. Place a tame lion in the cage, and apply a Majorana exchange operator[6] between it and a wild lion.

As a variant, let us suppose, to fix ideas, that we require a male lion. We place a tame lioness in the cage, and apply a Heisenberg exchange operator[7] which exchanges the spins.

13. A RELATIVISTIC METHOD. We distribute about the desert lion bait containing large portions of the Companion of Sirius. When enough bait has been taken, we project a beam of light across the desert. This will bend right round the lion, who will then become so dizzy that he can be approached with impunity.

## 3. METHODS FROM EXPERIMENTAL PHYSICS

14. THE THERMODYNAMICAL METHOD. We construct a semi-permeable membrane, permeable to everything except lions, and sweep it across the desert.

15. THE ATOM-SPLITTING METHOD. We irradiate the desert with slow neutrons. The lion becomes radioactive, and a process of disintegration sets in. When the decay has proceeded sufficiently far, he will become incapable of showing fight.

16. THE MAGNETO-OPTICAL METHOD. We plant a large lenticular bed of catnip (*Nepeta cataria*), whose axis lies along the direction of the horizontal component of the earth's magnetic field, and place a cage at one of its foci. We distribute over the desert large quantities of magnetized spinach (*Spinacia oleracea*), which, as is well known, has a high ferric content. The spinach is eaten by the herbivorous denizens of the desert, which are in turn eaten by lions. The lions are then oriented parallel to the earth's magnetic field, and the resulting beam of lions is focussed by the catnip upon the cage.

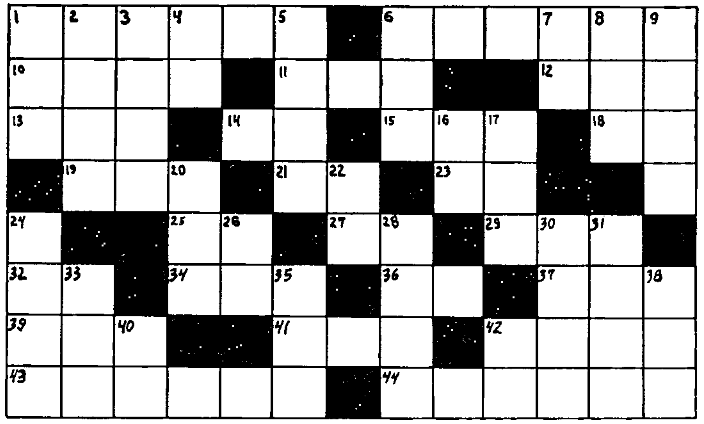
[6] See, for example, H. A. Bethe and R. F. Bacher, Reviews of Modern Physics, vol. 8, 1936, pp. 82-229; especially pp. 106-107.

[7] *Ibid.*

# Cross-Number Puzzle

by U. Clid

Fill in the blank spaces with the numbers corresponding to the definitions given. Some of the numbers may be ambiguous, but most are not.



HORIZONTAL

1. The first 6 figures of the Golden Ratio
6. The next 6 figures of the Golden Ratio
10. The 19th term of the series 1, 1, 2, 3, 5, . . .
11. The 7th term of the series 0, 57, 57, 114, 171 . . .
12. $30^2$
13. 664 (base 7) in base 9
14. One of the terms of the Fibonacci series of 10 Horizontal
15. The simplest integral Pythagorean triangle
18. XVI
19. This number, when turned upside down, becomes one-third greater
21. N in $a^6 = b^3 = c^2 = N$ where a, b and c are different integers
23. The 1st digit is twice the 2nd
25. The constant of a normal 3x3 magic square
27. Twice 22 Vertical
29. The 24th term of 10 Horizontal
32. The largest 2-digit prime
34. The 8th term of 11 Horizontal
36. A Perfect Number
37. A coefficient of the expansion of $(a + b)^{18}$
39. Largest prime permutation of 365
41. The boiling point of water in $^{\circ}$F
42. The smallest integer which can be expressed as the sum of two cubes in two different ways
43. $\pi$
44. e

VERTICAL

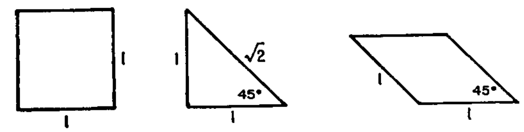
1. $n^2$ where $n^2$ is equal to the nth term of 10 Horizontal
2. A palindromic number
3. The box address of RMM
4. $p^4$ where p is an odd prime
5. The respective values of a, b, c and d in the simplest integral solution to $a^3 + b^3 + c^3 = d^3$
6. Days in a year
7. The 11th term of 10 Horizontal
8. A prime
9. One-half of a Perfect number
16. The second digit is half the first
17. A palindromic prime
20. The number 744 (base 10) in base 11
22. The square of a prime
24. The year RMM started publication
26. The vertex angle of an isosceles triangle with base angles of $63^{\circ}$
28. The number 17942 (base 10) in base 13
30. A multiple of 101
31. The number 1233102 (base 4) in base 10
33. $\lceil$(34 Horizontal) + N$\rceil$ where N is a common divisor of 34 Horizontal and 33 Vertical
35. x in $3x^2 + 4x = 45140$
38. A "selling" price-tag for a $7 item
40. One of a pair of permutable primes
42. Solve for n: $2^n = 2048$



# Parquetry Blocks

R. G. Buschman

Among the sets of children's blocks which are now on the market, one manufacturer has a set called Parquetry Blocks. All the blocks are of the same thickness and have vertical sides so that we can classify them and carry on our discussion in terms of the shapes of their bases. We can then use the terminology of plane figures and, lacking the set, models could easily be cut from cardboard.




In this particular set of blocks there are 8 squares, 8 isosceles right triangles and 16 rhombuses with acute angles equal to those of the triangles. Suppose that we let the length of a side of a square be our unit of length. The sides of the squares, the legs of the triangles, and the sides of the rhombuses are all then of unit length. The areas of squares, triangles, and rhombuses are respectively 1, 1/2, and $\sqrt{2}/2$; the total area of the set is $12+8\sqrt{2}$.

One, two, four, or eight persons could divide the blocks equally so that each has the same number of blocks of each shape. (Since the manufactured set is also colored, it is only possible for eight persons to divide them into equivalent sets if they are not particular about color.) The blocks are sold in a square box so that one person can construct a square; in fact, in several ways according to the diagrams supplied by the manufacturer. We next propose some problems involving these blocks. Suppose the set is equally divided, as described, and each person is to use all his subset in each case.

*Problem* 1. Can 2, 4, or 8 persons each construct a square with his set? (The case for 1 person was solved by the manufacturer.)

*Problem* 2. Can 1, 2, 4, or 8 each construct a triangle?

*Problem* 3. Can 1, 2, 4, or 8 each construct a rhombus?

*Problem* 4. Can 1, 2, 4, or 8 each construct a rectangle other than those discussed in problem 1?

*Problem* 5. One of the diagrams supplied by the manufacturer shows that a regular octagon can be obtained from one of the arrangements of the square by simply removing the blocks at the corners of the square, in this case by removing all of the triangles. Now if the blocks of this octagon are divided into equivalent sets can 2, 4, or 8 persons each construct an octagon? Is the solution altered by resubstituting the eight triangles for four of the squares?

---

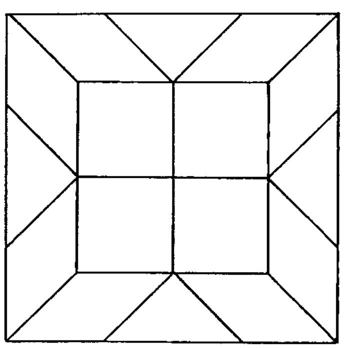
PARQUETRY BLOCKS are manufactured by *PLAYSKOOL*, Chicago, Illinois.

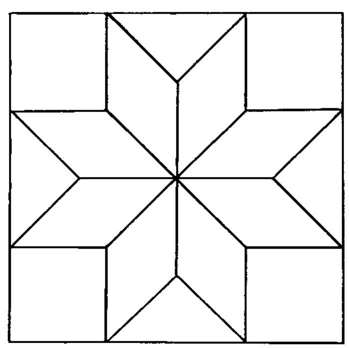
Certainly there are other interesting problems concerning this set, but we leave to problem-solvers the pleasure of inventing their own at this point.

One might have to experiment for some time before discovering one of the possible combinations. A trial and error program and sufficient time will even show that some of the combinations are impossible. It is interesting to note that for four of the problems the impossible cases will appear if one studies the relationships to the arithmetic properties of numbers of the form $a+b\sqrt{2}$, where $a$ and $b$ are non-negative integers. For problem 4, on the other hand, our solution is still a rather long subcase treatment, so perhaps someone else can find a shorter one.

In the solutions which we present we note that the edges of the figures to be constructed must be of the form $a+b\sqrt{2}$ since each of the edges of the component blocks is, with the exception of a hypotenuse of a triangle, of unit length. Thus to obtain a side of length $b\sqrt{2}$ we must put $b$ triangles (i.e. the hypotenuse of each) along that side. If we know the number of triangles which must be along the sides of the figures to be constructed, this is a helpful hint in the possible cases. If there are not enough triangles in the set, then that particular case is impossible. This is the key to our solutions.

*Solution* 1. The area of a square being $L^2$ and the area of an individual's square being $12+8\sqrt{2}$, $6+4\sqrt{2}$, $3+2\sqrt{2}$, or $3/2+\sqrt{2}$ respectively for 1, 2, 4, or 8 persons, we try to write these numbers as squares. For the case of one person $12+8\sqrt{2}=(2+2\sqrt{2})^2$ which was demonstrated by the original square and we note that we must always have 2 triangles on each edge. For the case of two persons, $6+4\sqrt{2}=(2+\sqrt{2})^2$. This can be demonstrated making use of the information that one triangle must be on each edge. For 4 persons, $3+2\sqrt{2}=(1+\sqrt{2})^2$ which requires 4 triangles for each of the 4 squares - which is impossible since only 8 triangles are supplied. For 8 persons $3/2+\sqrt{2}=(a+b\sqrt{2})^2$ for which there is no integer solution for a and b. Therefore, this construction is impossible.

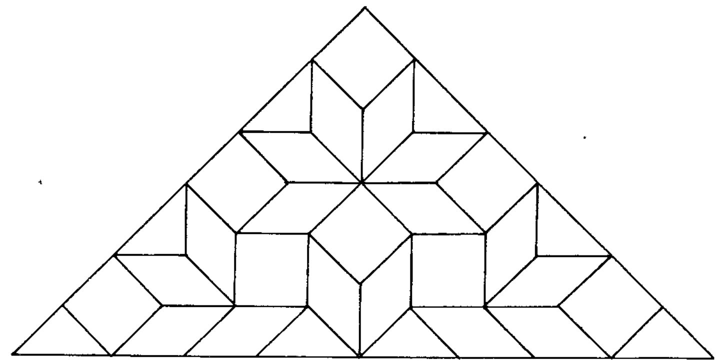








*Solution* 2. The area of an isosceles right triangle is $L^2/2$ and the length of the hypotenuse H is $L\sqrt{2}$. Since for 1 person the area is $12+8\sqrt{2}=(4+2\sqrt{2})^2/2$, we have $L=4+2\sqrt{2}$, $H=4+4\sqrt{2}$ which indicates that all triangles must be along the three sides of the triangle. This triangle can then be constructed. For case 2, $6+4\sqrt{2}=(2+2\sqrt{2})^2/2$, $L=2+2\sqrt{2}$, $H=4+2\sqrt{2}$ so we would need $(2)(6)=12$ triangles - impossible. For 4 or 8 similar reasoning indicates 16 or 24 triangles are needed respectively so that these constructions are also impossible.



*Solution* 3. The area of a rhombus is $L^2/\sqrt{2}$ and we do not even get to the argument about triangles. Let $B+C\sqrt{2}=A=(a+b\sqrt{2})^2/\sqrt{2}=2ab+[(a^2+2b^2)/2]\sqrt{2}$. The areas required were $12+8\sqrt{2}$, $6+4\sqrt{2}$, $3+2\sqrt{2}$, $3/2+\sqrt{2}$. Case 8 is impossible since $B=3/2=2ab$ cannot hold for $a$ and $b$ integers. Since $C$ is even, in the other cases, both $a$ and $b$ must be even, hence $B$ must be divisible by 8. Since $B=12$, 6, or 3 these cases are also impossible.

*Solution* 4. The area can be expressed as $A=LW=(a+b\sqrt{2})(c+d\sqrt{2})=(ac+2db)+(ad+bc)\sqrt{2}$. Thus for case 1 we have $ac+2bd=12$, $ad+bc=8$. We can further assume $b\geqq d$, which is no restriction. Also $b+d\leqq 4$ or we would not have a sufficient number of triangles. Thus we have 9 cases: $(b,d)=(4,0)$, $(3,0)$, $(2,0)$, $(1,0)$, $(0,0)$, $(3,1)$, $(2,1)$, $(2,2)$.

(i) $b=d=0$ does not satisfy the second equation.

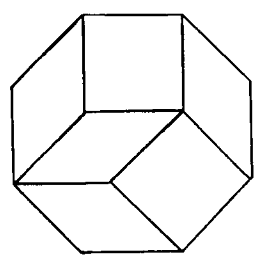
(ii) $b=d=2$ and $b=2$, $d=1$ gives the values respectively $a=c=2$ and $a=4$, $c=2$, both of which are already known from problem 1.

(iii) $d=b=1$ leads to $ac=10$, $a+c=8$ and $d=1$, $b=3$ leads to $ac=6$, $a+c=8$ neither pair of which can hold for $a$ and $c$ integers.

