ISSUE NO. 4
AUGUST 1961

# RECREATIONAL MATHEMATICS magazine

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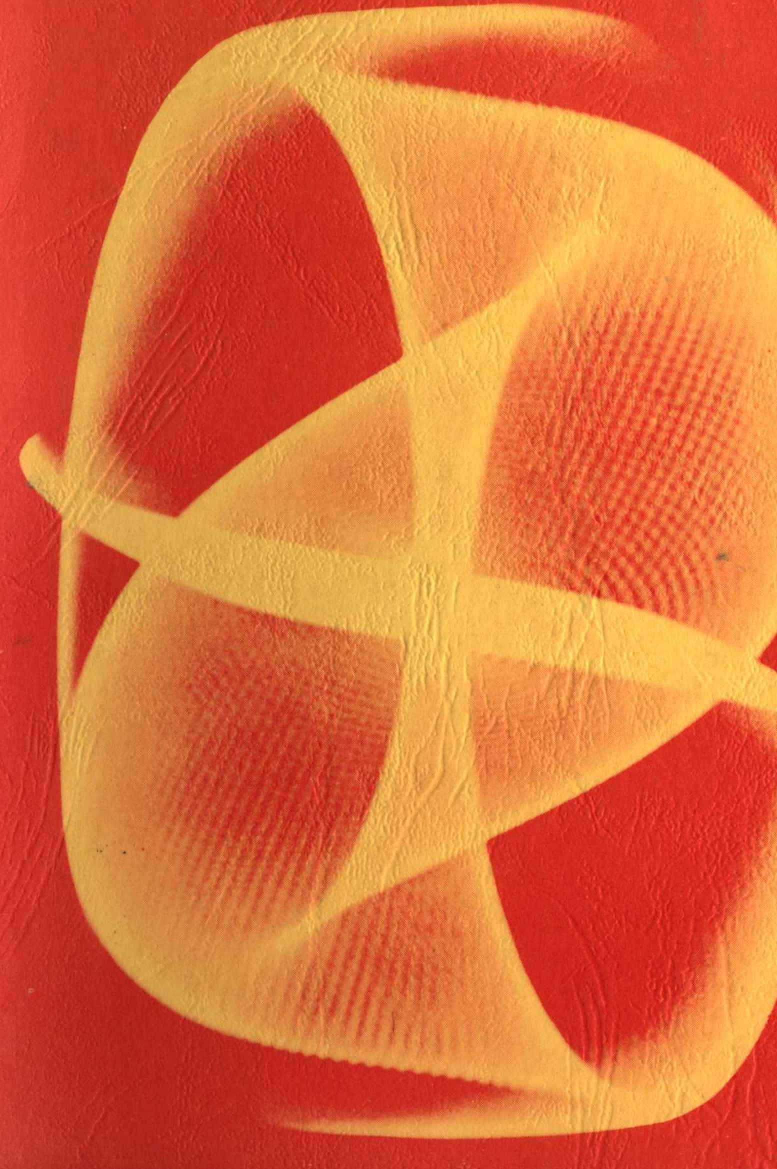

Cover: OSCILLON # 257 — by Ben F. Laposky


RECREATIONAL MATHEMATICS magazine

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AUGUST 1961 ISSUE NUMBER 4

PUBLISHED AND EDITED BY JOSEPH S. MADACHY


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## From the Editor

Regular readers of RMM will note the striking change in cover design - an idea which will be a regular feature from now on. Each issue will have a different cover which will feature, in most cases, some particular article inside.

* * *

The response to the editorial request to send in comments and answers has been most gratifying - as you can easily note by seeing the list of names of puzzle-solvers throughout the pages of this issue of RMM.

There were many responses to the request that those desiring a revised reprint of RMM No. 1 (February 1961) inform the editor and I hope that others who would like to see and purchase such a reprint will drop a postcard to the Editor, RMM, Box 1876, Idaho Falls, Idaho. The next issue (October) of RMM will carry a notice stating whether a reprint will be made or not - and how it might be obtained if reprinted. So, if you want RMM No. 1, let me know.

* * *

The next issue of RMM will carry Part 2 of S. W. Golomb's "General Theory of Polyominoes" which is called *Patterns and Polyominoes;* Ali R. Amir-Moez will divulge the proper method of spiral construction; Maxey Brooke will show how a Magic Tessarack is constructed - a magic four-dimensional cube!; of course, other articles and the usual features and departments.

* * *

Reader reaction to RECREATIONAL MATHEMATICS MAGAZINE as a new media for mathematics in the light and entertaining manner has been beyond the expectations of the editor - for which much thanks must go to all the writers and contributors of the material found within the pages of RMM.

1 August 1961 J.S.M.



# The General Theory of Polyominoes

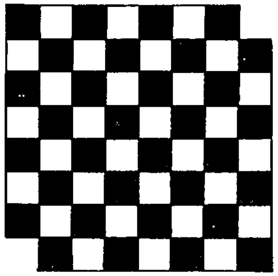
by Solomon W. Golomb

Editor's Introduction: *In this issue of RMM Solomon W. Golomb starts the first of a three-part series on Polyominoes - the succeeding articles to be published in the October and December issues. The term "Polyomino" was originated by Golomb and is defined as nothing more than a set of squares connected along one or more edges. RMM readers are guaranteed a full measure of enjoyment with the ideas, games and methods of analysis engendered by this simple idea.*

## Part 1 - Dominoes, Pentominoes and Checker Boards

Many readers of RECREATIONAL MATHEMATICS MAGAZINE may already be familiar with the following problem:

Given a checker board with a pair of opposite corners deleted (see diagram), and given a box of dominoes, where each domino covers exactly two squares of the checker board, is it possible to cover this checker board exactly with dominoes?

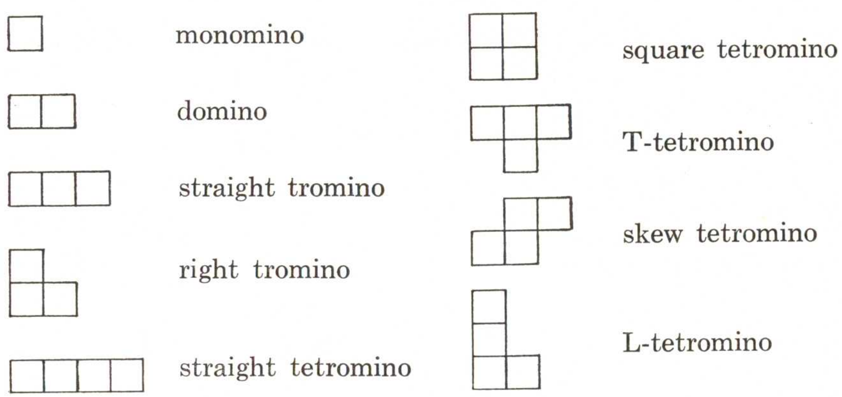


CHECKER BOARD with opposite corners deleted cannot be covered with dominoes.

The answer is "NO", and a remarkable proof can be given. Using the ordinary coloring of the checker board, each domino will cover one light square and one dark square. Thus n dominoes will cover n light squares and n dark squares; that is, an equal number of each. But our defective checker board has more dark squares than light squares, and so it cannot be covered with dominoes.

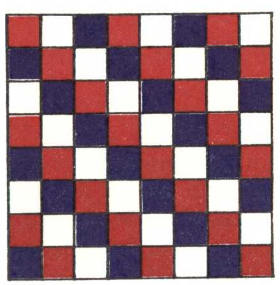
This result is really a theorem in *combinatorial geometry*, an important branch of mathematics which is frequently neglected because there seem to be few general methods, and because systematic rules will not replace ingenuity as the key to discovery. Many of the design problems in practical engineering are combinatorial in nature - especially where standard components or shapes must be fitted together in some optimum fashion. The aim of this article is two-fold: first, to illustrate some of the mathematical thinking which can be used effectively whenever problems in combinatorial geometry arise; and second, to serve as an introduction to the fascinating puzzles and mathematical recreations which can be constructed with dominoes and their more elaborate cousins, the *polyominoes*.

We define a *polyomino* to be a simply connected set of squares, with each square connected to at least one other square along an edge. (Chess players might call it "rook-wise connected" - that is, a rook placed on any square of the polyomino must be able to travel to any other square, in a finite number of moves.) The simpler polyominoes—all possible shapes covering fewer than five squares—are shown below.




It is impossible to cover an 8 x 8 checker board entirely with trominoes, because 64 is not divisible by 3. Instead we inquire: Can the 8 x 8 checker board be covered with 21 trominoes and one monomino?

First, suppose we use 21 *straight* trominoes. We color the checker board patriotically (see diagram), and observe that a straight tromino will cover one red square, one white square, and one blue square, no matter where on the board it is placed. Thus 21 straight trominoes will cover 21 red squares, 21 white squares, and 21 blue squares. By actual count, our patriotic coloring involves 22 red squares, 21 white squares, and 21 blue squares.



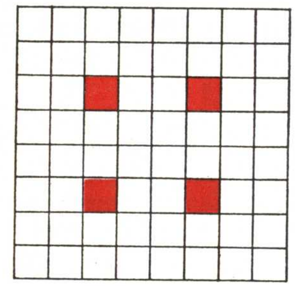
If a monomino is placed in the lower left hand corner, the remaining squares will consist of 22 red, 21 white, and 20 blue squares. Thus the checker board with lower left-hand corner removed cannot be covered with straight trominoes. By symmetry, no matter which corner is removed, the rest of the board cannot be covered with straight trominoes.



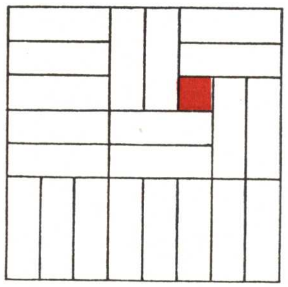


Symmetry arguments are very powerful tools in combinatorial geometry. For example, if a monomino is placed on any blue square, or on any white square, *or on any square symmetric to a blue square or a white square*, the rest of the board cannot be covered with straight trominoes!

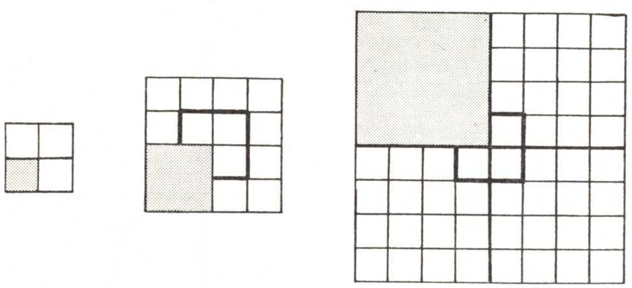
The only red squares not symmetric to blue or white squares are the four shown at the left. We have already proved that if a monomino is placed anywhere on the board *except* one of these four squares, the rest of the board cannot be covered with straight trominoes. The symmetry principle suggests that these four remaining squares *might* be exceptional.



The figure at the right shows that they actually are. It *is* possible to cover the the checker board with 21 straight trominoes and one monomino, *provided* that the monomino is placed on one of the four exceptional squares.