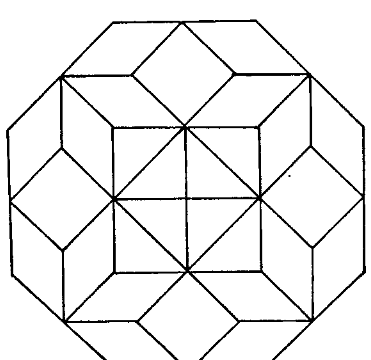
(iv) $d=0$, $b>0$ yields $ac=12$, $bc=8$, i.e. $a=3b/2$, $c=8/b$, so that $b=2$ or 4. Thus $LW=(3+2\sqrt{2})(4+0\sqrt{2})$ and $LW=(6+4\sqrt{2})(2+0\sqrt{2})$ must yet be investigated. First we note that the hypotenuses of triangles are restricted to one pair of opposite sides. Consider then the other pair of opposite sides. A rhombus cannot appear there unless there is an associated triangle somewhere along that side to complete the fit. (Try it.) A leg of a triangle cannot appear unless there is another triangle to be fitted to form a square. (Try it.) Thus we need only consider squares. If this pair of sides is composed of squares (and/or pairs of triangles making up squares), then the problem is delayed to the consideration of the next row. The argument is then repeated until we run out of squares thus showing these cases to be impossible.

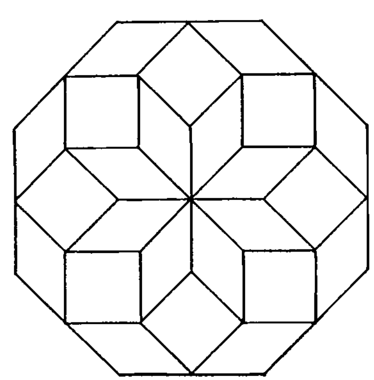
For case 2 we have $ac+2bd=6$, $ad+bc=4$, $b\geqq d$, $b+d\leqq 2$ and only $(b,d)=(2,0)$, $(1,0)$, $(0,0)$, $(1,1)$ to consider. $(0,0)$ is out as in case 1, $(1,1)$ leads to a known square, $d=0$ implies $b=2$ and we must test $(3+2\sqrt{2})(2+0\sqrt{2})$. If this construction were possible, then the two cases in (iv) would have been possible as they can be written in the form $2(3+2\sqrt{2})\ (2+0\sqrt{2})$, i.e. as a pair of rectangles of the present form. For case 4 we have $b+d\leqq 1$, but $d=0$ implies $b=2$; for 8, $ac+2bd={}^3/_2$, so in each case we are led to a contradiction and the constructions are impossible.

*Solution* 5. The area of an actagon is $(2+2\sqrt{2})L^2$. Therefore $L=2$, $\sqrt{2}$, 1, or $1/\sqrt{2}$ respectively for the cases 1, 2, 4, or 8. In the cases 1 and 4 no triangles are needed on the edges and we can construct the desired octagons without them*. In case 2 each edge of each of the octagons would have to be a triangle, requiring 16 triangles,



* However, we cannot construct an octagon using *all* of the blocks. There are various ways of building an octagon with 4 squares left over or with 8 triangles left over.









which is impossible. For 8, the edges are too short to be constructed at all.

To summarize then, we can only construct one or two squares, one triangle, no rhombuses, no rectangles except as combinations of squares, and one or four octagons with or without the use of triangles. Not very many possibilities.

Still, the blocks are not devoid of some interest. The reader is left with the following problems. The first is not difficult - the second is unknown as far as an answer is concerned.

(1) Construct an octagon with an octagonal hole, using all the pieces.

(2) All the constructions shown (and many others which have been tried) have at least one line of symmetry. Are there any constructions of symmetrical shapes that have no line of symmetry? That is, where the *pieces* are not symmetrically arranged.

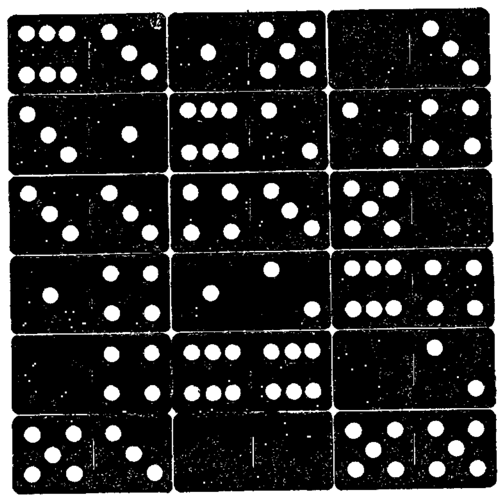

"SORRY, WE HAVE NO PLACE FOR HIM! NOW IF HE COULD ONLY WORK CALCULUS . . . . . "

# Strictly For Squares!

This little collection of Magic Squares is intended to show RMM readers that there are a variety of magic arrangements of numbers besides the ones invariably seen.

Magic Squares are easily constructed — see some of the references listed at the end of this article — but when restrictions are placed on the configuration used or the series of numbers to be arranged, the construction is not quite so simple.

Domino Magic Squares can be constructed — the problem being that a pair of numbers (dots) is to be properly handled. The arrangement to the right was sent in by H. V. Gosling who found it in a little puzzle book published (in English) in Moscow, U.S.S.R. in 1957 ("Figures For Fun" by Yakov Perelman). The sums of the dots in each row, column and two main diagonals are 18.



| | | | |
|---|---|---|---|
| ♠K | ♣A | ♦J | ♥Q |
| ♥J | ♦Q | ♣K | ♠A |
| ♣Q | ♠J | ♥A | ♦K |
| ♦A | ♥K | ♠Q | ♣J |

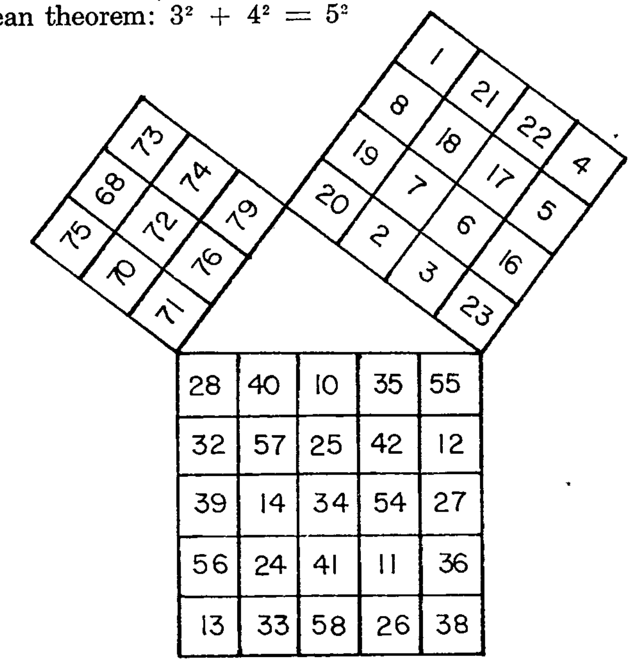
The arrangement of cards to the left, by Dorman Luke of Florida, is a Magic Square of cards that has some remarkable properties. Each row, column and the two main diagonals contain each of the four values (Ace, King, Queen, Jack) and four suits. Each set of four squares at the four corners of this Magic Square and the middle four squares hold each of the values and suits. The four corners of each 2x2 square that can be formed within the Magic Square holds the four values and suits. The four corners of the Magic Square hold all values and suits. The middle two squares in the top and bottom rows and the middle two squares in the first and fourth columns contain the four values and suits. There are other Magic arrangements which we leave to RMM readers to discover.

Here's an unusual-shaped Magic arrangement which Ben Laposky calls Pythagorean Magic Squares. The three squares together are made up of 50 different numbers. The magic constant (the sum of

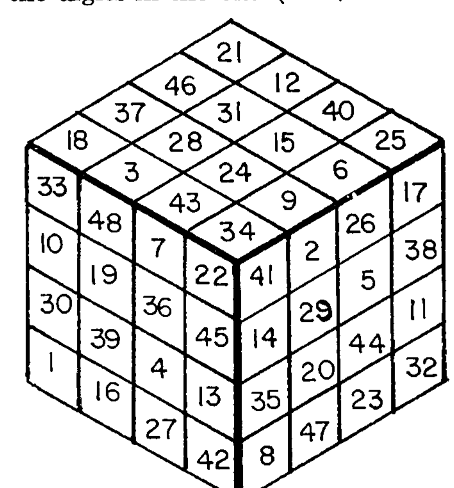




each row, column and two main diagonals) of the 3x3 square, 216, minus the magic constant of the 4x4 square, 48, equals the magic constant of the 5x5 square, 168. The total summation of the entire 3x3, 648 plus the total summation of the 4x4, 192, equals the total summation of the 5x5, 840. These summations are a parallel to the Pythagorean theorem: $3^2 + 4^2 = 5^2$




All the individual digits in the 3x3 plus all the digits in the 4x4 equal the sum of the digits in the 5x5 (99 + 84 = 183).




Mr. Laposky also gives us his "Odd and Even Sum and Difference Diamonds" above.

In the top diamond, the sum of the even numbers minus the sum of the odd numbers equals 6 all ways [e.g. (18 + 46) — (37 + 21) = 6]. In the left diamond, the difference of the even numbers plus the difference of the odd numbers is 52 all ways [e.g. (30 — 10) + (33 — 1) = 52]. In the right diamond, the difference of the evens minus the difference of the odds is zero all ways.

The sum of every "V" of 8 numbers (between any two diamonds) is 196 (e.g. 18 + 3 + 43 + 34 + 41 +14 + 35 + 8 = 196).

The three diamonds together are made up of the consecutive numbers 1 through 48.

Here is a Magic Square that looks conventional enough that was composed by no less a person than Benjamin Franklin. Though he thought he had wasted his time on such frivolities, he did produce some rather remarkable squares — the one shown and also a 16x16 which has the same set of properties to be described for the 8x8 shown here.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 52 | 61 | 4 | 13 | 20 | 29 | 36 | 45 |
| 14 | 3 | 62 | 51 | 46 | 35 | 30 | 19 |
| 53 | 60 | 5 | 12 | 21 | 28 | 37 | 44 |
| 11 | 6 | 59 | 54 | 43 | 38 | 27 | 22 |
| 55 | 58 | 7 | 10 | 23 | 26 | 39 | 42 |
| 9 | 8 | 57 | 56 | 41 | 40 | 25 | 24 |
| 50 | 63 | 2 | 15 | 18 | 31 | 34 | 47 |
| 16 | 1 | 64 | 49 | 48 | 33 | 32 | 17 |

The magic constant for this square is 260. However, any half-row or half-column has a constant of 130; the four corners plus the middle four squares total 260; bent diagonals (e.g. 52 to 54 then 43 to 45) total 260. Another curious property is the fact that *any* 2x2 smaller square inside has a total of 130.

The checkerboard affair shown on the next page is taken from the February 1935 issue of SPHINX, the now-extinct European counterpart of RECREATIONAL MATHEMATICS MAGAZINE. The sums of each of the rows and columns is equal to CEL × AA. We shall leave this magic checkerboard for RMM readers to solve — find the numbers corresponding to the letters in the squares. The whole problem can be approached quite logically.





| | | | | | | | |
|---|---|---|---|---|---|---|---|
| GUR | | CFR | | AEA | | GEO | |
| | LIO | | FOA | | FCR | | OOR |
| ARI | | URE | | UAL | | LLF | |
| | IIC | | OGG | | RAI | | RUE |
| FGI | | LCE | | AUL | | UFF | |
| | RRL | | IEF | | ILO | | OFA |
| AAG | | GLC | | GRU | | COU | |
| | IUU | | OCU | | LGC | | FIG |

Magic Squares made up of prime numbers only are of more than casual interest because of the restrictions placed on the numbers available. Here are four such Magic Squares — the concluding Magic Square of Primes being one of the most unique the Editor has seen.

Leo Moser of the University of Alberta gives us the pair of Prime Magic Squares below. The magic constant for the first square is 1496

| | | | |
|---|---|---|---|
| 29 | 1061 | 179 | 227 |
| 269 | 137 | 1019 | 71 |
| 1049 | 101 | 239 | 107 |
| 149 | 197 | 59 | 1091 |

| | | | |
|---|---|---|---|
| 31 | 1063 | 181 | 229 |
| 271 | 139 | 1021 | 73 |
| 1051 | 103 | 241 | 109 |
| 151 | 199 | 61 | 1093 |

and for the second square it is 1504. Examination by the sharp reader will disclose the fact that the first square is composed of the first member of sixteen prime pairs and the second square of the second member of the same prime pairs.

The 7x7 Magic Square shown on the next page is a remarkable one composed of prime numbers only. This is not only magic with a magic constant of 27627 along each row, column and two main diagonals, but

| | | | | | | |
|---|---|---|---|---|---|---|
| 111 | 3851 | 9257 | 1747 | 6481 | 881 | 5399 |
| 6397 | 827 | 5501 | 71 | 3779 | 9221 | 1831 |
| 3881 | 9281 | 1759 | 6361 | 911 | 5417 | 17 |
| 839 | 5381 | 101 | 3797 | 9227 | 1861 | 6421 |
| 9311 | 1777 | 6367 | 941 | 5441 | 29 | 3761 |
| 5387 | 131 | 3821 | 9239 | 1741 | 6451 | 857 |
| 1801 | 6379 | 821 | 5471 | 47 | 3767 | 9341 |

also along each broken diagonal (a pair of parallel diagonals which include 7 numbers) — e.g. $3881 + 827 + 9257 + 5471 + 1741 + 29 + 6421 = 27627$. However, this is not all! If we remove all the units places (i.e. make $9341 = 934$ or $6367 = 636$) the square remains a *Pan-diagonal Magic Square* — with the same properties as just mentioned for the primes. The magic constant along each row, column, and all diagonals is 2760 when the units' places are removed.