Our next result is surprisingly different: *No matter where on the checker board a monomino is placed, the remaining squares can always be covered with 21 right trominoes.*



*Progressive covering by right trominoes.*

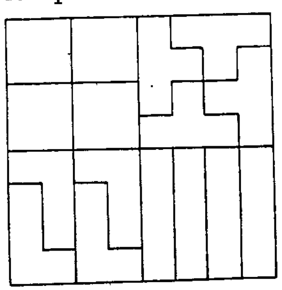
Consider first a 2 x 2 board. Wherever a monomino is placed, the rest can be covered by a right tromino.

Next consider a 4 x 4 checker board. Divide it into quarters. Each quarter is a 2 x 2 board. Let the monomino be placed in one of the quarters, say the lower left. The rest of this quarter can be covered with a right tromino, since the quarter is a 2 x 2 board. In each of the other three quadrants, if a single square is removed, the rest can be covered with a right tromino. But a tromino placed in the center removes one square from each quadrant, and makes it possible to complete the covering.

The 8 x 8 case is treated in the same way. First divide into quadrants, which will be 4 x 4 checker boards. The monomino must be in one of the four quadrants, which can be completed because it is a 4 x 4 board. The other quadrants can be covered if one square is removed from each, again referring back to the 4 x 4 case. And these three *extra* squares can be juxtaposed to form a right tromino in the center of the board.

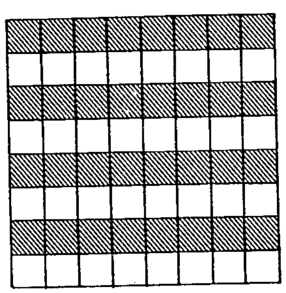
The proof just given proceeded by *mathematical induction.* The 2 x 2 case was very easy, and the $2^{n+1}$ x $2^{n+1}$ case followed readily from the $2^n$ x $2^n$ case. Such proofs are very valuable in combinatorial analysis. Geometrically, they suggest that complex patterns can be gotten by systematic repetition and combination of simple patterns.

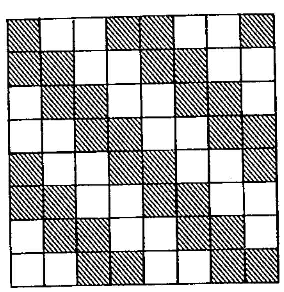
Some theorems about tetrominoes are worth mentioning, although detailed proofs will be omitted.



It is easy to cover the checker board entirely with straight tetrominoes, or with square tetrominoes, or with T-tetrominoes, or with L-tetrominoes. This is clear from the figure. On the other hand, it is impossible to cover the board, or even a single edge of the board, with skew tetrominoes.

It is impossible to cover the checker board with 15 T-tetrominoes and one square tetromino. This can be proved using the ordinary coloring for the checker board.





It is likewise impossible to cover the checker board with 15 L-tetrominoes and one square tetromino. Now, however, the most convenient proof uses the striped shading. It is also impossible to cover the checker board with one square tetromino and any combination of straight tetrominoes and skew tetrominoes. The proof in this case makes use of the jagged shading shown above.

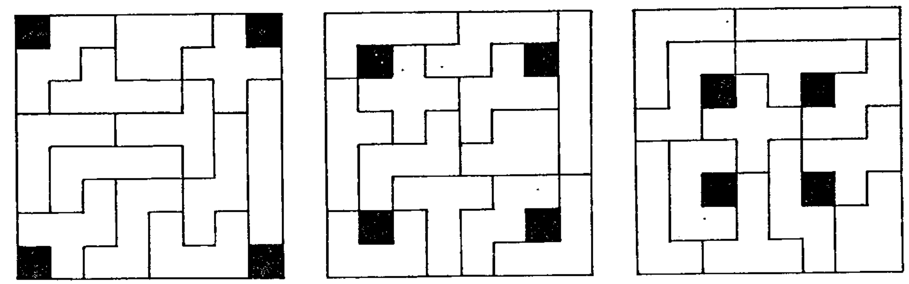




## PENTOMINOES

Now we come to the pentominoes. There are twelve distinct shapes each covering five squares, so that their total area is 60 squares.

There are numerous ways of placing all 12 distinct pentominoes on an 8 x 8 checker board at once, and there will always be four squares left over. Many interesting patterns can be formed by artistically specifying the positions of the four extra squares. Three of these patterns are illustrated below.

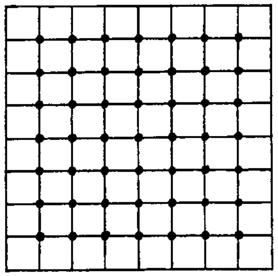


INTRICATE PATTERNS place all 12 distinct pentominoes on a single checker board.

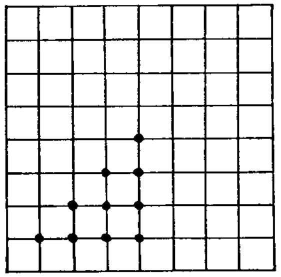
Another obvious possibility is to require that the four surplus squares form a 2 x 2 square (*a square tetromino*) in some specified position on the board. (Two favorite locations are the center and the corner.) This problem is completely solved by a very remarkable theorem, which can be proved using only *three* constructions!

*Wherever on the checker board a square tetromino is placed, the rest of the board can be covered with the twelve pentominoes.*

At first glance, there are 49 possible locations for the square tetromino. The bold dots in the first figure designate the 49 possible locations of the *center* of the 2 x 2 square. However, applying symmetry principles, the problem reduces to the ten positions indicated by the dots in the second figure.

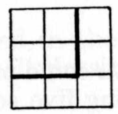


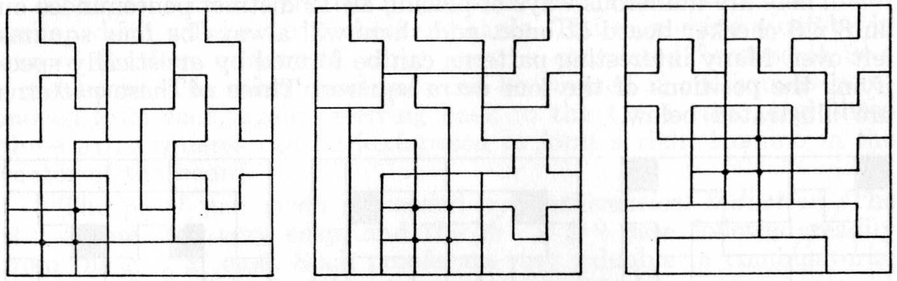
The 49 possible positions for the center of a square tetromino on the checker board



The 10 *non-symmetric* positions for a square tetromino on the checker board

A clever stratagem is to combine the square tetromino with the V-pentomino to form a 3 x 3 square as shown to the right.



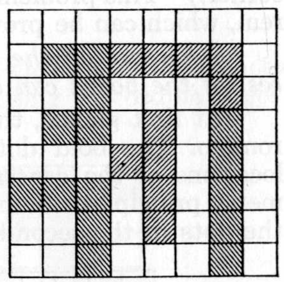


*THREE CONSTRUCTIONS which suffice to prove that no matter where a 2 x 2 square is removed from the checker board, the remaining* 60 *squares can be covered by the* 12 *distinct pentominoes.*

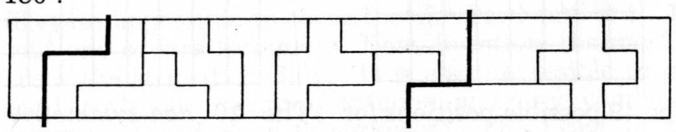
Then the three diagrams, shown above, complete the solution, because any of the ten positions for the square tetromino can be realized by first selecting the correct diagram, and then the correct position for the 2 x 2 square within the 3 x 3 square.

It is also natural to inquire: "What is the *least* number of pentominoes which will span the checker board?" That is, we attempt to place *some* of the pentominoes on the board in such a way that none of the remaining pentominoes can be added.

The minimum number needed to span the board turns out to be 5, and one such configuration is shown at the right.



Many other patterns can be formed using the twelve pentominoes, as the reader may attempt for himself. These include rectangles of 6 x 10, 5 x 12, 4 x 15, and 3 x 20. The hardest of these is the 3 x 20 rectangle, and the solution given here is believed to be unique, except for the possibility of rotating the central portion (between the dark lines) by 180°.



The 12 pentominoes form a 3 x 20 rectangle.
Is this construction unique?

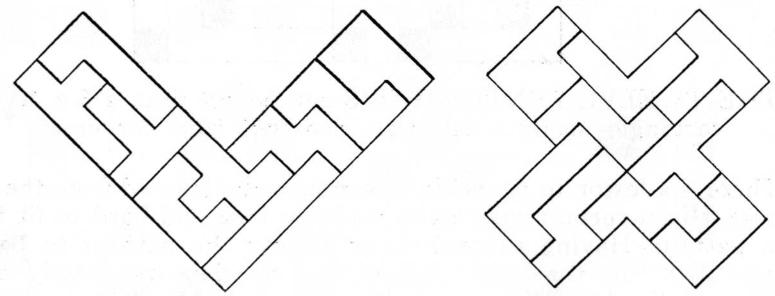




Professor R. M. Robinson, of the University of California at Berkeley, has proposed another fascinating pentomino problem, which he calls the "triplication problem." It goes as follows:

*Given a pentomino, use nine of the other pentominoes to construct a scale model, 3 times as wide and 3 times as high as the given pentomino.*

Solutions are shown here for the V-pentomino. and the X-pentomino.

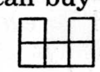
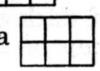


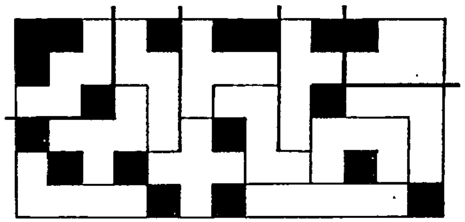
TRIPLICATION of the V-pentomino and the X-pentomino.

The reader is invited to test his ingenuity with the triplication of the other pentominoes.

Besides their fascination as a puzzle, pentominoes on the checker board also make an exciting competitive game. Two or more players take turns placing a pentomino of their choosing on an initially empty checker board. The first player who is unable to find room on the board for any of the unused pentominoes is the loser. (If all twelve pieces are played - rarely accomplished - the player who placed the twelfth piece wins.) The game will last at least five and at most twelve moves, can never result in a draw, has more possible openings than chess, and will intrigue players of all ages. It is difficult to advise what strategy should be followed, but there are two valuable principles:

1. Try to move in such a way that there will be room for an *even number* of pieces. (This assumes only two are playing).
2. If you cannot analyze the situation, do something to complicate the position, so that the next player will have even more difficulty analyzing it than you did.

A pentomino problem of a rather different nature is the following: A man wishes to construct the twelve pentominoes out of plywood. His saw will not cut around corners. What is the smallest plywood rectangle he can buy from which he can cut all 12 pentominoes? (The U-pentomino, , will require special effort. Assume that it must be cut as a  hexomino, and finished later). The *best* answer is not known, but a 6 x 13 rectangle may be used. In the illustration, the darker lines are to be cut first, starting from the sides and working inward.



THE TWELVE PENTOMINOES can be cut from a 6 x 13 rectangle, using a saw that does not turn corners.

There is a lesson in plausible reasoning to be learned from the pentominoes. Given certain basic data, we labor long and hard to fit them into a pattern. Having succeeded, we believe the pattern to be the *only* one that "fits the facts," indeed that the data are merely manifestations of the beautiful, comprehensive whole. Such reasoning has been used repeatedly in religion, in politics, even in science. The pentominoes illustrate that many different patterns may be possible from the same "data", all equally valid, and the nature of the pattern we end up with is determined more by the shape we were looking for than by the data at hand. It is also possible that for certain data, no pattern of the type we are conditioned to seek may exist. This will be illustrated presently, by the hexominoes.

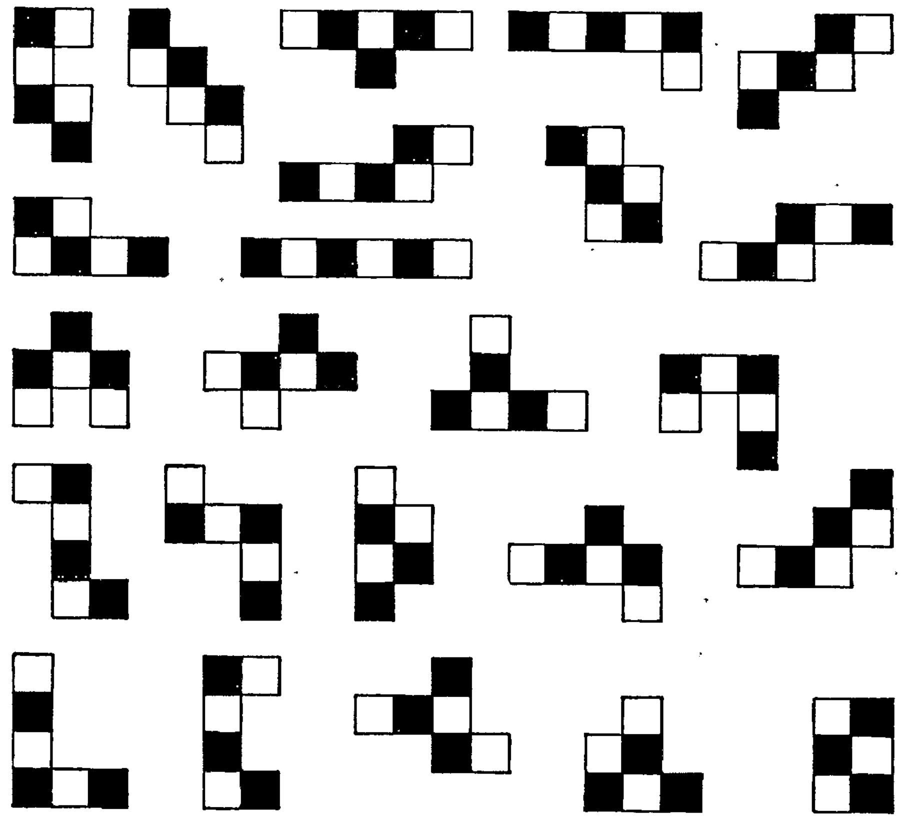
Beyond the 12 pentominoes, there are 35 distinct hexominoes and 108 distinct heptominoes. No one has yet succeeded in obtaining an expression or formula for the exact number of n-ominoes as a function of n. Combinatorial problems of this sort are often tantalizingly difficult.

The 35 hexominoes cover a total area of 210 squares. It is natural to attempt to arrange them in rectangles, either 3 x 70 or 5 x 42 or 6 x 35 or 7 x 30 or 10 x 21 or 14 x 15. All such attempts, however, are predestined to fail. For in each of the rectangles one could introduce a checker board coloring, with 105 light squares and 105 dark squares, an *odd* number of each. There are 24 hexominoes which will always cover three dark squares and three light squares (an odd number of each). The other 11 hexominoes always cover two squares of one color and four of the other, an *even* number of each. The 35 hexominoes are illustrated on the next page according to their checkerboard-covering characteristics.

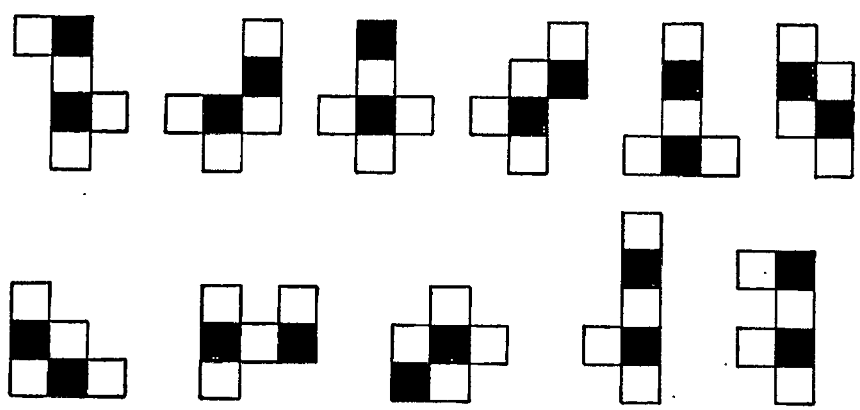
There are an even number of "odd" hexominoes and an odd number of "even" hexominoes. Since "even times odd = even" and "odd times even = even," the 35 hexominoes will always cover an *even* number of light squares and an *even* number of dark squares. But the number of light (or dark) squares is 105 for any of the rectangles in question, and 105 is *odd*.







The 24 "odd" Hexominoes



The 11 "even" Hexominoes

It is noteworthy that the same checker board coloring used to prove the simplest result about dominoes also serves to prove a far more complex theorem about hexominoes. The underlying theme of the checker board coloring is "parity check," a simple yet powerful mathematical tool based on the obvious fact that an even number is never equal to an odd number. The use of *colors* is a valuable aid to the intuition — objects colored differently will seldom be confused. And sometimes, as in the straight tetromino problem, the colors vividly proclaim a solution which might otherwise have been overlooked.

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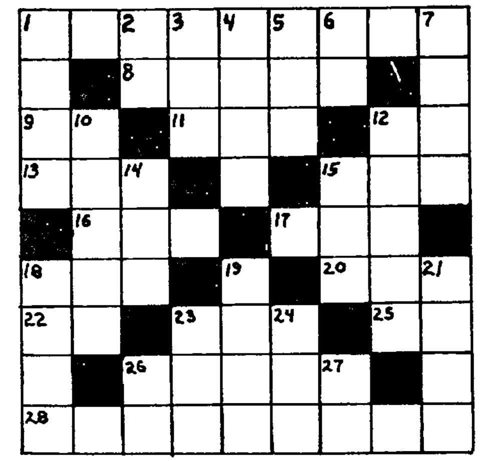
# Cross-Number Puzzle

by Brother Alfred

For whole numbers, it may be assumed that the number cannot begin with zero. For a decimal, the number could possibly start with one or more zeros. There should be just one possible answer. As a negative check, the sum of all the digits in the table should be 245.



HORIZONTAL

1. A square whose successive digits differ from each other by 1.
8. A cyclic arrangement of 04268.
9. Two consecutive digits greater than 5 which form a number which is twice a square.
11. The value of x (approximation) in $10^x = 3$.
12. A number which is the sum of all its factors that are less than the number (but including 1).
13. The number of the "beast" in the Apocalypse.
15. The sixth power of a number.
16. One hundred less than a square.
17. The number of permutations of 6 things taken three at a time.
18. An odd multiple of 5.
20. A multiple of five which is the sum of a two digit number whose digits are the reverse of each other
22. The number greater than 50 which is the product of three of the primes less than 10.
23. The number of spaces formed in a plane by 24 lines, no three of which go through a point and no two of which are parallel.
25. A prime number which is the number of sides of a regular polygon.
26. A cyclic arrangement of 0, 1, 2, 3, 4.
28. A factorial.

VERTICAL

1. The square of No. 2 vertical.
2. The reciprocal of the probability of throwing 2 with a pair of dice.
3. A perfect square.
4. The first two digits form a number which is a multiple of 5. The last two digits form a number three halves of the number formed by the first two digits.
5. A multiple of 149.
6. A cube has a small portion cut off each of its corners. The number of edges of the resulting figure.
7. A square such that the digits in reverse are also a square. The smaller of the two.
10. The number of seconds in a day.
12. The number formed by the first three digits is 24 less than four times the sum of the last two digits.
14. A perfect square.
15. One greater than a factorial.
18. A square with the first two digits the same and the last two digits the same.
19. A number for which the first and third, the second and fourth digits are the same.
21. The number has factors 2 and 5, while the first three digits are odd numbers in sequence.
23. The twenty-fourth term in the series 1, 3, 6, 10, 15 . . .
24. A square.
26. An odd square whose second digit is one more than twice the first digit.
27. The number of ways in which two dice can come up (considering them as distinct entities).

# *Electronic Abstractions: Mathematics in Design*

*by Ben F. Laposky*

Electronics, photography, art and mathematics are the unique combination shown in the *Electronic Abstractions* illustrated in these pages. The electronic abstractions are in a way an art form for the space age, both in their visual impact and in the means by which they are created.

The electronic abstractions, or *oscillons*, as they are also called, are actually abstract art forms in light. They appear as glowing traces on the screen of a cathode-ray oscilloscope, either as standing figures or in rhythmic motion. They are created by the combination and control of various basic electrical wave forms.

The oscilloscope is a widely-used electronic test and indicator instrument. It resembles a television receiver somewhat in appearance, with its fluorescent screen, and with many more controls. (A TV set is actually a form of an oscilloscope, so are radarscopes.) It is primarily used to display electrical waveforms, from which technicians, scientists, and others, can determine frequencies, voltages, and other functions of electrical circuits. Oscilloscopes are also used as indicators in certain types of analogue computers.

The basic wave forms the 'scope shows may be originated by so-called *oscillators*, or other wave form generators—mostly electronic in nature. These forms are mathematical curves. The fundamental one is the *sine wave*, a sinuous curve which is identical with the sine curve of trigonometry. The other wave forms are the square waves, sawtooth, symmetrical triangular, logarithmic, pulsed, and so on.

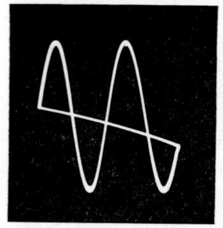
While the sine wave is a comparatively simple wave, electronically, the square waves and sawtooth waves are much more complex. By mathematical analysis, they are made up electronically from sine waves. The square wave is the resultant of all the odd harmonics composing it. (The harmonics are the integral multiples of the basic wave, increasing in number to infinity. When these are added together geometrically, they will form the more complex wave.) The sawtooth is the resultant of all the harmonics of the basic sine wave, and the symmetrical triangular wave of all the even harmonics.

Just as simple geometric elements, such as lines, angles and curves, may be built up into complex designs with an aesthetic appeal, so too these basic electrical waves may also be combined into more intricate shapes and figures.