The above 7x7 Prime Magic Square and the tremendous one below were sent to Francis L. Miksa of Aurora, Illinois from an inmate in prison who, obviously, must remain nameless.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1153 | 8923 | 1093 | 9127 | 1327 | 9277 | 1063 | 9133 | 9661 | 1693 | 991 | 8887 | 8353 |
| 9967 | 8161 | 3253 | 2857 | 6823 | 2143 | 4447 | 8821 | 8713 | 8317 | 3001 | 3271 | 907 |
| 1831 | 8167 | 4093 | 7561 | 3631 | 3457 | 7573 | 3907 | 7411 | 3967 | 7333 | 2707 | 9043 |
| 9907 | 7687 | 7237 | 6367 | 4597 | 4723 | 6577 | 4513 | 4831 | 6451 | 3637 | 3187 | 967 |
| 1723 | 7753 | 2347 | 4603 | 5527 | 4993 | 5641 | 6073 | 4951 | 6271 | 8527 | 3121 | 9151 |
| 9421 | 2293 | 6763 | 4663 | 4657 | 9007 | 1861 | 5443 | 6217 | 6211 | 4111 | 8581 | 1453 |
| 2011 | 2683 | 6871 | 6547 | 5227 | 1873 | 5437 | 9001 | 5647 | 4327 | 4003 | 8191 | 8863 |
| 9403 | 8761 | 3877 | 4783 | 5851 | 5431 | 9013 | 1867 | 5023 | 6091 | 6997 | 2113 | 1471 |
| 1531 | 2137 | 7177 | 6673 | 5923 | 5881 | 5233 | 4801 | 5347 | 4201 | 3697 | 8737 | 9343 |
| 9643 | 2251 | 7027 | 4423 | 6277 | 6151 | 4297 | 6361 | 6043 | 4507 | 3847 | 8623 | 1231 |
| 1783 | 2311 | 3541 | 3313 | 7243 | 7417 | 3301 | 6967 | 3463 | 6907 | 6781 | 8563 | 9091 |
| 9787 | 7603 | 7621 | 8017 | 4051 | 8731 | 6427 | 2053 | 2161 | 2557 | 7873 | 2713 | 1087 |
| 2521 | 1951 | 9781 | 1747 | 9547 | 1597 | 9811 | 1741 | 1213 | 9181 | 9883 | 1987 | 9721 |





This Magic Square of Primes is truly unique! It is a bordered 13x13 magic square. The whole 13x13 arrangement has a magic constant of 70681; the next inside 11x11 square has a magic constant of 59807; the next inside 9x9 square has a magic constant of 48933; the 7x7 has a constant of 38059; the 5x5 has a constant of 27185; and the small 3x3 has a constant of 16311. The common difference between each successive set of squares is 10874.

Elsewhere in this issue of RMM the reader will find several other remarkable magic squares and cubes.

Now — do any readers of RMM have their own extraordinary Magic Squares?

* * *

## REFERENCES

Here are a few references to Magic Squares and to some of those described above.


ANDREWS, W. S. *Magic Squares and Cubes*, Dover Publications, N. Y. 1960. A collection of magic squares, cubes and spheres. Methods of construction, theories, magic circles, magic stars, fourth dimensional magic configurations — in fact just about everything about magic squares can be found in this classic collection. Benjamin Franklin's squares are analyzed on pages 96-112.

KRAITCHIK, MAURICE *Mathematical Recreations*, Dover Publications, N. Y. 1953. Kraitchik was editor of the aforementioned SPHINX and any issue of this magazine (now collectors' items) devotes a page or two to magic squares. In his book, Kraitchik devotes a chapter to Magic Squares — a most concise and complete account of Magic Squares, their properties and methods of construction.

GARDNER, MARTIN Mathematical Games *Scientific American* January 1957, pages 138-142. An unusual magic square with remarkable predictive properties.

SCHAAF, WILLIAM L. *Recreational Mathematics — A Guide to the Literature* published by the National Council of Teachers of Mathematics, 1958. Over 200 references to magic squares on pages 79-88.

# *Addition-Multiplication Magic Squares*

*by W. W. Horner*

*"Mighty are numbers, joined with art resistless"* - Euripides

The purpose of this note is to show how to make a square of order 8 which is magic both for addition and multiplication. We begin with two arithmetic series:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| I. | $a$ | $a+d$ | $a+2d$ | $a+3d$ | $a+4d$ | $a+5d$ | $a+6d$ | $a+7d$ |
| II. | $a_1$ | $a_1+d_1$ | $a_1+2d_1$ | $a_1+3d_1$ | $a_1+4d_1$ | $a_1+5d_1$ | $a_1+6d_1$ | $a_1+7d_1$ |

The terms of series I can be arranged so as to form a magic rectangle as shown in figure 1. Similarly for series II we form the rectangle in figure 2. To simplify the procedure we will represent figure 1 by figure 3 and figure 2 by figure 4.

| | | | |
|---|---|---|---|
| $a$ | $a + 6d$ | $a + 5d$ | $a + 3d$ |
| $a + 7d$ | $a + d$ | $a + 2d$ | $a + 4d$ |

Figure 1

| | | | |
|---|---|---|---|
| A | B | C | D |
| E | F | G | H |

Figure 3

| | | | |
|---|---|---|---|
| $a_1 + 3d_1$ | $a_1 + 5d_1$ | $a_1 + 6d_1$ | $a_1$ |
| $a_1 + 4d_1$ | $a_1 + 2d_1$ | $a_1 + d_1$ | $a + 7d_1$ |

Figure 2

| | | | |
|---|---|---|---|
| J | K | L | M |
| N | P | Q | R |

Figure 4

The following square is an 8x8 Eulerian square composed of the 16 letters of the above rectangles. The letter pairs are to be interpreted as products, i.e. FN = (F)(N). Now if we substitute the terms

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| FN | GL | EM | HP | AQ | DJ | BK | CR |
| AJ | DQ | BR | CK | FL | GN | EP | HM |
| BL | CN | AP | DM | EJ | HQ | FR | GK |
| EQ | HJ | FK | GR | BN | CL | AM | DP |
| HK | ER | GQ | FJ | CM | BP | DN | AL |
| CP | BM | DL | AN | HR | EK | GJ | EQ |
| DR | AK | CJ | BQ | GP | FM | HL | EN |
| GM | EP | HN | EL | DK | AR | CQ | BJ |





of the two arithmetic series above for the letters in figure 5 we find that the sum of the products in each orthogonal and main diagonal is constant and is equal to $8aa_1 + 28ad_1 + 28a_1d + 98dd_1$. Hence the square is magic for addition. Furthermore, since each orthogonal and main diagonal contains the same set of factors, the square is magic for multiplication.

| | | | |
|---|---|---|---|
| 1 | 7 | 6 | 4 |
| 8 | 2 | 3 | 5 |

Figure 6

| | | | |
|---|---|---|---|
| 12 | 14 | 15 | 9 |
| 13 | 11 | 10 | 16 |

Figure 7

Now for a numerical example. As indicated above, the arithmetic series 1, 2, 3, 4, 5, 6, 7, 8 will yield the magic rectangle in figure 6. The second series 9, 10, 11, 12, 13, 14, 15, 16 will yield the magic rectangle in figure 7. The corresponding letter values can be assigned:

| | | | |
|---|---|---|---|
| A = 1 | E = 8 | J = 12 | N = 13 |
| B = 7 | F = 2 | K = 14 | P = 11 |
| C = 6 | G = 3 | L = 15 | Q = 10 |
| D = 4 | H = 5 | M = 9 | R = 16 |

Substituting these values in the square of figure 5 we obtain figure 8.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 26 | 45 | 72 | 55 | 10 | 48 | 98 | 96 |
| 12 | 40 | 112 | 84 | 30 | 39 | 88 | 45 |
| 105 | 78 | 11 | 36 | 96 | 50 | 32 | 42 |
| 80 | 60 | 28 | 48 | 91 | 90 | 9 | 44 |
| 70 | 128 | 30 | 24 | 54 | 77 | 52 | 15 |
| 66 | 63 | 60 | 13 | 80 | 112 | 36 | 20 |
| 64 | 14 | 72 | 70 | 33 | 18 | 75 | 104 |
| 27 | 22 | 65 | 120 | 56 | 16 | 60 | 84 |

Figure 8

The magic constant for addition is 450 and the magic constant for multiplication is 16! or 20,922,789,888,000. It is noted that the square in figure 8 does not contain 64 distinct numbers. This can be corrected by choosing different series.

Let us take the following series of numbers:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| III. | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 9 |
| IV. | 13 | 15 | 17 | 19 | 23 | 25 | 27 | 29 |

These series were selected with the following criteria in mind:

A. At most there can be but one term common to both series.

B. There can be no common ratio between any two terms of one series and any two terms of the other.

The square shown in figure 9 can then be constructed. The magic constant for addition is 840 and the magic constant for multiplication is 2,058,068,231,856,000.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 46 | 81 | 117 | 102 | 15 | 76 | 200 | 203 |
| 19 | 60 | 232 | 175 | 54 | 69 | 153 | 78 |
| 216 | 161 | 17 | 52 | 171 | 90 | 58 | 75 |
| 135 | 114 | 50 | 87 | 184 | 189 | 13 | 68 |
| 150 | 261 | 45 | 38 | 91 | 136 | 92 | 27 |
| 119 | 104 | 108 | 23 | 174 | 225 | 57 | 30 |
| 116 | 25 | 133 | 120 | 51 | 26 | 162 | 207 |
| 39 | 34 | 138 | 243 | 100 | 29 | 105 | 152 |

Figure 9

This square contains six remarkable constellations whose terms are arranged symmetrically about the center as shown in figure 10. If figure 10 is superimposed on figure 9, the eight numbers under the same letter will have the same sum and product as the square itself.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| A | B | | F | F | | B | A |
| B | A | E | | | E | A | B |
| | E | C | D | D | C | E | |
| F | | D | C | C | D | | F |
| F | | D | C | C | D | | F |
| | E | C | D | D | C | E | |
| B | A | E | | | E | A | B |
| A | B | | F | F | | B | A |

Figure 10



Editor's Note: *Ali R. Amir-Moez of the University of Florida, formerly of Purdue University, has written and drawn an excellent little booklet called* Ruler, Compasses, and Fun. *It is intended as an introduction to the use of ruler and compass for students in the lower grades. ..The emphasis is on the construction of mathematical forms and leads to the formation of mosaics and patterns of beauty. The booklet may be purchased for* 65 *cents from the Lafayette Printing Co.,* 511-17 *Ferry Street, Lafayette, Indiana.*

*The following article, not included in the above booklet, shows RMM readers how to construct mathematically correct spirals.*

# Circles and Spirals

by Ali R. Amir-Moez

Usually a spiral is drawn free hand, and it has to be touched and retouched. In this note we would like to discuss construction of tangent circles and apply the results to construction of spirals.

1. *Tangent circles*: An important property of two tangent circles is that the line connecting the two centers passes through the point of tangency (Fig. 1). This idea enables us to draw practically an approximation of any curve with arcs of circles, tangent to one another. For example we shall describe the curve in (Fig. 2). The curve consists of the arc AB with center $O_1$, the arc BC with center $O_2$, the arc CD with center $O_3$, the arc DE with center $O_4$, and the arc EF with center $O_5$.

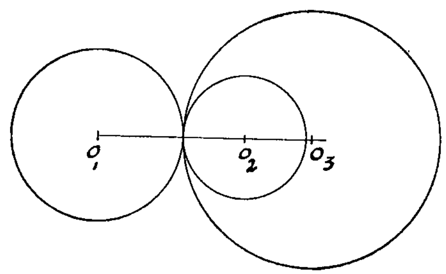


Fig. 1

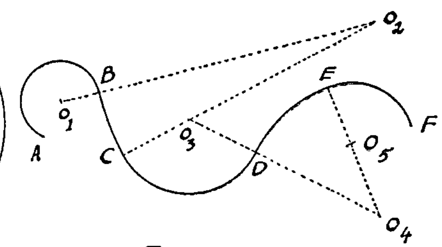


Fig. 2

2. *Spirals*: First we describe a two-center spiral. Let two points 1 and 2 be marked on the paper as centers of the spiral (Fig. 3). We draw a half circle with center 1 and radius 12. Then we choose 2 to be the center. We double the radius and draw another half circle starting with the end of the half circle which is already drawn. Again we choose 1 as the center and we double the radius. We draw another half circle as a continuation of the previous half circle. Continuing this way we get a two center spiral.

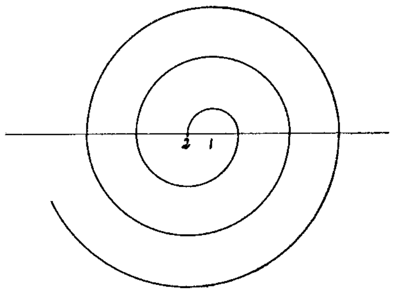



Fig. 3

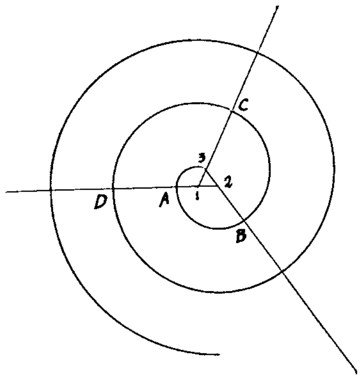



Fig. 4

Now for a three center spiral we draw an equilateral triangle. Then we extend the sides only in one direction (Fig. 4). Choose 1 as the center and the distance between 1 and 3 as the radius. Draw the arc of the circle until it meets the line 21 at A. Now we move the center to 2 and open the compasses up to A. We draw the arc AB. Then we choose 3 as the center and 3B as the radius. We draw the arc BC. Again we move the center to 1 and draw the arc CD. Continuing this way we get a three-center spiral.

For a four center spiral we draw a square, and extend its sides in one direction (Fig. 5). The technique of drawing a spiral is the same as before. Here we have added another spiral to suggest other possibilities. We draw the half circle with diameter 14. Then we start the other spiral with center at 2.