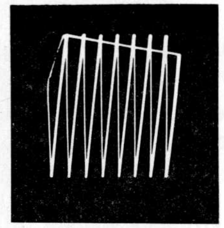
For example, the sine wave may be woven into the attractive Lissajous figures. In the oscilloscope there are two inputs, the vertical and the horizontal. By putting sine waves of different frequencies



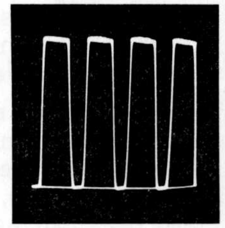




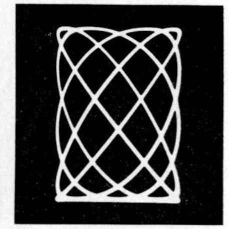
Sine Wave



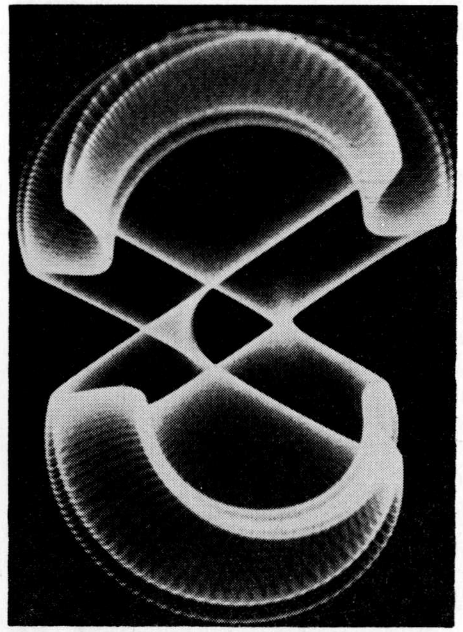
Sawtooth Wave



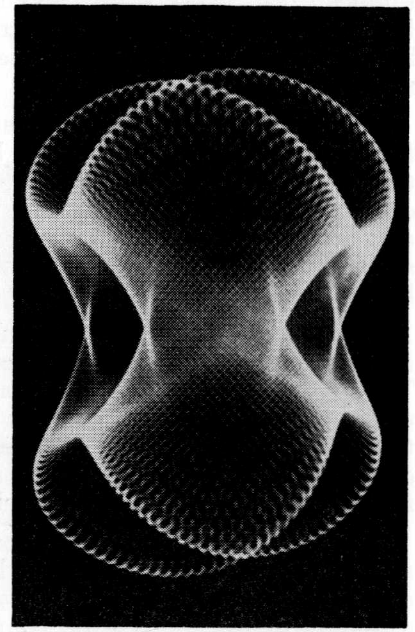
Square Wave



Lissajous Figure



OSCILLON # 39



OSCILLON # 11

(numbers of waves per second) into the H and V sections, these basket-weave figures result. Sawtooth waves will give patterns with a diamond-like effect.

Using other electrical circuit combinations, we can get ellipses, circles, cycloids, rosettes, roulettes, spirals, and so on. All of these are similar or identical to mathematical or geometrical curves, and may also be simple elements of decorative design, too.

The tracings of pendulums are related to Lissajous figures, but with a periodic time change in their swings. The sand bob, a pendulum with a cavity for sand, and a tiny hole at the bottom, was used even in ancient times to trace these figures. Later, pen machines were invented to produce them. Now they can easily be traced by photography—a swinging light over a camera or photographic paper (in a darkened room) will produce a variety of pendulum patterns. The oscilloscope may also show similar forms.

Other mathematical curves resembling those traced by the so-called *geometric lathe* may be displayed on the oscilloscope screen. The geometric lathe is an intricate and costly machine used to engrave the beautiful lacework seen on currency and securities (to foil counterfeiters, incidentally). It produces these designs by means of a combination of curve and rectilinear motions, in precise arithmetical ratios.

By the use of a wide variety of electronic setups an almost infinite number of patterns may be created. However, in order to get those with a definite art value, or appeal as abstract design, selective control and combination must be exercised by the operator or the artist doing this work. Some of the simpler patterns, of course, are similar to those obtainable in other ways, or which might be drawn by hand.

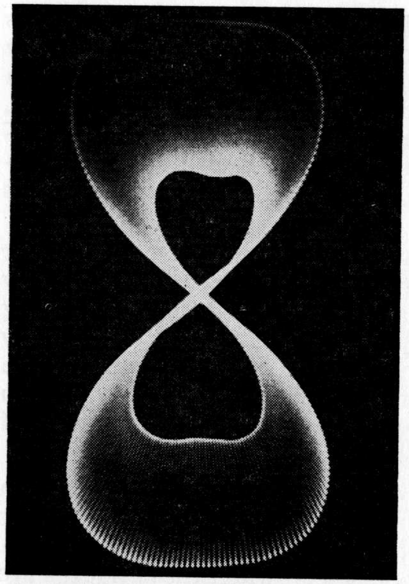
In themselves, the electronic abstractions or oscillons developed by this writer represent a more advanced technique as an art form. They employ some specialized circuits and electrical arrangements not used in normal testing, research or other electronic procedures, besides the basic wave form generators, oscillators, amplifiers, modulation circuits, deflection circuits, and so on. For some of the more involved figures a large number of these elements may be combined.

When a number of factors, such as frequencies, voltages, currents and magnetic field strengths are concerned in the creation of an electronic abstraction, a variation of any of them may produce a quite different figure. It may be changed in over-all shape, or in texture. It may present a solid appearance (the result of using high-frequency curves) or a more open effect of lines, as with low frequency waves. In some cases as many as 75 different factors as these may be involved in the creation of an oscillon, or electronic design.

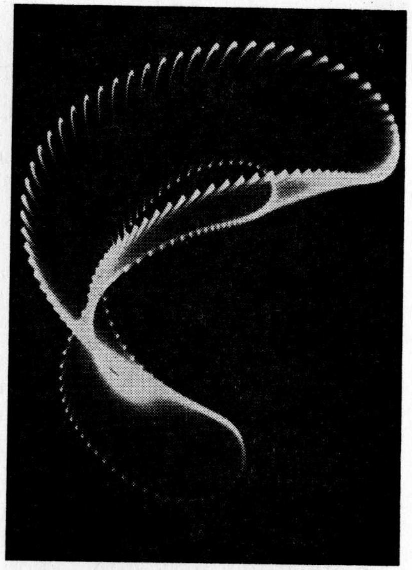
Some of the oscillons have an almost sculptural or third-dimensional appearance, as in a projection of a geometric solid or higher order



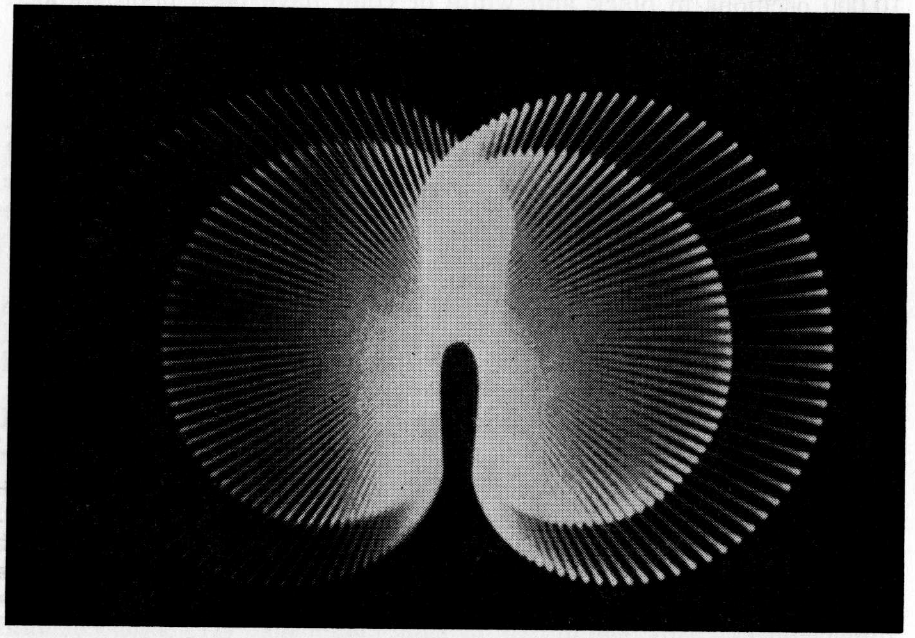




OSCILLON # 10



OSCILLON # 18



OSCILLON # 41

curved surface. This may be due to what are called phase differences of the sine waves composing them. Two sine waves of equal frequency and voltage, if put into the oscilloscope 90° out of phase will produce a circle on the screen; if put into the oscilloscope 45° or 135° out of phase, they will result in an ellipse. The more intricate combinations of additional waves result in other figures with an apparent depth dimension, if phase differences like these (and others) are introduced as factors.

Normally the figure may be seen as a glowing image, as a white (or green, or blue) trace on the oscilloscope screen. It may be stationary or in motion, parts of it pulsating in and out, or rotating, or moving back and forth on the screen. Actually, the entire figure is traced by one point of light on the screen, from the end of the electron beam within the cathode ray tube, just as in a television set. (However, in some patterns, parts of this trace may be darkened or deflected off the screen, so they will not appear in the finished composition.)

After a figure is finally composed on the screen by the connection of the different input circuits and the setting or changing of the various controls on them, it is then photographed, if of possible artistic value. No other lighting is directed on the 'scope screen when these figures are photographed. Fast films and high speed camera lenses are used; special development techniques are also sometimes necessary. Because of these factors, and others, the photographs of the traces are not always as sharp and well-defined as pen drawings would be. (Over 10,000 oscillons in black and white or color have been photographed by this writer so far.)

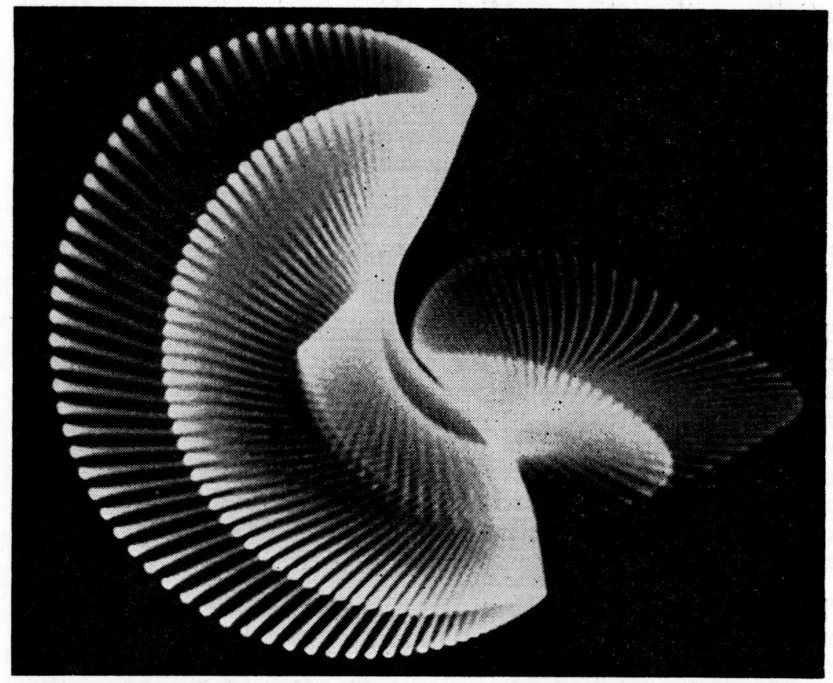
Color may be added to the oscillons by means of special filters ahead of the white trace cathode ray tube. To produce the multi-color effects in a pattern, these filters may be in motion. Generally, a circular transparent disc with two or three color segments is rotated ahead of the screen. This is somewhat similar to one of the early color television systems. As the different color segments move across the face of the screen, the lines or masses in the patterns take on different tints or hues.

The oscillons might be called a form of *visual music* as they are created by means of electrical wave forms in light as music is created by means of sound wave forms in air. The designs are as abstract and mathematical as music is, for the most part, abstract and mathematical. The pulsations of the moving patterns on the screen have a graceful and rhythmic quality reminding one of the gyrations of ballet dancers.

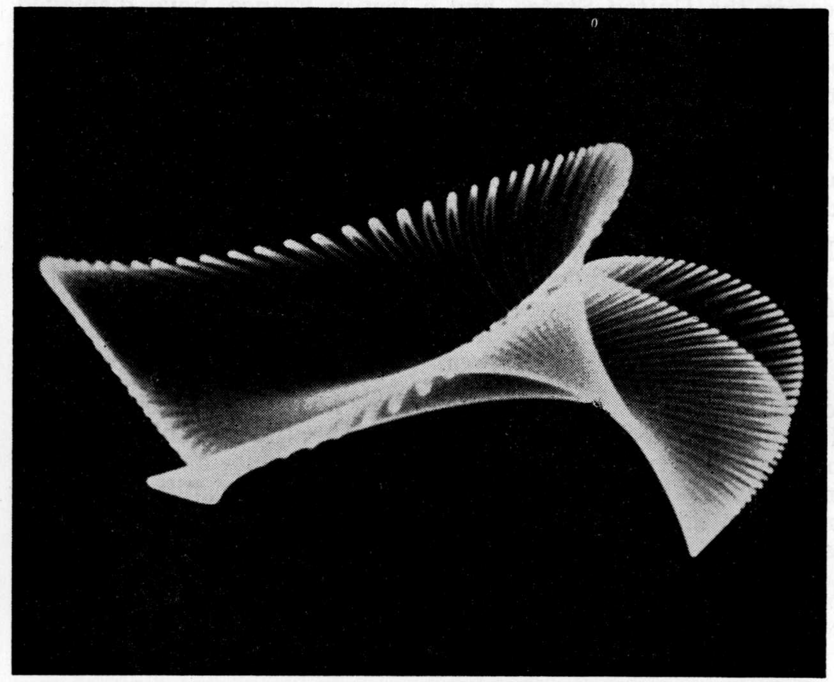
Another interesting aspect of the oscillons is that they could possibly be called a fourth-dimensional art form, more than any other. In the pattern the horizontal and vertical factors ($x$ and $y$) represent two dimensions. The beam tracing the pattern (which also may be varied for light and dark effects, as in television), is the third space dimension ($z$). But, the fourth, or time dimension ($t$), is also of vital importance in the creation of oscillons. The basic frequencies composing them have time as a factor, measured as so many cycles per second.







OSCILLON # 38



OSCILLON # 16

In relativity mathematics or Minkowski geometry time, the fourth dimension, is directly related to the other three of space as $x$, $y$, $z$, $t$.

The electronic abstractions composed by this writer represent the most varied and complex development of the technique of using electronic oscillograms for art yet shown in America or abroad. However, the possible use of oscilloscope patterns for applied design was first suggested by an engineer in 1937 (C. E. Burnett in *Electronics* magazine). A few others have experimented to a limited degree with the idea, including the production of a couple of abstract movie shorts using 'scope traces, some even with music.

This writer actually first became interested in the possibilities of using electronic wave forms for creating design as the result of studies of various mathematical curves, including polar curves, pendulum tracings, magic square line patterns, as well as many other geometric forms in nature - crystals, diatoms, etc. While the creation and application of this new art form is not in itself just a hobby or recreation, it did indirectly have its beginnings for him in experiments with decorative designs based on magic line patterns from magic squares.

The primary appeal of the abstractions is similar to that of any art work or decorative design. They have been widely shown in travelling art exhibits in black and white and color photographs which have been circulated by Sanford Museum of Cherokee, Iowa. More than 50 places in the United States, and a few in France, have displayed them since 1953; more are scheduled for the future. Many of these have been college and university art departments, as well as museums and art centers. The mathematics department of Vassar College, a science seminar at Colorado College, the Cranbrook Institute of Science, and the Institute of Design at the Illinois Institute of Technology (Chicago) as well as Cooper Union, New York, and others, have shown these displays.

The electronic abstractions have been used in several ways in applied art, especially in advertising layouts. One national advertisement for electronics in copper featured an oscillon in color. Others were used in advertisements for typewriters, electrical generators, drugs, and even among other things - perfumes!

For the mathematician or mathematical hobbyist, the abstractions should have some appeal. Their precision of line, their use of the curves of geometry and trigonometry, and their various harmonies and rhythmic sequences, all have definite mathematical significance. Yet, they still have an emotional attraction beyond the cold tracings of an algebraic curve, as they are deliberately composed for a truly aesthetic appeal, as with all art creations.





# *Circles and* $\pi$

How can a circle know it must -
Obedient to a cosmic trust -
Keep radius and circumference
As fixed related measurements?

For crowns and pies and wheels and rings
And other wholly rounded things
Diameter we multiply
By that strict ancient wizard $\pi$
To equal the perimeter
Or else an error we infer.

And circles thin or circles square
Cannot be circles anywhere.
Even an oval will not do.
It must be round. It must be true.
It must recall three point one four
And endless lines of digits more -
The $\pi$ to which eternally
All circles owe their fealty.

And if you disobey this rule
We have to keep you after school
Until like Euclid you have found
That circles know their way around.

— Thomas John Carlisle
(from New York Herald Tribune
with permission)

# *The Christian, Mohammedan, and Jewish Calendars*

*by Sidney Kravitz*

The purpose of this article is to mathematically describe, and compare, the Christian, Mohammedan, and Jewish Calendars, with emphasis on those topics which are of interest in Recreational Mathematics. The author hopes that the reader will gain, in addition, an appreciation of the "other fellow's" point of view.

Relatively speaking, the sun and the moon may be regarded as the hour and minute hands of a huge sky clock. The accuracy with which the three calendars follow the motions of the sun and the moon are indicated in the table below.

| *SOURCE* | *Average length of the year in days* *Exact* | *Decimal* | *Average length of the month in days* *Exact* | *Decimal* |
|---|---|---|---|---|
| Astronomical Observations of the Sun and the Moon | | 365.2422 | | 29.5306 |
| Christian Calendar | $365^{97}/_{400}$ | 365.2425 | $30^{2097}/_{4800}$ | 30.4369 |
| Mohammedan Calendar | $354^{11}/_{30}$ | 354.3667 | $29^{191}/_{360}$ | 29.5306 |
| Jewish Calendar | $365^{24311}/_{98496}$ | 365.2468 | $29^{13753}/_{25920}$ | 29.5306 |

Except for the date of Easter, the Christian Calendar does not follow the motions of the moon. The Mohammedan Calendar, on the other hand does not follow the motions of the sun.

The Christian calendar contains a regular year of 365 days and a leap year of 366 days, the latter occurring 97 times every 400 years. Leap years are spaced every fourth year except when the year is divisible by 100 but not by 400. (Thus 1900 was not a year year, but 2000 will be).

The number of days in a 400 year cycle is (400)(365)+97 = 146,097 and this is exactly 20,871 weeks. The Christian calendar therefore repeats itself every 400 years; but 400 is not divisible by 7, with the result that a given day of the year cannot occur with equal frequency on every day of the year. As an example, the table below contains an actual count of December 25 over each 400 year cycle. The table shows that the probability that Christmas will fall on a Sunday is $^{58}/_{400}$=0.1450 which is greater than 1 in 7 (=0.1429; See Reference 1). An even more interesting result is the fact that the 13th

*Day of the Week Frequencies for Each 400 Year Cycle*

| | *December 25* | *The 13th of the Month* | *February 29* |
|---|---|---|---|
| Sunday | 58 | 687 | 15 |
| Monday | 56 | 685 | 13 |
| Tuesday | 58 | 685 | 15 |
| Wednesday | 57 | 687 | 13 |
| Thursday | 57 | 684 | 14 |
| Friday | 58 | 688 | 14 |
| Saturday | 56 | 684 | 13 |
| TOTAL | 400 | 4800 | 97 |





of the month falls on a Friday more than on any other day of the week (2). On the other hand one textbook on statistics (3) claims that the probability of a leap year containing 53 Sundays is $^{2}/_{7}$. This is incorrect, a leap year contains 53 Sundays if February 29 falls on either a Tuesday or a Wednesday. The table above shows the probability to be $^{15+13}/_{97} = {}^{28}/_{97}$.

The Mohammedan calendar contains either a regular year of 354 days or a Leap year of 355 days. The twelve months of the year each have 30 days and 29 days alternately except that during a leap year the last month contains 30 days instead of 29. In every cycle of 30 years there at 11 leap years. These fall on the 2nd, 5th, 7th, 10th, 13th, 16th, 18th, 21st, 24th ,26th and 29th years o fthe cycle. The first day of the Mohammedan year 1381 was the first day of a new cycle of 30 years and corresponded to June 15, 1961.

The day of the week on which an Mohammedan date occurs may be found as follows. First calculate

$$Q = 4Y + [11(Y - 1)/30 + 1.49] + [(1.5)(M - 1) + 0.5] + D$$

where Y is the Year, M is the Month (in numerical order) and D is the day of the month. In this equation the brackets denote the largest integer function, that is to say, the fractional part of the number must be discarded and only the integer part retained. (For example [2.73]=2.) Then divide Q by 7 and note only the remainder. If the remainder is zero the day falls on Saturday; if the remainder is one, the day falls on Sunday; etc . . . if six it falls on Friday.

As an example, consider the 13th day of the fifth month of the Mohammedan year 1382. We have Y = 1382, M = 5, D =13. So that Q = 6054 by using the formula above. When Q is divided by 7, we get a remainder of 6 which corresponds to Friday.

We now proceed to find the date in the Christian calendar on which the above Mohammedan date occurs. For this purpose we use the following general formula (4)

$$C = 0.97022298\ Y + 621.57736$$

where C and Y denote the Christian and Mohammedan years respectively including the fractional part represented by the month and the day of the month. 1382 is the second year in a cycle of 30 years, and is a leap year of 355 days. The 13th day of the fifth month is the 131st day of the year, thus $M=1382^{131}/_{355}=1382.369014$. Substituting this value in the formula we get C = 1962.78354. Now (365)(0.78354) = 285.992 or the 286th day of 1962. The 286th day is Oct 13th, but this day falls on a Saturday in 1962. Due to leap year variations in the two calendars the above formulas may be in error by a day or two, hence we adjust our calculation by one day and conclude that the 13th day of the 5th month of the Mohammedan year 1382 will fall on Friday, October 12, 1962.

Since the Mohammedan year is numerically smaller than the Christian year but only 97% as long it will eventually catch up. This will occur on the first day of the fifth month of 20,874 at which time the date in both calendars will be the same.