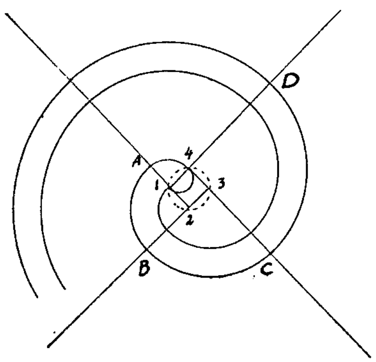



Fig. 5

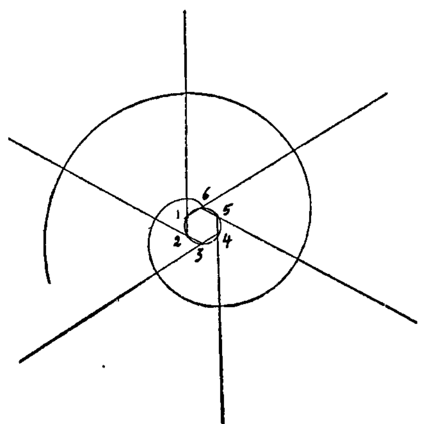



Fig. 6





Now for a six center spiral we only give a diagram (Fig. 6). In all cases it is possible to make multiple spirals.

As the reader already guessed it is possible to have any multi-center spiral. We have to draw a regular polygon and extend the sides in one direction. Then we follow the methods mentioned previously. The more centers used the more accurate the spiral becomes.

Eventually, by using the infinite number of points on the circumference of a circle as successive centers, we can draw a true spiral called an *involute of the circle* (Fig. 7). This is the spiral generated by the end of a piece of string as it is unound from a round dowel.

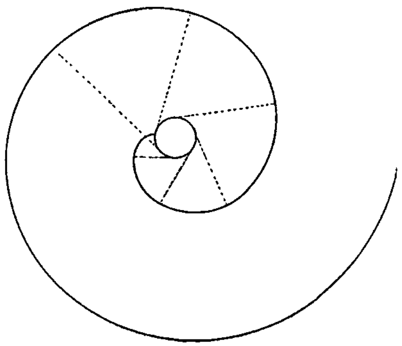


Fig. 7

# Puzzles and Problems

The number of RMM readers who sent in their answers is most heartening to the Editor. The answers to the August set of puzzles is given on pages 59-60.

1. A One-Truck Mind

It was a mistake to put an accountant on a dispatcher's job, but Mr. Smith found himself doing just that due to a rash of flu which had disabled half the working force. He knew he'd miss supper tonight! Hours ago he had dispatched a truck with the wrong merchandise. Now he knew the truck could only go 35 miles per hour and his car could sustain 60 miles per hour - he just wondered when he'd catch up to the truck which left at Noon. It was already 4:30.

(Gerard Mosler)

2. Christmas Bonus

"Mom!" protested Paul, as her sleeve swept his money off the table. "There goes the Christmas bonus!"

"I'm sorry, dear." His mother helped recover the coins which had scattered over the rug. "But only these few cents?"

'It's for the kids in my troop," explained the boy. "The money comes from the chores we do and I share it out equally amongst us every month."

"Like a treasurer," commented his mother.

Paul nodded. "We've done it three months and each time we've had a bit left over. The first month there was \$23.30 and that gave a few cents left over after we divided it out. The second month there was \$31.65 with twice as much left over and the last month we got \$37.24 with three times as much left over as the first month."

Retrieving the last coin, Paul slipped them into his pocket. "I save these over as a bonus for us for Christmas."

There had been no changes in Paul's troop, so how many boys were in it? And how much had accumulated in three months for their Christmas bonus?

(J. A. H. Hunter)

3. The Pyramid

"There is a vacant square plot on the palace grounds," said the Pharoah to his chief architect, "and I want you to set up a pyramid on it. Make it something unusual."

In due time the architect reported all was ready for inspection.

Viewing the pyramid carefully Pharoah remarked, "I don't see anything special about all this, except, perhaps, that it has an equal margin between its base and the sides of the plot."

"That is the important point," replied the architect. "You see, each dimension of the pyramid - the base length, slant edge and vertical height - is a whole number of yards in length and, with this margin which is 2 yards wide, the apex is also a whole number of yards distant





from the corners of the plot. This is the only pyramid under 30 feet high having these properties."

What is its vertical height and the area of the plot?

(Sinclair Grant - Perth, Scotland)

4. The Golden Spheres

"With this great pile of glittering coins," the Pharoah ordered the trembling goldsmith, "you are to make for me this night four golden spheres. Failure to follow my every instruction will result in instant death."

The poor craftsman managed to find voice enough to ask, "How large do you wish the spheres to be?"

"I have consulted the priests. The radius of each sphere measured in groods must be a whole number smaller than 12 and each sphere must be different in size. The sum of the diameters of the smallest and largest spheres must be 26 groods. This rule applies also to the sum of the other two diameters. Furthermore, you must use every scrap of gold. Now set to work. If you are not finished before the rising of the sun tomorrow, you will die."

As soon as the Pharoah left, the goldsmith began feverishly to work by his furnace, weighing the gold and making his calculations. He finally arrived at a solution correct in all respects but one: the largest sphere had a radius of 12 groods. In desparation, the goldsmith decided to risk the Pharoah's anger. He completed his work just before the light of day. At dawn he was executed for failure to follow the exact instructions of the priests.

What size should the sphere have been?

(D. C. Cross - Birmingham, England)

5. Turning Corners

What is the longest piece of board one inch thick and three inches wide that can be carried around a right-angled corridor which is 4 feet wide and 8½ feet high? Naturally, we can't bend the board.

What would be the general solution to this problem where the board is A thick and B wide, the corridor is C wide and D high?

(J.S.M.)

6. Souvenir Bargaining in Octavia

In Octavia, the inhabitants only have eight fingers and count in the scale of eight. Their monetary system has 20 ziggles to 1 zaggle. The price of a souvenir was 6 zaggles 11 ziggles but I was offered a reduction of one-fifth if I bought 6. How much were the six souvenirs?

(Norman J. Hill - Picton, Ontario)

7. Major Perkins

Only bleary-eyed military men ever try to shave with a toothbrush.
No bleary-eyed military men are at all musical.
Men who never try to shave with a toothbrush never play golf.
Major Perkins plays the tuba on Wednesdays.
Only musical people who are golfers forget to change their socks.

(Assuming the truth of the above statements, wha can you deduce about the Major?) (J. A. Lindon - Surrey, England)

8. A Travelling Man

A traveller sets out to cross a desert. On the first day he covers 1/10 of the journey; on the second day he goes 2/3 of the distance already travelled. He continues on in this manner, alternating the days on which he does 1/10 of the distance still to be done, with days on which he travels 2/3 of the total distance already covered. At the end of the seventh day he finds that another 22.5 miles will see the end of his journey. How wide is the desert?

(Noel A. Longmore - Kent, England)

9. The Two-Car Problem

Two cars set out at noon to go from Town A to Town B. One car does the journey in 6 hours and the other does it in 8 hours. Assuming that they follow the same route and travel at uniform speeds, at what time will the faster car be three times further from B than the slower car is from A? At what time will the slower car be three times further from A than the faster car is from B?

(Noel A. Longmore - Kent, England)

10. Construction Problem

CDEF is a square inscribed in a right triangle ABC. If DE is 60 units long and AB is 221 units long find AC and BC knowing that AC>BC. See figure 1 below. (Remy Landau - Montreal, Quebec)

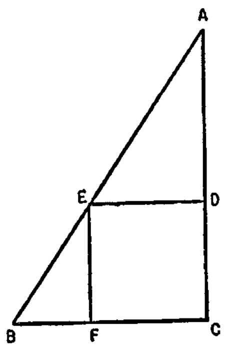



Figure 1

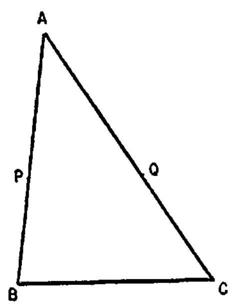



Figure 2

11. Another Construction

Construct points P and Q on sides AB and AC, respectively, of triangle ABC such that AP = PQ = QC. See Figure 2 above.

(H. E. Tester - Middlesex, England)



## *A Look at the Fourth Dimension*

*Dr. Jan S. Paul*

Man thinks of himself as living in a three dimensional world; one of length - how long is the road ahead; width - how wide is the ocean; and height - how high is the mountain. Yet he lives with another dimension - time.

Suppose it were possible to agree on an arbitrary unit of time measure, so that we could determine actual distance in time between, say, today, and a month from today. If we can do this, then time can also be used as a tangable unit of measurement, just as length, width, and height.

Let us put this theory into practice by first constructing a box to represent today, (figure 1). Let us then construct an identical box to represent one month from today, (figure 2).

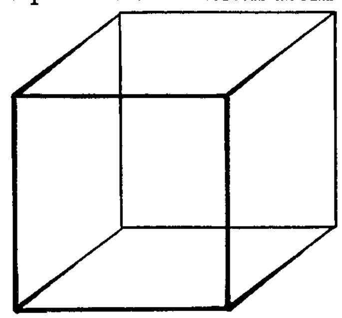


Figure 1

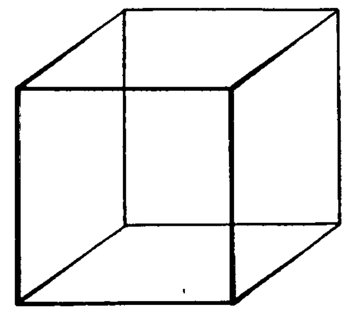


Figure 2

Each of these boxes are three dimensional objects, having length, width, and height.

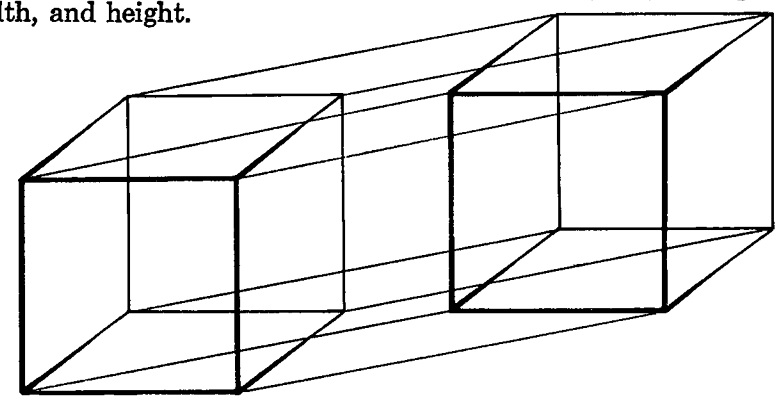


Figure 3

When we connect our two boxes with "time lines", the result is a true fourth dimensional object, having length, width, height, and the added dimension of time. It also contains 24 faces, 32 edges, 16 vertices, and untold angles.

The resulting figure is a true fourth dimensional figure, having length, and width, and height, plus the added dimension of time.

Thus we see that in reality, man does live in a fourth dimensional world; and each of us, like our box, has a fourth dimension - for our very lifetime, from birth until death, is a dimension in time.

# How to Make a Magic Tessarack

*By MAXEY BROOKE*

The origin of the magic square is lost in antiquity. Tradition has it that the first one was discovered by the Chinese emperor Yu about 2200 B.C. on the back of a sacred turtle.

During the first of the 18th century, one Joseph Sauvier presented the French Academy of Science with the first magic cube.

And V. Schlegel (*Bull. Soc. Math. France*, 1892, p. 97) invented the magic tessarack which we propose to develop here, perhaps for the first time in English.

Now, a few words about the general properties of the tessarack, (sometimes called a four-dimensional cube) after a few statements relative to one, two and three dimensions.

A line segment consists of the two endpoints and the line itself, a total of three elements.

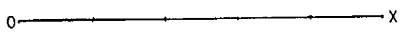
A square consists of four corners, four sides, and the square itself; a total of nine elements.

A cube consists of 8 corners, 12 sides, 6 faces, and the cube itself; a total of 27 elements.

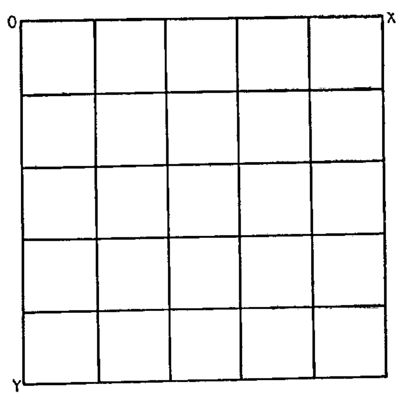
The number of elements are 3, $3^2$, and $3^3$ for 1, 2, and 3 dimensional figures. By analogy, the tessarack of 4 dimensions contains $3^4$ or 81 elements; 16 corners, 32 faces, 24 sides, 8 cubes, and the tessarack itself.

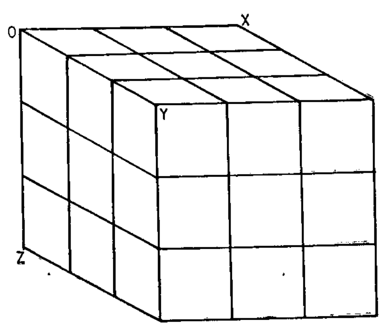
Next, we will take up modular construction.

Consider a line segment OX, divided into $p$ equal segments.




This one dimensional figure is a modular line. The value of the module is $p$ ; in this case, $p = 5$.




A line segment OY, equal to OX, is drawn perpendicular to OX at O and is divided into the same number of segments as OX. Lines parallel to OX are drawn from each modular point on OY; lines parallel to OY are drawn from each modular point on OX.





The resulting two dimensional figure is a modular square and again the modular value is $p = 5$.

A line segment OZ, equal to OX is drawn perpendicular to the plane OX-OY and is divided into the three segments equal to the three segments of OX. Planes equal to and parallel to OX-OY are drawn from each modular point on OZ. Planes parallel to OX-OZ are drawn at each modular point OX-OY, and planes parallel to OY-OX are drawn through the modular points on OX.