The Jewish calendar is divided into cycles of 19 years in which the 3, 6, 8, 11, 14, 17 and 19th years of the cycle contain 13 months and the others contain 12 months. A twelve month year may contain either 353, 354 or 355 days and a thirteen month year may contain either 383, 384, or 385 days. The months contain either 29 or 30 days following a pattern that depends on the number of days in the year as shown in the table below.

| *Days in the Year* | | 353 | 354 | 355 | 383 | 384 | 385 |
|---|---|---|---|---|---|---|---|
| Month | I | 30 | 30 | 30 | 30 | 30 | 30 |
| | II | 29 | 29 | 30 | 29 | 29 | 30 |
| | III | 29 | 30 | 30 | 29 | 30 | 30 |
| | IV | 29 | 29 | 29 | 29 | 29 | 29 |
| | V | 30 | 30 | 30 | 30 | 30 | 30 |
| | VI | 29 | 29 | 29 | 30 | 30 | 30 |
| (Leap Month) | VIa | | | | 29 | 29 | 29 |
| | VII | 30 | 30 | 30 | 30 | 30 | 30 |
| | VIII | 29 | 29 | 29 | 29 | 29 | 29 |
| | IX | 30 | 30 | 30 | 30 | 30 | 30 |
| | X | 29 | 29 | 29 | 29 | 29 | 29 |
| | XI | 30 | 30 | 30 | 30 | 30 | 30 |
| | XII | 29 | 29 | 29 | 29 | 29 | 29 |

The length of the month is taken as exactly $29^{13753}/_{25920}$ days. Thus twelve months would be $354^{9516}/_{25920}$ days, and thirteen months would be $383^{23269}/_{25920}$ days, and a full cycle of 19 years would be 235 months or $6{,}939^{17875}/_{25920}$ days. The year 1 is assumed to have begun on October 7, 3761 B.C., a Sunday night, at $^{204}/_{1080}$ths of an hour after 11 p.m. Three hundred and one cycles later brings us to the year 5720 which began a new cycle of 19 years on October 2, 1959, a Friday, at $^{19}/_{1080}$ of an hour after 1 p.m.

The New Year will often be observed not on the day on which the "calendar" new moon occurs, but one or two days later. This postponement is based on religious considerations and follows these rules:

1) All "calendar" new moons which fall on Sunday, Wednesday, or Friday are postponed to the next day.

2) All "calendar" new moons which fall after 12 noon on any day are postponed to the next day, but if the next day is Sunday, Wednesday, or Friday it is postponed two days.

3) All "calendar" new moons which fall on a Tuesday of a non-leap year at $^{204}/_{1080}$th of an hour after 3 a.m. or later are postponed to Thursday.





4) All "calendar" new moons which fall on a Monday in the year following a leap year at $^{589}/_{1080}$th of an hour after 9 a.m. or later, postpone to Tuesday.

With these rules we see that although the "calendar" new moon of the year 5720 took place on October 2, 1959, this was a Friday, and also after 12 noon, and consequently its observance was postponed to Saturday, October 3, 1959.

## REFERENCES


(1) Mathematics Magazine, Sept. 1957, P. 51; Problem proposed by the Author.

(2) Otto Dunkel, Memorial Problem Book, American Mathematical Monthly, Aug. 1957, Part II, p. 53; problem proposed by B. H. Brown.

(3) "Mathematical Analysis of Statistics" by C. H. Forsyth, John Wiley and Sons, 1924.

(4) This formula differs but slightly from the formula originally proposed in the Calendar and Chronology articles in the Encyclopedia Britannica.

(5) The Jewish Encyclopedia, article on Calendar.

(6) "Elementary Number Theory", Uspensky and Heaslet, McGraw-Hill, 1939. This book contains an excellent account of the determination of the date of Easter.

## *The Next 495 Prime Numbers - 14741 to 19571*

No errors were found in the June listing of Prime numbers. The following list was taken from D. N. Lehmer's "List of Prime Numbers from 1 to 10,006,721" and then proofread against that list and computer calculated lists supplied by Vernon J. Shipley of Kitchener, Ontario and Sidney Kravitz of Dover, New Jersey.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 14741 | 15161 | 15569 | 15973 | 16447 | 16921 | 17359 | 17789 | 18211 | 18661 | 19183 |
| 14747 | 15173 | 15581 | 15991 | 16451 | 16927 | 17377 | 17791 | 18217 | 18671 | 19207 |
| 14753 | 15187 | 15583 | 16001 | 16453 | 16931 | 17383 | 17807 | 18223 | 18679 | 19211 |
| 14759 | 15193 | 15601 | 16007 | 16477 | 16937 | 17387 | 17827 | 18229 | 18691 | 19213 |
| 14767 | 15199 | 15607 | 16033 | 16481 | 16943 | 17389 | 17837 | 18233 | 18701 | 19219 |
| 14771 | 15217 | 15619 | 16057 | 16487 | 16963 | 17393 | 17839 | 18251 | 18713 | 19231 |
| 14779 | 15227 | 15629 | 16061 | 16493 | 16979 | 17401 | 17851 | 18253 | 18719 | 19237 |
| 14783 | 15233 | 15641 | 16063 | 16519 | 16981 | 17417 | 17863 | 18257 | 18731 | 19249 |
| 14797 | 15241 | 15643 | 16067 | 16529 | 16987 | 17419 | 17881 | 18269 | 18743 | 19259 |
| 14813 | 15259 | 15647 | 16069 | 16547 | 16993 | 17431 | 17891 | 18287 | 18749 | 19267 |
| 14821 | 15263 | 15649 | 16073 | 16553 | 17011 | 17443 | 17903 | 18289 | 18757 | 19273 |
| 14827 | 15269 | 15661 | 16087 | 16561 | 17021 | 17449 | 17909 | 18301 | 18773 | 19289 |
| 14831 | 15271 | 15667 | 16091 | 16567 | 17027 | 17467 | 17911 | 18307 | 18787 | 19301 |
| 14843 | 15277 | 15671 | 16097 | 16573 | 17029 | 17471 | 17921 | 18311 | 18793 | 19309 |
| 14851 | 15287 | 15679 | 16103 | 16603 | 17033 | 17477 | 17923 | 18313 | 18797 | 19319 |
| 14867 | 15289 | 15683 | 16111 | 16607 | 17041 | 17483 | 17929 | 18329 | 18803 | 19333 |
| 14869 | 15299 | 15727 | 16127 | 16619 | 17047 | 17489 | 17939 | 18341 | 18839 | 19373 |
| 14879 | 15307 | 15731 | 16139 | 16631 | 17053 | 17491 | 17957 | 18353 | 18859 | 19379 |
| 14887 | 15313 | 15733 | 16141 | 16633 | 17077 | 17497 | 17959 | 18367 | 18869 | 19381 |
| 14891 | 15319 | 15737 | 16183 | 16649 | 17093 | 17509 | 17971 | 18371 | 18899 | 19387 |
| 14897 | 15329 | 15739 | 16187 | 16651 | 17099 | 17519 | 17977 | 18379 | 18911 | 19391 |
| 14923 | 15331 | 15749 | 16189 | 16657 | 17107 | 17539 | 17981 | 18397 | 18913 | 19403 |
| 14929 | 15349 | 15761 | 16193 | 16661 | 17117 | 17551 | 17987 | 18401 | 18917 | 19417 |
| 14939 | 15359 | 15767 | 16217 | 16673 | 17123 | 17569 | 17989 | 18413 | 18919 | 19421 |
| 14947 | 15361 | 15773 | 16223 | 16691 | 17137 | 17573 | 18013 | 18427 | 18947 | 19423 |
| 14951 | 15373 | 15787 | 16229 | 16693 | 17159 | 17579 | 18041 | 18433 | 18959 | 19427 |
| 14957 | 15377 | 15791 | 16231 | 16699 | 17167 | 17581 | 18043 | 18439 | 18973 | 19429 |
| 14969 | 15383 | 15797 | 16249 | 16703 | 17183 | 17597 | 18047 | 18443 | 18979 | 19433 |
| 14983 | 15391 | 15803 | 16253 | 16729 | 17189 | 17599 | 18049 | 18451 | 19001 | 19441 |
| 15013 | 15401 | 15809 | 16267 | 16741 | 17191 | 17609 | 18059 | 18457 | 19009 | 19447 |
| 15017 | 15413 | 15817 | 16273 | 16747 | 17203 | 17623 | 18061 | 18461 | 19013 | 19457 |
| 15031 | 15427 | 15823 | 16301 | 16759 | 17207 | 17627 | 18077 | 18481 | 19031 | 19463 |
| 15053 | 15439 | 15859 | 16319 | 16763 | 17209 | 17657 | 18089 | 18493 | 19037 | 19469 |
| 15061 | 15443 | 15877 | 16333 | 16787 | 17231 | 17659 | 18097 | 18503 | 19051 | 19471 |
| 15073 | 15451 | 15881 | 16339 | 16811 | 17239 | 17669 | 18119 | 18517 | 19069 | 19477 |
| 15077 | 15461 | 15887 | 16349 | 16823 | 17257 | 17681 | 18121 | 18521 | 19073 | 19483 |
| 15083 | 15467 | 15889 | 16361 | 16829 | 17291 | 17683 | 18127 | 18523 | 19079 | 19489 |
| 15091 | 15473 | 15901 | 16363 | 16831 | 17293 | 17707 | 18131 | 18539 | 19081 | 19501 |
| 15101 | 15493 | 15907 | 16369 | 16843 | 17299 | 17713 | 18133 | 18541 | 19087 | 19507 |
| 15107 | 15497 | 15913 | 16381 | 16871 | 17317 | 17729 | 18143 | 18553 | 19121 | 19531 |
| 15121 | 15511 | 15919 | 16411 | 16879 | 17321 | 17737 | 18149 | 18583 | 19139 | 19541 |
| 15131 | 15527 | 15923 | 16417 | 16883 | 17327 | 17747 | 18169 | 18587 | 19141 | 19543 |
| 15137 | 15541 | 15937 | 16421 | 16889 | 17333 | 17749 | 18181 | 18593 | 19157 | 19553 |
| 15139 | 15551 | 15959 | 16427 | 16901 | 17341 | 17761 | 18191 | 18617 | 19163 | 19559 |
| 15149 | 15559 | 15971 | 16433 | 16903 | 17351 | 17783 | 18199 | 18637 | 19181 | 19571 |





# *The World of Large Numbers*

*by Brother Alfred*

Today, we are living in an age of large numbers. The average citizen peruses statements about an eighty billion dollar budget and a close to three hundred billion dollar debt; he reads about the billions of years the universe has been in existence and the countless miles it extends into space.

And yet, it is very difficult for him to form any adequate idea of just what these numbers mean. What, for example, is a billion dollars? One illustration that seems to bring the matter home rather neatly starts with a million dollars worth of one-thousand-dollar bills. Neatly pressed together, one thousand of them would form a pile 8 inches high. Using the same type of thousand-dollar bill, a billion dollars would constitute a column 666⅔ ft. high which is taller than the Washington Monument by more than a hundred feet! Yet, mathematically speaking we are talking about $10^9$ which is quite a moderate figure and hardly deserves the distinction of being called large.

The poet has his idea of the large number: as numerous as the stars of the heavens or the sands of the seashore. The former is really a misconception since the human eye can only see about 3,000 stars at any one time even with good visibility. The sands of the seashore offer greater possibilities. But why stop with the seashore? Let us consider how many grains of sand could be formed from the entire earth. Sand may be defined as a particle ranging in size between 2 mm. and 0.1 mm. in diameter. Let our sand grain be 0.5 mm. in diameter. Its volume would then be .06545 cubic mm. Considering the earth as a sphere of approximately 4,000 miles radius, the number of grains of sand that could be formed from its substance would be $1.7 \times 10^{31}$.