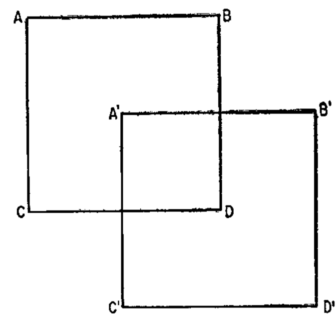
The resulting three dimensional figure is a modular cube. Here the modular value is $p = 3$.

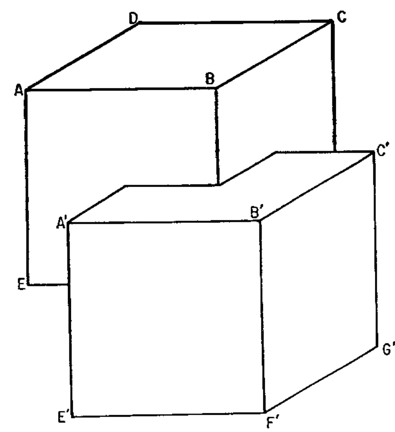
A line segment OQ, equal to OX is drawn perpendicular to cube OX-OY-OZ at O and is divided into the same number of segments as OX (This line is imaginary in the fourth dimension). Cubes equal to and parallel to OX-OY-OZ are drawn from each modular point on OQ. Cubes parallel to OX-OY-OQ are drawn from each modular point on OZ; cubes parallel to OX-OZ-OQ are drawn from each modular point on OY; and cubes parallel to OY-OZ-OQ are drawn from each modular point on OX.

The resulting four dimensional figure is a modular tessarack. It cannot be drawn on a two dimensional plane with any degree of accuracy.

Of course, still higher hyper-dimensional figures can be constructed in an analogous manner.

A cube can be represented two-dimensionally by drawing two intersecting squares ABCD and A′B′C′D′. By connecting A and A′, B and B′, C and C′, D and D′, a skeleton cube is represented. The other bounding faces are AA′B′B, BB′D′D, DD′C′C, and CC′A′A.




In a like manner, the tessarack can be represented by two intersecting cubes ABCDEFGH and A′B′C′D′E′F′G′H′. Corners F,G,H,D′ and H′ cannot be seen. By connecting A and A′, B and B′, C and C′, D and D′, E and E′, F and F′, G and G′, H and H′, a skelton tesserack is represented. The other bounding cubes are ABEFA′B′E′F′, BFGCB′F′G′C′ ,DCGHD′C′G′H′, ADHEA′D′H′E′,ABCDA′B′C′D′, and EFGHE′F′G′H′.

At this point, a few mathematical conventions must be introduced.

All numbers are to the base $p$, where $p$ is the modular value.

Numbers smaller than $p$ are written with $n$ digits, where $n$ is the number of dimensions adding zeros to the left if there are fewer than $n$ digits.

The sum of two numbers $aa'a'' + bb'b'' = cc'c''$ are obtained thus; $a + b = c$, $a' + b' = c'$, $a'' + b'' = c''$. (base $p$). (We do not "carry" numbers from position to position). For example:

$$214 + 243 = 1012 \text{ (base 5)}$$

but by adopting the convention just outlined we have

$$214 + 243 = 402 \text{ (base 5)}$$

Then $$214 + 563 = 000 \text{ (base 7)}$$

A number series $(r)_p$ is produced thus:

$$(r)_p = 0, r, 2r \ldots . (p - 1)r$$

For example, by following the addition convention above:

$$(410)_5 = 0, 410, 320, 230, 140.$$

[The series would normally be written:

$$(410)_5 = 0, 410, 1320, 2230, 3140.]$$

Now to develop a magic square. First draw a modular square. In the first row insert the series $(11)_5$. Now choose a number that does not appear in the series, say 23. In the left column put the series $(23)_5$. The rest of the rows are filled by adding 11, 22, 33, and 44 to the numbers in the left column.

| | | | | |
|---|---|---|---|---|
| 0 | 11 | 22 | 33 | 44 |
| 23 | 34 | 40 | 01 | 12 |
| 41 | 02 | 13 | 24 | 30 |
| 14 | 20 | 31 | 42 | 03 |
| 32 | 43 | 04 | 10 | 21 |

| | | | | |
|---|---|---|---|---|
| 0 | 6 | 12 | 18 | 24 |
| 13 | 19 | 20 | 1 | 7 |
| 21 | 2 | 8 | 14 | 15 |
| 9 | 10 | 16 | 22 | 3 |
| 17 | 23 | 4 | 5 | 11 |

Converting to a decimal base, we have a familiar panmagic square whose constant is 60. (Besides having all the rows, columns and two main diagonals equal to the magic constant, a panmagic square — or





pandiagonal magic square — contains broken diagonals with the magic constant. For example, in the diagram we find 13 + 6 + 4 + 22 + 15 = 60 or 17 + 6 + 20 + 14 + 3 = 60.)

Next, a magic cube. Insert the series $(121)_3$ along the axis OX. series $(211)_3$ along OY, and series $(112)_3$ along OZ. The modules are filled by adding pairs of squares together in a given plane. E.g. in the plane OX-OY add the values along the OX axis to those along the OY axis (121 + 211 = 022; 121 + 122 = 210; etc.). In the plane OX-OZ add the values along the OX axis to those along the OZ axis (121 + 112 = 200; etc.).

In order to make each row and column of squares in each of these portions equal to 000, we do a bit of switching to get our final figure.

This magic figure converts to the decimal system in figure 2.

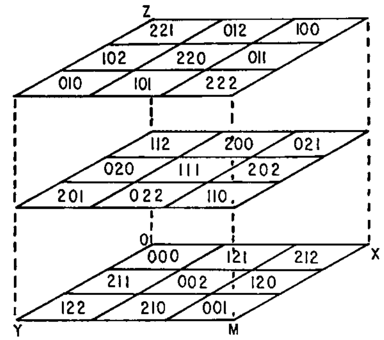



Figure 1

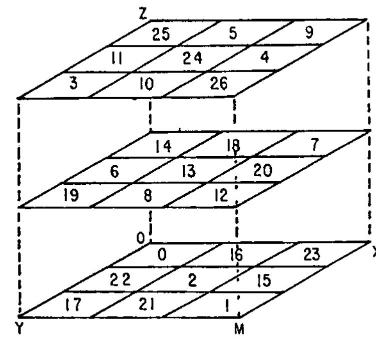



Figure 2

The cube is magic because each face, each section parallel to a face, the six diagonals, and certain broken diagonals form squares which are magic.

The six faces are semi-magic, (i.e. the rows and columns equal the magic constant, but the two main diagonals do not), the three interior sections are magic.

There are 27 ways of lining up three consecutive squares, the sum of which is 39. With a little study, a number of other magic properties, such as a line from Z to M, can be discovered.

The magic tessarack requires both imagination and ingenuity. We use four series which we designate as $(r)_p = (2212)_3$; $(s)_p = (2122)_3$; $(t)_p = (1222)_3$; $(u)_p = (2221)_3$. They are used on the OX, OY, OZ, and OQ axes. The schematic diagram is set up thus;

| | o | r | 2r | s | r+s | 2r+s | 2s | r+2s | 2r+2s |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0000 | 2212 | 1121 | 2122 | 1001 | 0210 | 1211 | 0120 | 2002 |
| t | 1222 | 0101 | 2010 | 0011 | 2220 | 1102 | 2100 | 1012 | 0221 |
| 2t | 2111 | 1020 | 0202 | 1200 | 0112 | 2021 | 0022 | 2201 | 1110 |
| u | 2221 | 1100 | 0012 | 1010 | 0222 | 2101 | 0102 | 2011 | 1220 |
| t+u | 0110 | 2022 | 1201 | 2202 | 1111 | 0020 | 1021 | 0200 | 2112 |
| 2t+u | 1002 | 0211 | 2120 | 0121 | 2000 | 1212 | 2210 | 1122 | 0001 |
| 2u | 1112 | 0021 | 2200 | 0201 | 2110 | 1022 | 2020 | 1202 | 0111 |
| t+2u | 2001 | 1210 | 0122 | 1120 | 0002 | 2211 | 0212 | 2121 | 1000 |
| 2t+2u | 0220 | 2102 | 1011 | 2012 | 1221 | 0100 | 1101 | 0020 | 2222 |

Converted to the decimal system, this becomes

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 77 | 43 | 71 | 28 | 21 | 49 | 15 | 56 |
| 53 | 10 | 57 | 4 | 78 | 38 | 63 | 32 | 25 |
| 67 | 33 | 20 | 45 | 14 | 61 | 8 | 73 | 39 |
| 79 | 36 | 5 | 30 | 26 | 64 | 11 | 58 | 51 |
| 12 | 62 | 46 | 74 | 40 | 6 | 34 | 18 | 68 |
| 29 | 22 | 69 | 16 | 54 | 50 | 75 | 44 | 1 |
| 41 | 7 | 72 | 19 | 66 | 35 | 60 | 47 | 13 |
| 55 | 48 | 17 | 42 | 2 | 76 | 23 | 70 | 27 |
| 24 | 65 | 31 | 59 | 52 | 9 | 37 | 3 | 80 |

You will note that this array is a magic square with a constant of 360. It consists of 8 semi-magic squares and one central magic square with a constant of 120. These make up the faces and sections of the cubes (three in a row) that bound the tessarack. These sets are:

1. The cubes composed of the linear rows and of vertical columns.
2. The cubes made up of the 1st, 4th, 7th; the 2nd, 5th, and 8th; and the 3rd, 6th, and 9th horizontal lines and vertical rows.

There are 12 diagonal sets of three cubes each, one of which is;

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 77 | 43 | 4 | 78 | 58 | 8 | 73 | 39 |
| 79 | 36 | 5 | 74 | 40 | 6 | 75 | 44 | 1 |
| 41 | 7 | 72 | 42 | 2 | 76 | 37 | 3 | 80 |

And finally 24 broken diagonals of cubes. This makes the tessarack magic.



# *Word Games* by *S. Baker*

## "7" LETTER SCRAMBLE

In this issue we are going to give the SEVEN LETTER SCRAMBLE a new twist. Here you have a list of the seven sets of six letters only. To solve this puzzle you must add a letter to make a list of seven-letter words - but the letter to be added is always the same. After compiling this list, go back to the original set of letters and this time add one letter again - but this time all the letters must be different.

E I O F N N
E E I C N G
E Y N D R A
E A O M S T
E E A L C T
E E L G A N
E E T G A N

## CHANGE A LETTER

1. B A L C O N Y
2.
3.
4.
5.
6. C O L L A R S

You must go from BALCONY to COLLARS in exactly five changes. Each change is to consist of a new word which differs from the one above by only one letter change.

E.g. we could go from START to DRESS in four changes as follows: START-STATE-DATES-DARES-DRESS.

---

The Answers to the August Issue *Word Games.*

Here is a list of the various solutions to Mr. Longmore's BISHOP puzzle.

| | | | | | | |
|---|---|---|---|---|---|---|
| 1. | BISHOP | IMPALE | SPINET | HANGAR | OLEATE | PETREL |
| 2. | BISHOP | ILLUME | SLIDES | HUDDLE | OMELET | PESETA |
| 3. | BISHOP | INTIRE | STADDA | HIDDER | ORDEAL | PEARLS |
| 4. | BISHOP | INTONE | STOMAT | HOMAGE | ONAGER | PETERS |
| 5. | BISHOP | INHUME | SHORES | HURDLE | OMELET | PESETA |
| 6. | BISHOP | INHERE | SHARES | HERMIT | OREIDE | PESTER |
| 7. | BISHOP | INCOME | SCOPES | HOPPLE | OMELET | PESETA |
| 8. | BISHOP | ILLUME | SLICES | HUCKLE | OMELET | PESETA |
| 9. | BISHOP | IMPALA | SPINED | HANGAR | OLEATE | PADRES |
| 10. | BISHOP | IMPALE | SPIREA | HARBIN | OLEINE | PEANED |
| 11. | BISHOP | INHERE | SHARES | HERONS | ORENDA | PESSAS |

And the list of solvers, with the solutions submitted in parentheses:

W. A. Robb, Ottawa, Ontario (1,2,3,4,6); Arthur B. Gardner, Baltimore, Maryland (8,9,10); Jack Halliburton, Los Angeles (11); Norvin Pallas, Cleveland, Ohio (5); J. Charles Clapham, Vancouver, B.C. (7); Joseph D. E. Konhauser, State College, Pa. (1). Noel A. Longmore (the originator of the problem) submitted solution 1.

The common English word which has a negative and double negative form, but no positive form is UNINHIBITED. No one solved this problem.

Also, no one was able to come up with any other words containing the five vowels in their correct or reverse order. (The known words, given in the August issue, are ABSTEMIOUS, FACETIOUS and SUBCONTINENTAL.)

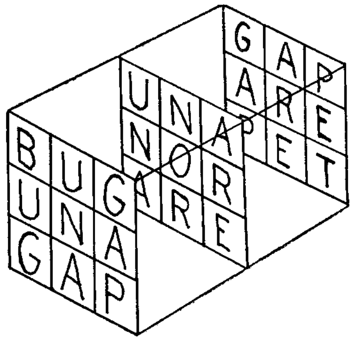
# *Word Cubes and 4-D Hypercubes*

*J. A. Lindon, Surrey, England*

A word-square has symmetry about its main diagonal: the *nth* row and the *nth* column are alike:

| | | |
|---|---|---|
| B | U | G |
| U | N | A |
| G | A | P |

In a word-cube the corresponding diagonal of symmetry goes right through the solid, so that we have in effect layers of word-squares such that every *nth* one, numbered from one of the three coordinate planes, is the same:



The above plane sections are by cuts made transversely to one of the axes. It will be seen that a cut made transversely to one of the other axes will give the same result, except that some sections will have to be read from the reverse side of the plane — we cannot arrange to have our B at the same corner of all three external faces. We can write this schematically:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 2 | 4 | 5 | 3 | 5 | 6 |
| 2 | 4 | 5 | 4 | 7 | 8 | 5 | 8 | 9 |
| 3 | 5 | 6 | 5 | 8 | 9 | 6 | 9 | 10 |

and if these are turned into word-squares, a particular digit representing the same letter throughout, they will be found to combine to form a word-cube. The corresponding scheme for 4th-order squares is:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 2 | 5 | 6 | 7 | 3 | 6 | 8 | 9 | 4 | 7 | 9 | 10 |
| 2 | 5 | 6 | 7 | 5 | 11 | 12 | 13 | 6 | 12 | 14 | 15 | 7 | 13 | 15 | 16 |
| 3 | 6 | 8 | 9 | 6 | 12 | 14 | 15 | 8 | 14 | 17 | 18 | 9 | 15 | 18 | 19 |
| 4 | 7 | 9 | 10 | 7 | 13 | 15 | 16 | 9 | 15 | 18 | 19 | 10 | 16 | 19 | 20 |

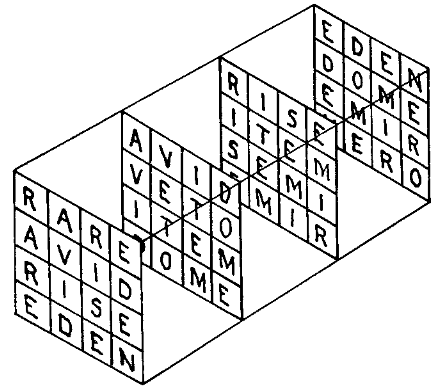
and here is an example of a 4th-order word-cube made to this pattern:

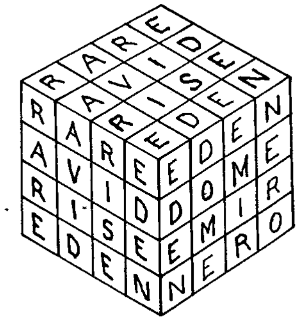
| | | | |
|---|---|---|---|
| RARE | AVID | RISE | EDEN |
| AVID | VETO | ITEM | DOME |
| RISE | ITEM | SEMI | EMIR |
| EDEN | DOME | EMIR | NERO. |





Or in 3-D form:





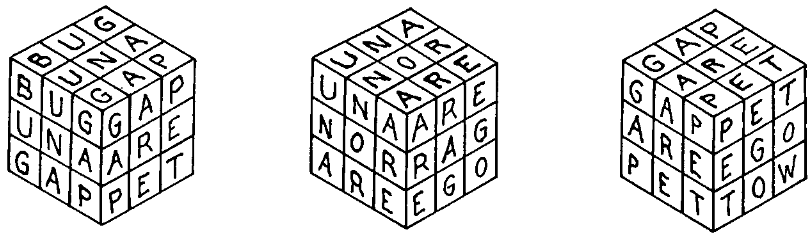
A word-cube, then, can be represented by a single row of word-squares . To represent a 4-D word-hypercube of order *n* we must have a complete set of *n* such rows, forming in effect a 2-D array of appropriate word-squares. In this array each row will represent a different word-cube, and each such cube will be one of the 3-D sections of the 4-D hypercube by a cut made transversely to one of the four coordinate axes. Here is the numerical scheme for a 3rd-order hypercube:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 2 | 4 | 5 | 3 | 5 | 6 |
| 2 | 4 | 5 | 4 | 7 | 8 | 5 | 8 | 9 |
| 3 | 5 | 6 | 5 | 8 | 9 | 6 | 9 | 10 |
| 2 | 4 | 5 | 4 | 7 | 8 | 5 | 8 | 9 |
| 4 | 7 | 8 | 7 | 11 | 12 | 8 | 12 | 13 |
| 5 | 8 | 9 | 8 | 12 | 13 | 9 | 13 | 14 |
| 3 | 5 | 6 | 5 | 8 | 9 | 6 | 9 | 10 |
| 5 | 8 | 9 | 8 | 12 | 13 | 9 | 13 | 14 |
| 6 | 9 | 10 | 9 | 13 | 14 | 10 | 14 | 15 |

The squares to the left of and below the zigzag line are repetitions obtainable from symmetry, and could have been omitted . Taking the 3rd-order cube already given as the first in the above scheme, we can complete the latter thus:

| | | |
|---|---|---|
| BUG | UNA | GAP |
| UNA | NOR | ARE |
| GAP | ARE | PET |
| | NOR | ARE |
| | OVA | RAG |
| | RAG | EGO |
| | | PET |
| | | EGO |
| | | TOW |

and this in fact represents a 4-D word-hypercube of order 3. Cuts made transversely to any of the four coordinate axes give the following set of word-cubes as solid sections:



The 4th-order scheme is:

| | | | |
|---|---|---|---|
| 1 2 3 4 | 2 5 6 7 | 3 6 8 9 | 4 7 9 10 |
| 2 5 6 7 | 5 11 12 13 | 6 12 14 15 | 7 13 15 16 |
| 3 6 8 9 | 6 12 14 15 | 8 14 17 18 | 9 15 18 19 |
| 4 7 9 10 | 7 13 15 16 | 9 15 18 19 | 10 16 19 20 |
| | 5 11 12 13 | 6 12 14 15 | 7 13 15 16 |
| | 11 21 22 23 | 12 22 24 25 | 13 23 25 26 |
| | 12 22 24 25 | 14 24 27 28 | 15 25 28 29 |
| | 13 23 25 26 | 15 25 28 29 | 16 26 29 30 |
| | | 8 14 17 18 | 9 15 18 19 |
| | | 14 24 27 28 | 15 25 28 29 |
| | | 17 27 31 32 | 18 28 32 33 |
| | | 18 28 32 33 | 19 29 33 34 |
| | | | 10 16 19 20 |
| | | | 16 26 29 30 |
| | | | 19 29 33 34 |
| | | | 20 30 34 35 |

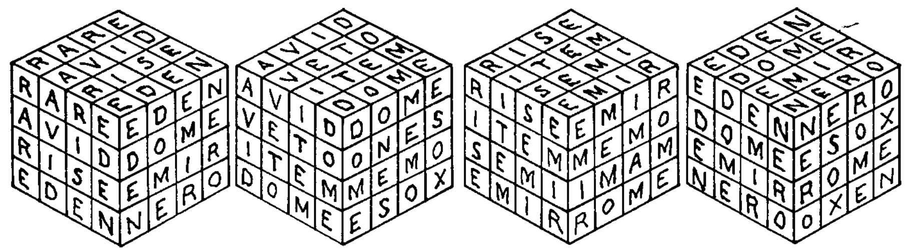
And here is an actual example (not quite so easy to fabricate as our 3rd-order one):

| | | | |
|---|---|---|---|
| RARE | AVID | RISE | EDEN |
| AVID | VETO | ITEM | DOME |
| RISE | ITEM | SEMI | EMIR |
| EDEN | DOME | EMIR | NERO |
| | VETO | ITEM | DOME |
| | ERIN | TIDE | ONES |
| | TIDE | EDAM | MEMO |
| | ONES | MEMO | ESOX |
| | | SEMI | EMIR |
| | | EDAM | MEMO |
| | | MAYA | IMAM |
| | | IMAM | ROME |
| | | | NERO |
| | | | ESOX |
| | | | ROME |
| | | | OXEN |





And here are the four principal solid sections of the above 4-D hypercube:



The internal lettering of the various layers is shown in the foregoing array scheme. Every one of the sixty-four letters comprising the cube on the left begins a word whose remaining three letters are to be found in corresponding positions in the other three cubes. The hypercube itself contains 256 letters and consists of 256 interlocking words, though many of these are of course the same.

Any reader who succeeds in putting the above four cubes *symmetrically* together to form the actual 4-D hypercube will receive one genuine superdried 4-D hyperprune, with his initials engraved on the inside of the stone. (From the Editor — he lives on them!)

* * *

# *Alphametics*

It is the intent to publish *Alphametics* with unique answers. However, sharp RMM readers sometimes find two or more answers to these puzzles - and we like this! The answers to the August *Alphametics* follow this month's selection.

This may be a very appropriately-worded little puzzle! (J. A. H. Hunter)

(Afterthought by the Editor: A year's subscription, or extension, to RMM for anyone who finds a solution to this *Alphametic* if the word TIRED is substituted for TRIED.)

```
  T H E S E
  T E A S E
  T R I E D
 -----------
R E A D E R
```

---

```
    S E E
    T H E
   -------
  B x x x
  I x x
x x R x
-----------
x x x D S x
```

This *Alphametic* is for the BIRDS! But try to solve it anyway.

(Alan Gold)

---

When Edward, the big game hunter, followed the trail of the man-eating tiger a shade too far into the jungle, the local reporter filed the following cable to the news agencies:

ED : IN : : DEN : END

(It will be noted that the punctuation is somewhat unconventional - though mathematical.)

With such information what can be DENIED?

(Derrick Murdoch)

---

BBG for Canadians needs no explanation - but for U. S. readers of RMM it means Board of Broadcast Governors and this *Alphametic* is appropriately worded.

(Alan Gold)

```
    O H
    N O
   -----
    N E W
  B B G
  -------
  B L O W
```

* * *

The RMM readers who correctly solved the August *Alphametics* are listed after the answers here.

```
(1)    S E N D          9567
       M O R E    =     1085
     ---------        ------
     M O N E Y         10652
```





(2) $\dfrac{\text{E V E}}{\text{D I D}} = .\text{TALKTALKTALK}\ldots\ldots$

(a) $\dfrac{212}{606} = .349834983498\ldots$

(b) $\dfrac{242}{303} = .798679867986\ldots$

```
(3)     C A T S      (a)   3462      (b)   3470
          E A T             546             947
      -----------          -----           -----
      x x x x x            20772           24290
    x x x x x            13848      or   13880
  x x x x x             17310           31230
  -------------         -------         -------
  x M O U S E S         1890252         3286090
```

```
(4) W H E A T   (a) 9 5 3 0 7   (b) 9 7 3 0 5
      F I E L D       1 X 3 2 8       1 8 3 2 X
    -----------     -----------     -----------
    F A R M E R     1 0 4 6 3 4     1 0 4 6 3 4
```

Where X stands for 10 in the base 11 system. H - I and T - D are interchangeable.

```
(5)      D A N       542
         A N D       425
       E D N A   =  3524
           A T        41
       -------      ----
       A D E N      4532
```

(6) $\sqrt{\text{CAREER}} = \text{NOW}$

(a) $\sqrt{531441} = 729$

(b) $\sqrt{561001} = 749$

The correctly solved puzzles are in parentheses after the names of RMM readers who submitted their answers to the above *Alphametics*.

**Merrill Barneby, Grand Forks, N. Dakota (1,4a,4b,6a); Corine Bickley, St. Louis, Mo. (1,6a,6b); Alan Bostrom, Claremont, Calif. (1,5,6a,6b); J. Charles Clapham, Vancouver, British Columbia (1,2b,3a,4a,4b,5,6a); Howard Cohodas, Geneva, Ohio (1,6a); Jerry Dice, Tallmadge, Ohio (1,4a,5); John Eckelman, Westport, N. Y. (5,6a,6b); Ronald L. Enyeart Los Angeles, Calif. (6a,6b); Harry M. Gehman, Buffalo, N. Y. (6a,6b); Alan Gold, Downsview, Ontario (1,5); Jack Halliburton, Los Angeles, Calif (1,2b,5,6b); Pete Hoyle, Williamsburg, Va. (1,2b,3a,-4b,6a); Philip Greco, Brooklyn, N.Y. (1,2b,4b,5); Mark Green, Beverly Hills, Calif. (5,6a); Joe Haseman, Lakeland, Florida (1,2b,3b,4a,5,6a,6b); Charles Jones, Orlando, Florida (6a); David Kaplan, Bronx ,N. Y. (1,3b,5,6b); Helen Lawrence, Bowling Green, Kentucky (1,2a,2b,3a,3b,4b,5,6a,6b); Bertha McDaniel, Stayton, Oregon (1,2b,3a,6a); Remy Landau, Montreal, Quebec (1); Harry Nelson, Livermore, Calif. (1,2a,2b,3a,4a,5,6a,6b); William P. O'Neill, Butte, Montana (1,2b,3a,4a,5,6a,6b); Norvin Pallas, Cleveland, Ohio (5) Dianne Pickering, Natick, Mass. (1,5); Marion S. Prindle, Johnstown, N. Y. (1,2a,3b,4b,5,6a); Lucille K. Roberts, Pasadena, Calif. (1); Walter Steinbruch, Wynnewood, Pa. (1,2a,3a,4b,5,6a); Donald V. Trueblood, Bellevue, Wash. (1,2b,3b,4a,4b,5,6a); Anneliese Zimmermann, Montreal, Quebec (1,3b,4b,5,6a,6b).**

# *Numbers, Numbers, Numbers*

It is with a bit of pleasure and pride that the Editor reports that the list of 18 Perfect numbers given in the August 1961 issue of RMM came through WITHOUT any errors.

* * * * *

Brother Alfred has made further investigations into the realm of prime numbers in number systems other than base 10. It was pointed out in the June 1961 RMM (page 12) that there are three digits in base 11 which form six numbers all of which are prime. It appears that there are two such sets of digits in base 13 but none for base 15 or base 17. (X represents 10)

| Base 13 | | Base 10 | Base 13 | | Base 10 |
|---|---|---|---|---|---|
| 247 | = | 397 | 78X | = | 1297 |
| 274 | = | 433 | 7X8 | = | 1321 |
| 427 | = | 709 | 87X | = | 1453 |
| 472 | = | 769 | 8X7 | = | 1489 |
| 724 | = | 1213 | X78 | = | 1789 |
| 742 | = | 1237 | X87 | = | 1801 |

The question that suggests itself is whether there are four digits in some number system with all their permutations forming primes. The results for other number systems investigated thus far are as follows:

(1) In base 11, fifteen of the 24 permutations of 1457 are primes.