This is large. But suppose we were to make sand of all the material in the universe or even better consider the number of atoms or electrons to be found there. Evidently these figures would be educated guesses. As one example of the order of magnitude involved, we may quote Eddington's estimate of $10^{74}$ as the number of electrons in the universe. This is certainly a sizable number, but it does not arrive at what is known as the googol or $10^{100}$.

If we think of the age of the universe, we have present-day estimates of the order of 10 billion years. Even if we were to translate this very long time into seconds, the number we obtain is only $3.16 \times 10^{17}$.

Consider finally the size of the universe. With revised distance scales, one estimate is that we have now explored out to 1.6 billion light years. A light year being the distance light travels at 186,284 miles a second in one year and a year being approximately 365.26 days, the distance in miles in a light year is approximately $5.88 \times 10^{12}$. The present radius of the known universe, 1.6 billion light years, would thus be a figure of $9.4 \times 10^{21}$ miles. Going down to the very small unit of distance known as the Angstrom ($10^{-8}$ centimeter) in terms of which light waves are measured, this number would become $1.5 \times 10^{35}$ Angstroms.





Astronomical numbers are, of course, impressive, especially in our space age when man has ideas of touring the universe. But consider the problem involved. Our fast planes today are moving about 600 miles per hour which is a mere 1/6 mile per second! We would have to go much faster in space to get anywhere, especially since the nearest star after the sun is 4.3 light years distant. This is about 25.3 trillion miles. Suppose we have a space ship which is doing 1,000,000 miles an hour, the equivalent of 278 miles a second. It would still take about 2886 years to go one way to the close star, Alpha Centauri.

The numbers pertaining to the universe, whether they concern time, quantity of matter or distance are hard to visualize. But as mathematical quantities we may refer to them as moderately large. This, of course, is purely a matter of relationship. From the mathematical standpoint, it would be very difficult to say what is large and what is not. For no matter to what proportions we build a number, as long as it is finite, its relation to infinity is zero!

## MATHEMATICALLY LARGE NUMBERS

We shall now try to range beyond our astronomical numbers into what may be called mathematically large numbers. One such quantity has already been mentioned, namely, the googol. A still larger number is the googolplex which is defined as 10 to the googol power, written as $10^{\text{googol}}$.

It has been pointed out that the largest number one can form using *only* three digits is $9^{9^9}$ which is equal to $9^{387420489}$. In spite of the huge size of this number (it has 369,693,100 digits) a few things are known about it. The logarithm of this number has been calculated to 80 decimal places and the first 60 and last 26 digits of the number are known (the middle 369,693,014 are unknown - and who cares?).

Another interesting approach is the following. We make use of repeated logs and antilogs to "tame" these large numbers and bring them down to convenient representations. To simplify our notation, let us agree to write:

$$\log [\log (\log N)]$$

as $\log^3 N$ and

$$\text{antilog} [\text{antilog} (\text{antilog } N)]$$

as $\log^{-3} N$. With this notation, we see that:

$$\log^2 (\text{googol}) = 2$$

or

$$\text{googol} = \log^{-2} 2$$

Evidently, with this notation we can write unimaginably large numbers with the greatest of ease. For example, $\log^{-20} 25$. (Maybe I shouldn't have written this. Somebody may try to expand this number and end up in an institution for the mentally deranged.)

The numbers from one point of view may be compared to the molecules of the atmosphere that surrounds the earth. The molecules of the number world are the primes. Just as our atmosphere becomes rarer and rarer the higher we ascend until we get out into space where particles are really sparse, so likewise as we go out farther and farther in the world of numbers, primes become more of a rarity. Here are some interesting statistics:

At 100,000 there are 8 or 9 primes per hundred numbers
At 10,000,000,000 there are 4 or 5 per hundred
At the Googol there are only 4 or 5 primes per thousand
At about $10^{434}$ there is one prime per thousand
At $10^{43400}$ there is but one prime per hundred thousand

In other words, we can arrive at a point in the number universe where the primes are just as rare as we may care to have them: one in a million, one in a billion, and so on.

As a matter of fact, it is not hard to find the interval of, say, 1000 consecutive whole numbers which contains *no* prime number at all. We go about it this way:

$1001! = 1\times 2\times 3\times \ldots \times 1000\times 1001$
$1001!+2$ is divisible by 2, because 1001! is
$1001!+3$ is divisible by 3, because 1001! is
and so on until
$1001!+1001$ is divisible by 1001.

So there is a sequence of 1000 consecutive whole numbers, *none* of which is prime. The only problem involved in actually writing out these 1000 numbers is the fact that 1001! itself has 2571 digits!

## LARGE PRIMES

Mathematicians over the ages have given a great deal of attention to the problem of determining which numbers are prime. In spite of all the efforts of the greatest mathematicians, however, there is still no easy way of solving this problem. Apart from special methods which apply to numbers of certain forms, to ascertain whether a given number is a prime, one must divide it by all the primes up to its square root or use other processes which involve much the same quantity of calculation. Suppose, for example, we have available the List of Prime Numbers by D. N. Lehmer from 1 to 10,006,721. For simplicity, let us call this latter figure $10^7$. The largest prime that could be verified with the aid of this table would then be $10^{14}$. To make this verification, the straightforward method would be to divide the number being tested by all of the 665,000 primes in Lehmer's table !

Perhaps with this background, one can appreciate the magnitude of the accomplishment of determining that a number of the order of $10^{968}$ has been found to be a prime. How was it possible to accomplish





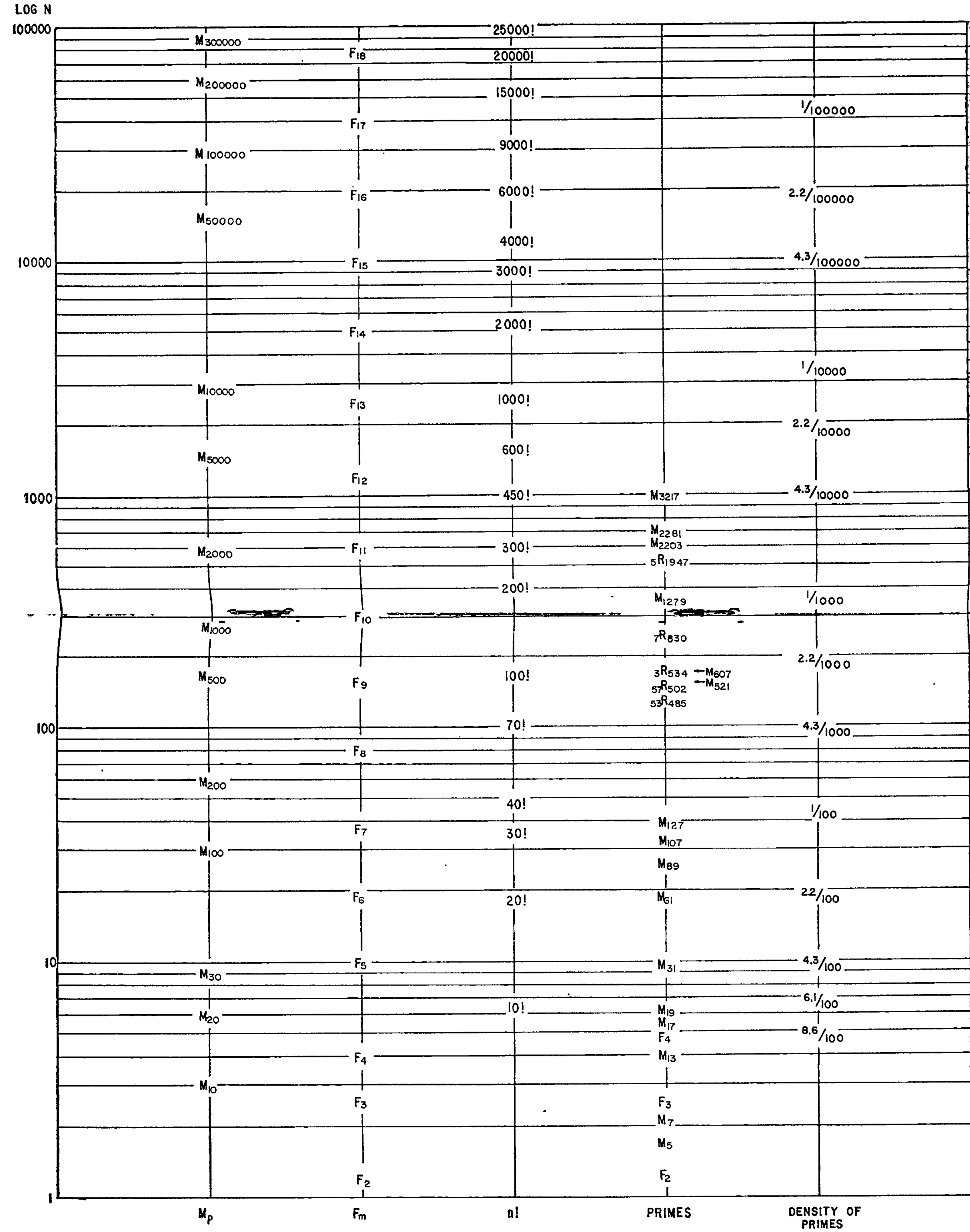
this feat? For one thing, special numbers were considered, in this case, a Mersenne number $2^{3217}-1$. Mathematicians by studying the nature of these numbers have been able to delimit the type of factor they may have and thus simplify the calculations. For another, modern calculating equipment was employed. To give some idea of the gain in power, let it be noted that in one investigation, the author asserts that the machine did in one minute what it would take a person using a desk calculator over a year to accomplish. Yet, to discover the very large prime with 969 places, the computing machine required 5½ hours.

It has been pointed out that with available equipment and methods we have for the moment come to the practical limit of determining large primes even of a specialized nature. Need it be said that mathematicians have not as yet discovered a formula that will always yield a prime number. If we had such a formula, there would no longer be a largest prime. But as matters stand today, it appears that the concept of a largest prime will be with us for a long time to come.

## CHART OF THE NUMBER UNIVERSE (On The Next 2 Pages)

Finally, to provide a picture of our universe of large numbers, a chart has been prepared which compresses numbers into a small space by means of a logarithmic scale. Along vertical lines, the following interesting sequences are displayed:

(1) The Mersenne numbers: $M_p = 2^p - 1$
(2) The Fermat numbers: $F_n = 2^{2^n} + 1$
(3) Factorials: $n! = (1)(2)(3)(4)\ldots(n-1)(n)$
(4) The large primes, M standing for Mersenne numbers, F for Fermat numbers and R for Robinson numbers of the form ${}_kR_n = (k)(2^n)+1$
(5) The density of the primes.