(2) In base 13 (where $\mathcal{E} = 11$, $\gamma = 12$), the digits $278\gamma$ have 16 primes amongst the 24 permutations; the digits $475\mathcal{E}$ have 17 primes amongst the 24 permutations; and $134\mathcal{E}$ has 19 primes.

* * * * *

It is well known that there is an infinity of Pythagorean triangles whose sides differ by 1:

$$3^2 + 4^2 = 5^2$$
$$20^2 + 21^2 = 29^2$$
$$119^2 + 120^2 = 169^2$$
$$696^2 + 697^2 = 985^2$$
$$4059^2 + 4060^2 = 5741^2$$
etc.

W. P. Whitlock, Jr. of New York observes that in conjunction with this series of Pythagorean triangles there is the following phenomenon: the first 5, or the first 29, or the first 169, etc., *consecutive odd cubes* equal a square.

$$1^3 + 3^3 + 5^3 + 7^3 + 9^3 = 35^2$$
$$29 \text{ consecutive odd cubes} = 1189^2$$
$$169 \text{ consecutive odd cubes} = 40391^2$$
$$985 \text{ consecutive odd cubes} = 1372105^2$$
$$5741 \text{ consecutive odd cubes} = 46611179^2$$

* * * * *





And now a miscellaneous collection of number relationships:

$$37(3 + 7) = 3^3 + 7^3$$
$$48(4 + 8) = 4^3 + 8^3$$
$$111(11 + 1) = 11^3 + 1^3$$
$$147(14 + 7) = 14^3 + 7^3$$
$$148(14 + 8) = 14^3 + 8^3$$

(Maxey Brooke)

* * * * *

$$4913 = (4 + 9 + 1 + 3)^3 = 17^3$$
$$5832 = (5 + 8 + 3 + 2)^3 = 18^3$$
$$17576 = (1 + 7 + 5 + 7 + 6)^3 = 26^3$$
$$19683 = (1 + 9 + 6 + 8 + 3)^3 = 27^3$$

(J. A. H. Hunter)

* * * * *

Etude in Repeating Digits

(39526741) (4188901) (2906161) (27961) (9901) (9091) (3541) (2161) (271) (241) (211) (101) (61) =

786782848151848938631000000000786782848151848938631 × 41 =
32258096774225806483871000000032258096774225806483871 × 37 =
1193549580646354839903227000001193549580646354839903227 × 31 =
37000037000037000037000037000037000037000037000037000037 × 13 =
481000481000481000481000481000481000481000481000481000481 × 11 =
5291005291005291005291005291005291005291005291005291005291 × 7 =
37037037037037037037037037037037037037037037037037037037037 × 3 =
111111111111111111111111111111111111111111111111111111111111

(Alan L. Brown)

## The Next 550 Prime Numbers - 19577 to 25097

No errors were reported for the August 1961 listing of primes.

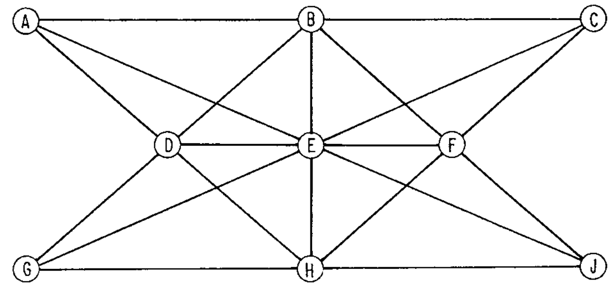
| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 19577 | 20089 | 20563 | 21089 | 21569 | 22051 | 22549 | 23039 | 23563 | 24019 | 24527 |
| 19583 | 20101 | 20593 | 21101 | 21577 | 22063 | 22567 | 23041 | 23567 | 24023 | 24533 |
| 19597 | 20107 | 20599 | 21107 | 21587 | 22067 | 22571 | 23053 | 23581 | 24029 | 24547 |
| 19603 | 20113 | 20611 | 21121 | 21589 | 22073 | 22573 | 23057 | 23593 | 24043 | 24551 |
| 19609 | 20117 | 20627 | 21139 | 21599 | 22079 | 22613 | 23059 | 23599 | 24049 | 24571 |
| 19661 | 20123 | 20639 | 21143 | 21601 | 22091 | 22619 | 23063 | 23603 | 24061 | 24593 |
| 19681 | 20129 | 20641 | 21149 | 21611 | 22093 | 22621 | 23071 | 23609 | 24071 | 24611 |
| 19687 | 20143 | 20663 | 21157 | 21613 | 22109 | 22637 | 23081 | 23623 | 24077 | 24623 |
| 19697 | 20147 | 20681 | 21163 | 21617 | 22111 | 22639 | 23087 | 23627 | 24083 | 24631 |
| 19699 | 20149 | 20693 | 21169 | 21647 | 22123 | 22643 | 23099 | 23629 | 24091 | 24659 |
| 19709 | 20161 | 20707 | 21179 | 21649 | 22129 | 22651 | 23117 | 23633 | 24097 | 24671 |
| 19717 | 20173 | 20717 | 21187 | 21661 | 22133 | 22669 | 23131 | 23663 | 24103 | 24677 |
| 19727 | 20177 | 20719 | 21191 | 21673 | 22147 | 22679 | 23143 | 23669 | 24107 | 24683 |
| 19739 | 20183 | 20731 | 21193 | 21683 | 22153 | 22691 | 23159 | 23671 | 24109 | 24691 |
| 19751 | 20201 | 20743 | 21211 | 21701 | 22157 | 22697 | 23167 | 23677 | 24113 | 24697 |
| 19753 | 20219 | 20747 | 21221 | 21713 | 22159 | 22699 | 23173 | 23687 | 24121 | 24709 |
| 19759 | 20231 | 20749 | 21227 | 21727 | 22171 | 22709 | 23189 | 23689 | 24133 | 24733 |
| 19763 | 20233 | 20753 | 21247 | 21737 | 22189 | 22717 | 23197 | 23719 | 24137 | 24749 |
| 19777 | 20249 | 20759 | 21269 | 21739 | 22193 | 22721 | 23201 | 23741 | 24151 | 24763 |
| 19793 | 20261 | 20771 | 21277 | 21751 | 22229 | 22727 | 23203 | 23743 | 24169 | 24767 |
| 19801 | 20269 | 20773 | 21283 | 21757 | 22247 | 22739 | 23209 | 23747 | 24179 | 24781 |
| 19813 | 20287 | 20789 | 21313 | 21767 | 22259 | 22741 | 23227 | 23753 | 24181 | 24793 |
| 19819 | 20297 | 20807 | 21317 | 21773 | 22271 | 22751 | 23251 | 23761 | 24197 | 24799 |
| 19841 | 20323 | 20809 | 21319 | 21787 | 22273 | 22769 | 23269 | 23767 | 24203 | 24809 |
| 19843 | 20327 | 20849 | 21323 | 21799 | 22277 | 22777 | 23279 | 23773 | 24223 | 24821 |
| 19853 | 20333 | 20857 | 21341 | 21803 | 22279 | 22783 | 23291 | 23789 | 24229 | 24841 |
| 19861 | 20341 | 20873 | 21347 | 21817 | 22283 | 22787 | 23293 | 23801 | 24239 | 24847 |
| 19867 | 20347 | 20879 | 21377 | 21821 | 22291 | 22807 | 23297 | 23813 | 24247 | 24851 |
| 19889 | 20353 | 20887 | 21379 | 21839 | 22303 | 22811 | 23311 | 23819 | 24251 | 24859 |
| 19891 | 20357 | 20897 | 21383 | 21841 | 22307 | 22817 | 23321 | 23827 | 24281 | 24877 |
| 19913 | 20359 | 20899 | 21391 | 21851 | 22343 | 22853 | 23327 | 23831 | 24317 | 24889 |
| 19919 | 20369 | 20903 | 21397 | 21859 | 22349 | 22859 | 23333 | 23833 | 24329 | 24907 |
| 19927 | 20389 | 20921 | 21401 | 21863 | 22367 | 22861 | 23339 | 23857 | 24337 | 24917 |
| 19937 | 20393 | 20929 | 21407 | 21871 | 22369 | 22871 | 23357 | 23869 | 24359 | 24919 |
| 19949 | 20399 | 20939 | 21419 | 21881 | 22381 | 22877 | 23369 | 23873 | 24371 | 24923 |
| 19961 | 20407 | 20947 | 21433 | 21893 | 22391 | 22901 | 23371 | 23879 | 24373 | 24943 |
| 19963 | 20411 | 20959 | 21467 | 21911 | 22397 | 22907 | 23399 | 23887 | 24379 | 24953 |
| 19973 | 20431 | 20963 | 21481 | 21929 | 22409 | 22921 | 23417 | 23893 | 24391 | 24967 |
| 19979 | 20441 | 20981 | 21487 | 21937 | 22433 | 22937 | 23431 | 23899 | 24407 | 24971 |
| 19991 | 20443 | 20983 | 21491 | 21943 | 22441 | 22943 | 23447 | 23909 | 24413 | 24977 |
| 19993 | 20477 | 21001 | 21493 | 21961 | 22447 | 22961 | 23459 | 23911 | 24419 | 24979 |
| 19997 | 20479 | 21011 | 21499 | 21977 | 22453 | 22963 | 23473 | 23917 | 24421 | 24989 |
| 20011 | 20483 | 21013 | 21503 | 21991 | 22469 | 22973 | 23497 | 23929 | 24439 | 25013 |
| 20021 | 20507 | 21017 | 21517 | 21997 | 22481 | 22993 | 23509 | 23957 | 24443 | 25031 |
| 20023 | 20509 | 21019 | 21521 | 22003 | 22483 | 23003 | 23531 | 23971 | 24469 | 25033 |
| 20029 | 20521 | 21023 | 21523 | 22013 | 22501 | 23011 | 23537 | 23977 | 24473 | 25037 |
| 20047 | 20533 | 21031 | 21529 | 22027 | 22511 | 23017 | 23539 | 23981 | 24481 | 25057 |
| 20051 | 20543 | 21059 | 21557 | 22031 | 22531 | 23021 | 23549 | 23993 | 24499 | 25073 |
| 20063 | 20549 | 21061 | 21559 | 22037 | 22541 | 23027 | 23557 | 24001 | 24509 | 25087 |
| 20071 | 20551 | 21067 | 21563 | 22039 | 22543 | 23029 | 23561 | 24007 | 24517 | 25097 |



## Readers' Research Department

Solutions to both of the August Research Problems were received.

The first problem was to determine whether more than three distinct integers could be used in the diagram below so that the sums of any three integers along any of the indicated lines are equal.




By using the symbol [D] as meaning "using D as a common point" and utilizing the same meaning when other letters are put in brackets we have the following analysis:

$$\begin{array}{rl} [D] & A + H = E + F \\ \text{Subtract } [F] & \underline{C + H = D + E} \\ \text{To give} & A - C = F - D \\ \text{Add } [E] & \underline{C + G = D + F} \\ \text{To give} & A + G = 2F \end{array} \qquad \begin{array}{rl} [C] & A + B = E + G \\ \text{Subtract } [G] & \underline{B + D = E + C} \\ \text{To give} & A - D = G - C \\ \text{Add } [E] & \underline{D + F = C + G} \\ \text{To give} & A + F = 2G \end{array}$$

This last pair of equations yield the fact that $A = F = G$. By substituting A for F and G in the original diagram and using E as a common point we will find that $C = D = J$. By substituting C for D and J in the diagram we will eventually find that $B = E = H$. Substituting B for E and H in the diagram yields a diagram having only the integers A, B and C. Therefore, no more than 3 integers can be used in the figure with the stated conditions.

* * * * *

The second problem required a method of construction to bisect the area of a regular pentagon given any point on one of the sides. The solution given applies in the same manner to any regular polygon with an odd number of sides.

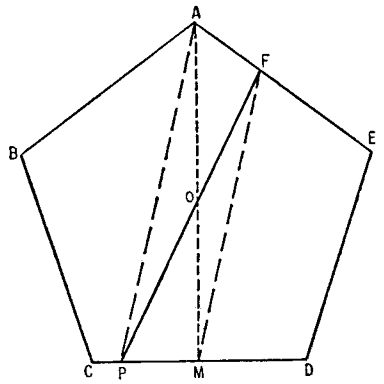
Let ABCDE be a regular pentagon and P a point on side CD. Draw MA, the axis of symmetry which bisects the pentagon. Then draw PA and MF parallel to PA. The line PF is the required line through point P which bisects the pentagon.

The proof: Draw FX perpendicular to AM (omitted in the drawing). APM and MFX are similar. Then PM:FX: :AM:XM or

$$(PM)(XM) = (FX)(AM) \qquad (1)$$

Triangles APM and MFX are similar. Then PM:FX::AM:XM or

$$(PM)(OX) = (FX)(MO) \qquad (2)$$




Subtracting equation (2) from (1):

$$(PM)(XM - OX) = (FX)(AM - MO)$$
$$(PM)(MO) = (FX)(AO)$$

i.e. Triangles MO Pand OAF are equal in area.

Since quadrilateral ABCP + △AOP + △POM = ½ the pentagon, then quadrilateral ABCP + △AOP + △AOF = ½ the pentagon.

* * * * *

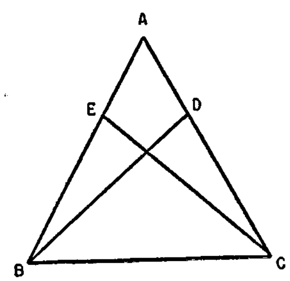
Analysis of the first problem was submitted by J. Charles Clapham, Vancouver, B.C.; Micky Earnshaw, Berkeley, California; Alan Sutcliffe, Yorkshire, England; Alan Gold, Downsview, Ontario; Marvin Lebow, Tulsa, Oklahoma

Analysis of the second problem was submitted by Marvin Lebow, Tulsa, Oklahoma; Walter W. Horner, Pittsburgh, Pennsylvania; J. Charles Clapham, Vancouver, B. C.; Maxey Brooke, Sweeny, Texas.

* * * * *

Here is the Research Problem for this issue:

ABC is an isosceles triangle. BD and CE are equal and meet AC and AB in D and E, respectively. Prove that points B, C. D and E are concyclic - lie on a circle. Submitted by Remy Landau of Montreal, Quebec.






# Letters to the Editor

Dear Mr. Madachy:

In the February issue (RMM # 1) two primes of the form $(10^n - 1)/9$ were given:

1,111,111,111,111,111,111 and 11,111,111,111,111,111,111,111

for n = 19 and 23, respectively.

Are there other primes composed of ones only, aside from these and 11?

For n up to 50, there are 35 cases where the number is composite. However, for the following 13 cases, the character of the resulting number is unknown: 34, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89 and 97. Is it a condition that n be a prime to produce a prime whose digits are all ones?

Rudolph Ondrejka
NAFEC
Atlantic City, N. J.

The Editor can add a reference that may help some RMM readers. Oscar Hoppe proved that the form for n = 19 was prime and his results were published in the PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY, 14 February 1918. The Editor does not have a copy of this reference and if there are any RMM readers who can supply a copy it would be appreciated.

* * * * *

Dear Sir:

Malcolm Tallman, in the August 1961 RMM (page 61), rendered a mnenomic for $\pi$ to 22 decimal places. The Editor of RMM submitted the following mnenomic for e to 37 decimal places:

> It appears a maneuver to remember e although it shouldn't. When truly justified, O just carry it out until you falter. O, it possibly induces pain. However, I can truly do thirty places or more correctly without slipping.

(i.e. e = 2.71828 18284 59045 23536 02874 71352 66249 78)

I wonder if there are any passages in classical prose or poetry which can be used as mnenomics for various constants?

U. Clid
Cleveland, Ohio

The Editor admits perpetrating the above. For readers who like classical literature, the following 50 decimal-place values of $\pi$, e and the Golden Ratio are supplied for reference:

$\pi$ = 3.1415926535 8979323846 2643383279 5028841971 6939937511
e = 2.7182818284 5904523536 0287471352 6624977572 4709369996
Tau=1.6180339887 4989484820 4586834365 6381177203 0917980576

Dear Sir:

BINGO appears to be the rage in various parts of the country and I wonder if some readers would know something about the odds against the player.

The total number of different cards $1365(3003)^4$ (i.e. 111,007,923,-832,370,565) which is so large that the following questions come to mind:

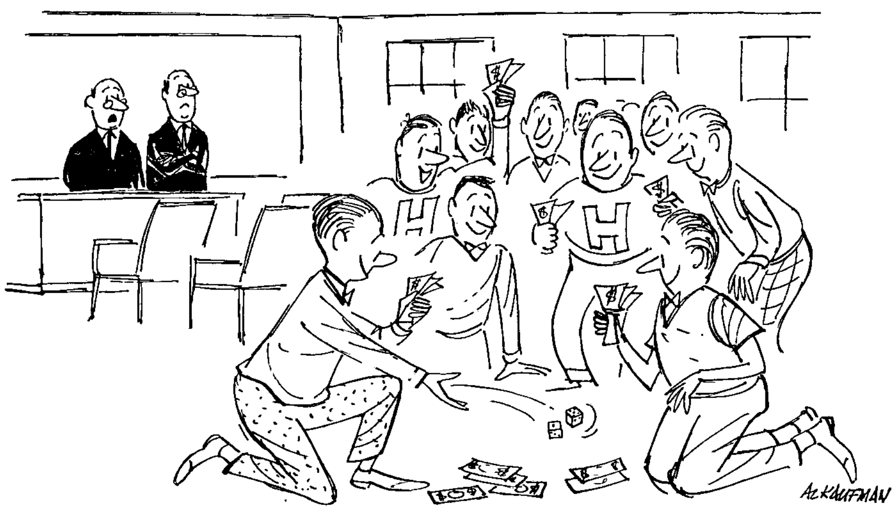
(1) How is a selection of, say, 5000 cards printed? What selective process is used by the printer?

(2) What are the odds against getting a "two-line" BINGO in 24 numbers?

(3) What are the odds against gettting a full card in 50 numbers?

The above is all based on the usual card where the numbers 1-15 are under B., 16-30 under I, 31-45 under N, 46-60 under G, and 61-75 under O; and where the middle square, under the N, is free.

E. S. Price
1696 East 57th Ave.
Vancouver, British Columbia,
Canada




"I WAS USING A PAIR OF DICE TO ILLUSTRATE MATHEMATICAL PROBABILITY AND THE LESSON GOT OUT OF HAND!"





ANSWERS TO THE *PUZZLES AND PROBLEMS* IN THE AUGUST ISSUE OF RMM (Pages 46 to 48)

1. Ladder Legs and Cross-Stays: A little geometry will disclose some congruent triangles and the distance from the top of the ladder to the crossing-point of the cross-stays is exactly 6 feet.

2. River-Crossing Dilemma: Many readers who submitted answers to this puzzle forgot the stipulation that "None of the explorers was willing to remain on either side of the river, *even momentarily*, while outnumbered by natives." Hence all answers which stated that an explorer remain in the boat while a quick exchange took place on one of the shores are incorrect. The crossing can be made in 13 crossings (not less) as follows: Let the natives be A, B, C and the explorers X, Y, Z with A being the native that can maneuver the boat.

A and B cross - A returns; A and C cross - A returns; X and Y cross - X and B return; X and A cross - X and C return; X and Z cross - A returns; A and B cross - A returns; A and C cross. Now all are on the other side of the river.

3. Tricky Window: The area of the moon-shaped opening is five square feet - the same area as that through which the sash moved.

4. Ladder and Barrel: There were arguments about this! A slight omission in the wording of the problem made it possible for ANY answer to be valid. However, if the diameter of the barrel is an integral number of feet, then the ladder must be 41 feet long.

5. Oracle of Three Gods: The variety of possible solutions to this puzzler amazed the editor. Some were so complicated in their phrasing that the editor accepted them on faith! However, here are two questions the pilgrim could ask, the answers to which would tell him whether his wife was faithful or not:

(a) If I asked you if my wife is faithful, would you say Yes?
(b) Are you as honest as my wife is faithful?

A *Yes* answer by any of three gods would mean the pligrim's wife is faithful. A *No* answer means his wife is not faithful.

6. The Prolific Author: 28 volumes were given away.

7. Sharing the Ride: A fair share of the fare is 75 cents. At \$3.00 for a roundtrip for one person, then one-half the distance would mean \$1.50. However, for this half-distance, there are *two* persons sharing the cab, therefore each should only pay half the fare for that distance - 75 cents.

8. The Stamp-Collecting Kid: Peter paid 49 cents for the Cape triangle and 55 cents and 58 cents for the other two stamps.

9. A Magic Hexagon: Fill the outer ring of hexagons with 4's, the inner ring with 2's.

RMM Readers who submitted correct answers, with the correctly solved puzzles in parentheses after their names, were:

**Jack Abad, San Francisco, Calif. (1,3,4,5,6,8,9); Lawrence H. Bailey, So. Dixbury, Mass. (1,4); Merrill Barnebey, Grand Forks, N. Dakota (1,2,3,4,5,6,8); Richard Beck Athens, Ohio (1,2,3,6); Corine Bickley, St. Louis, Mo. (2,9); Jeannette Bickley, St. Louis, Mo. (1,6,7,8); Donald K. Bissonnette, Tallahassee, Fla. (1,5,6); Dennis Bosch, Gowrie, Iowa (6); Alan Bostrom, Claremont, Calif. (1,2,3,5,6,8,9); Sam Cannella, Atlanta, Ga. (1,3,6); J. Charles Clapham, Vancouver, B. C. (1,2,3,4,5,6,8,9); Charles Davant III, Woodberry Forest, Va. (1,2,3,6,7); Howard Cohodas, Geneva, Ohio (1,2,6,8); John Eckelman, Westport, N.Y. (1,6,9); Ronald L. Enyeart, Los Angeles, Calif. (1,2,3,6,7,8,9); Philip Greco, Brooklyn, N. Y. (1,6,8);Alan Gold, Downsview, Ontario (1,6,); Mark Green, Beverly Hills, Calif. (3,4,6,7,9); Jack Halliburton, Los Angeles, Calif. (1,4,5,6,7,8,9); Ruth Harris, East Sullivan, N.H. (1,3,6); Joe Haseman, Lakeland, Fla. (1,2,4,6,8,9); Peter Heichelheim, Ann Arbor, Mich. (1,2,3,5,6,7,9); J. A. H. Hunter, Toronto, Ontario (4,5); Robert S. Johnson, Town of Mount Royal, Quebec (1,2,3,4,6,8); Charles Jones, Orlando, Fla. (6,7); Michael Keyes, Fremont, Ohio (1,4,5,6,7,8,9); Remy Landau, Montreal, Quebec (1,4); Helen Lawrence, Bowling Green, Ky. (1,3,4,6,8); Sanford S. Leffingwell, Billings, Mont. (1,2,5,6,9); Bertha McDaniel, Stayton, Oregon (1,2,4,5,6,8,9); Harry L. Nelson, Livermore, Calif. (1,3,4,5,6,8,9); William P. O'Neill, Butte, Mont. (1,2,3,4,6,8,9); Norvin Pallas, Cleveland, Ohio (2,3,6,8,9); Dianne Pickering, Natick, Mass. (1,6); Lucille K. Roberts, Pasadena, Calif. (6); Jerry Roger, Tulsa, Okla. (5,6,9); P. Schill (2,9); Don J. Singleton, Pasadena, Texas (1,3,6,7); Pvt. Michael Steiner, Ft. Bliss, Texas (5,6,7); Spencer Stopa, Chicago, Ill. (1,2,6,8); H. E. Tester, Middlesex, England (1,3,4); Donald V. Trueblood, Bellevue, Washington (1,2,3,4,6); Amos Voil, Los Angeles, Calif. (1,3,6,7,9); Anneliese Zimmermann, Montreal, Quebec (1,2,3,4,6,8).**

ANSWER TO THE CROSS-NUMBER PUZZLE IN THE AUGUST RMM (Page 13)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 4 | 3 | 2 | 1 |
| 2 | | 6 | 8 | 0 | 4 | 2 | | 0 |
| 9 | 8 | | 4 | 7 | 7 | | 2 | 8 |
| 6 | 6 | 6 | | 5 | | 7 | 2 | 9 |
| | 4 | 2 | 9 | | 1 | 2 | 0 | |
| 7 | 0 | | | 0 | 1 | | 1 | 7 |
| 4 | | 4 | 0 | 1 | 2 | 3 | | 9 |
| 4 | 7 | 9 | 0 | 0 | 1 | 6 | 0 | 0 |

A few minor imperfections in the wording of the clues were noticed by many solvers. Here are the solvers of the puzzle:

**Anneliese Zimmermann, Montreal, Quebec; Fred Weiner, Erlton, N. J.; Lt. Theodore E. Vitori, Austin Texas; Don J. Singleton, Pasadena, Texas; Gerald Schiele, Bellingham, Washington; Marion S. Prindle, Johnstown, N. Y., N. Y.; Norvin Pallas, Cleveland, Ohio; William P. O'Neill, Butte, Mont.; Harry L. Nelson, Livermore, Calif.; Helen Lawrence, Bowling Green, Ky.; Michael Keyes, Fremont, Ohio; Charles Jones, Orlando, Fla.; Robert S. Johnson, Town of Mount Royal, Quebec; Ruth Harris, East Sullivan, N .H.; Judith Hanerfeld, So. Ozone Park, N. Y.; Mark Green, Beverly Hills, Calif.; Philip Greco, Brooklyn, N. Y.; Alan Gold, Downsview, Ontario; Leonard Fellman, Milwaukee, Wisc.; Ronald L. Enyeart, Los Angeles, Calif.; John Eckelman, Westport, N. Y.; Howard Cohodas, Geneva, Ohio; J. Charles Clapham, Vancouver, Br. Columbia; Brother D. Joseph F.S.C., Canton, Ohio; Alan Bostrom, Claremont, Calif.; Gerard Blais, Schaghticoke, N. Y.; Jeanette Bickley, St. Louis, Mo.; Merrill Barnebey, Grand Forks N. Dakota.**



# ☆ RMM No. 1 February 1961 ☆

A reprint can be made upon receipt of sufficient orders with payment.

However, before we go into details, let's see what was in the first RECREATIONAL MATHEMATICS MAGAZINE:

*Some Inferential Problems* by J. A. H. Hunter

A fascinating little article showing how one can solve logic puzzles by the use of Boolean Algebra. Clearly explained by Mr. Hunter and accompanied by worked-out examples - and some that are *not* worked out!

*Wager Problems - Old and New* by Mel Stover

A collection of chess, checker and bridge problems and puzzles accompanied by sidelights on one of man's many weaknesses - gambling.

*Conics by Paper-Folding* by C. Stanley Ogilvy

Ellipses, parabolas and hyperbolas - by folding paper! With proofs for skeptics.

*Some Absolutely Amazing Afghan Bands* by Maxey Brooke and the Editor

An original discovery by the authors, published for the first time in RMM: How to cut out a paper chain of *any* number of links with a *single* cut of a pair of scissors.

And, of course, many other features already familiar to RMM readers.

The number of advance requests for a reprint was very encouraging - however, more would have been needed to justify reprinting.

A reprint of any magazine costs money — obviously. To make it possible the editor must have sufficient demand *and* funds. All those who would like to get RMM # 1 should *immediately* send a check, money order or cash for 65 cents for each copy desired. If sufficient demand is received, reprinting will be done and the copies mailed out.

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*There are at least* 3000 *subscribers to RMM who lack issue # 1 to make their collection of RMM's complete. If only* 1000 *decide to submit payment, a reprint can be easily made!*

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