LOG N
100000
10000
1000
100
10
1
M300000
M200000
M100000
M50000
M10000
M5000
M2000
M1000
M500
M200
M100
M30
M20
M10
F18
F17
F16
F15
F14
F13
F12
F11
F10
F9
F8
F7
F6
F5
F4
F3
F2
25000!
20000!
15000!
9000!
6000!
4000!
3000!
2000!
1000!
600!
450!
300!
200!
100!
70!
40!
30!
20!
10!
M3217
M2281
M2203
5R1947
M1279
7R830
3R534 ← M607
57R502 ← M521
53R485
M127
M107
M89
M61
M31
M19
M17
F4
M13
F3
M7
M5
F2
1/100000
2.2/100000
4.3/100000
1/10000
2.2/10000
4.3/10000
1/1000
2.2/1000
4.3/1000
1/100
2.2/100
4.3/100
6.1/100
8.6/100
Mp
Fm
n!
PRIMES
DENSITY OF PRIMES

# *The Hexahedra Problem*

*by John McClellan*

If one were to interrupt a group of crap-shooters during a game to ask whether their dice could be made in some other form, he would undoubtedly receive the treatment that such an off-beat question merited. However, if one of the players happened to be interested in mathematics - aside from the mathematics of chance - he might pause to think it over: for the ordinary die, or cube, with its six faces is only one member of a group of solids characterized by the fact that each of them has six faces. The group is called the Hexahedra, and the question of our imaginary busy-body can be stated more elegantly: How many plane-faced solids exist with just six faces, irrespective of type of polygonal face, and number of vertices?[1]*

This paper proposes an approach to the problem of how many six-faced variations there are. Some readers may enjoy applying this method to other groups of polyhedra, each group having the same number of faces, and some may be able to obtain a general answer for n-hedra.

The schematic drawings of the solids which are used throughout are 'Schlegel diagrams', named for the 19th century German mathematician who invented them. They are a most convenient way of showing the three components of face, vertex, and edge of polyhedra, and their relation to one another. They do *not*, however, give us information of a 'metrical' nature, such as length of line and degree of angle. If we imagine a solid to be made of sheet-rubber and remove one face, we may stretch out the remainder onto a plane. All the edges and vertices are preserved, and all the faces except one, which we imagine to surround the others.[2] We refer to them here simply as 'diagrams'.

When the number of faces (F) is given and even, the number of vertices (V) ranges from $F/2 + 2$ at minimum to $2F-4$ at maximum: or in all, there are $(3F-10)/2$ solids with the same number of faces, but differing as to number of vertices.[3] V and F are interchangeable in these formulae. When V is given and even, F ranges from $V/2 + 2$ at a minimum, to $2V-4$ at maximum.

Applying the above to the hexahedra, or F6, we find that six faces occur in four different V-families which may be described thus[4].

| | | | |
|---|---|---|---|
| (a) | V8 | F6 | e24 |
| (b) | V7 | F6 | e22 |
| (c) | V6 | F6 | e20 |
| (d) | V5 | F6 | e18 |

in which 'e' is the sum of the edges of the polygons - twice the sum of the polyhedral edges (E)[5]. These polyhedra differing as to number of

*Superior numbers refer to Notes.





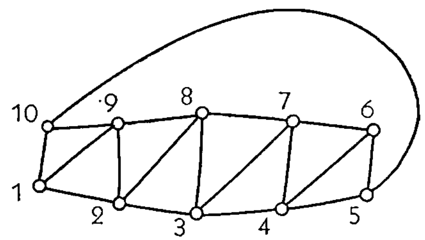
vertices, but retaining the same 'F' value are called, arbitrarily, 'members' of the F6 family.

These four cases, however, do not exhaust the possible hexahedra, for under certain circumstances, changes can be made in the polygonal faces without changing the numerical values of V, F, and e. These polyhedra which change their 'facial' character only are called 'allomorphic variations', or 'allomorphs'[6].

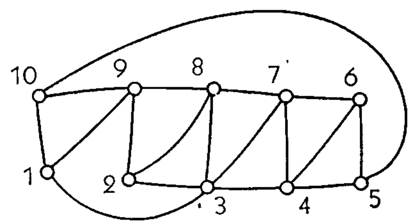
Let us take a polyhedron such as

(e) V10 F10 e36

and examine it. The mathematical phrase describes a solid of 10 vertices, and 10 faces. The sum of the edges of its faces is 36.



(a)



(b)

Figure 1

In the diagram of this solid (Fig. 1a) we see line 1-2 to be the mutual border of a 3-gon and a 6-gon. If we detach this line from 2 and rotate it to 3, the new line becomes the border between a 4-and a 5-gon. The number of vertices, faces, and edges remains the same - only the polygons making up the faces have been changed. We have created an allomorph by this operation.

We may draw certain conclusions from the above: for the rotation of an edge to be significant and to lead to the creation of an allomorph, a difference of at least 2 must exist between the sides of adjacent polygons. Also, the sum of the sides of the adjacent polygons must be at least 8 for a significant rotation of the border between them: for the rotation of the line dividing a 3-gon and a 4-gon can only result in the same polygons reserved. We were able to make the rotation above (Fig. 1) because the border-line separated a 6- and a 3-gon.

Another restriction whose importance will be seen when we examine the Hexahedra is that the rotation which leaves a vertex with less than 3 edges concurrent at it, requires a second rotation to fill the deficiency. This is true in the 'earliest' member of an V family - i.e., that member which has the least number of faces for a given number of vertices. A charactaristic of this member is that all its vertices are trilinear - hence, if any line is detached from a vertex, another line must be taken from a 4-linear vertex and joined to the other to give it its necessary quota of 3 lines. To illustrate,

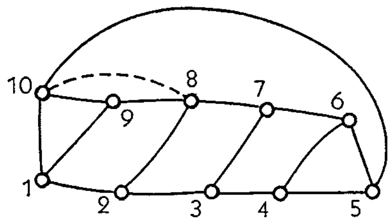



Fig. 2

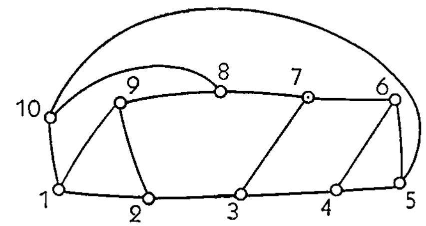



Fig. 3

if edge 10-9, the mutual boundary between a 3- and a 6-gon, is detached from 9 and rotated to 8 (dotted), vertex 9 lacks a necessary edge. However, vertex 8 now has 4 lines concurrent at it and can therefore give up one of them. When 2-8 is rotated to 2-9, vertex 9 is again trilinear. (Fig. 3).

We may say, therefore, that in this earliest member, two border rotations are necessary to accomplish a single significant change, or allomorph. In subsequent members, the increase of faces results in more and more 4-linear vertices, and the chance is thereby diminished of stripping vertices of their necessary quota of edges.

It will be noted that two of the four members which contain six faces are earliest members of their respective V families, the V8 F6 and the V7 F6.

If we wish to deal with this problem more analytically, and to put diagrams to one side, we may think of e as the sum of certain digits, corresponding to the sides of the polygonal faces, and may recast the problem thus: In how many ways can we reach the total, e, using certain digits F times?

In the case of the familiar cube, described as

(f) V8 F6 e24

we are saying, in effect, that 24 is the sum of certain digits - in this case, 4's - used 6 times. It is possible to reach the same total, e, with different groups of digits F times and these constitute the allomorphic variations of this member.

The 'certain' digits referred to are the polygons possible within the limitations of the polyhedron in question. It is found that when $V \leqq F$, the largest polygon possible has $2V-F-1$ sides; when $V \geqq F$, as in the case of the cube, the largest polygon may have $F-1$ sides.

Returning to the F6 family and examining the first solid of the list of V families containing six faces, the V8 F6 e24, we analyze it in this way: Its maximum polygonal face is a 5-gon, its minimum a 3-gon, and there are six faces in all. The sum of the edges, e, of these

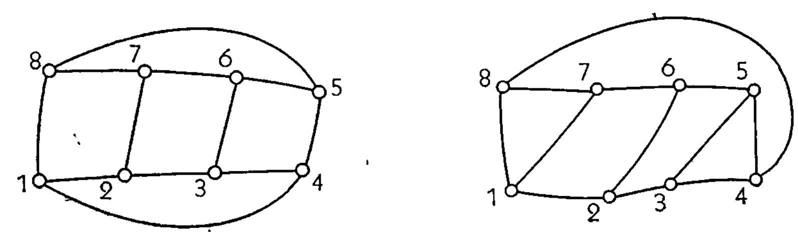




polygons is 24. The question is, In how many ways may 24 be totalled using any of the digits 5, 4, and 3 six times only, and allowing repetitions? The following table shows the four possibilities:

| (1) | (2) | (3) | (4) |
|---|---|---|---|
| 4 | 5 | 5 | 5 |
| 4 | 4 | 5 | 5 |
| 4 | 4 | 4 | 5 |
| 4 | 4 | 4 | 3 |
| 4 | 4 | 3 | 3 |
| 4 | 3 | 3 | 3 |
| 24 | 24 | 24 | 24 |

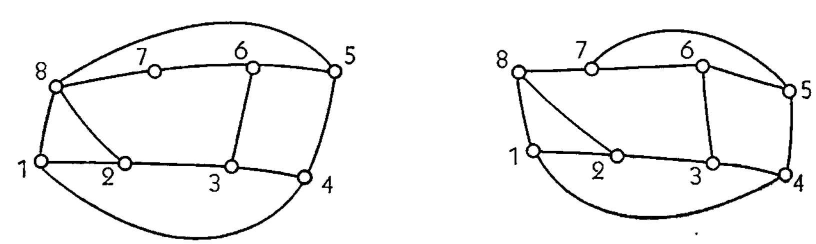
The '4's' (or, 4-gons) of column 1 are raised and lowered successively by 1 until in column 4 this is no longer possible. When we try to draw diagrams of these we find that only *two* are constructable - the arrangements of columns 1 and 3. This is not surprising, for we have already seen that in the earliest member of any V family, two rotations are necessary to accomplish one allomorph. The two possible constructions are shown below.




(a) (b)

Fig. 4

When we rotate edge 2-7 around 2, between the two 4-gons, to produce the 3-gon and 5-gon of column 2, vertex 7 loses a necessary edge (Fig. 5, a and b). The deficiency is supplied by the rotation of 5-8 to 5-7, and this results in the variation of column 3.




(a) (b)

Fig. 5

An analysis of the next member of the F6 family, the V7 F6 e22 member, shows the three arrangements of the digits representing the polygonal faces. The maximum-sided polygon is a 5-gon, as before. We restate the proposition to read, In how many ways can we total 22, using any of the digits 5, 4, and 3 six times, and allowing repetitions?

| (1) | (2) | (3) |
|---|---|---|
| 4 | 5 | 5 |
| 4 | 4 | 5 |
| 4 | 4 | 3 |
| 4 | 3 | 3 |
| 3 | 3 | 3 |
| 3 | 3 | 3 |
| 22 | 22 | 22 |

In column 1 the sum is attained with the maximum number of minimum-sided polygons - 3's and 4's. The 4's of this column are increased in the two succeeding columns with the corresponding decreases, to keep the balance, until no further raising and lowering of 4 is possible.

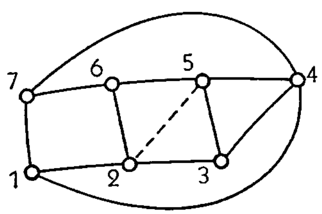



Fig. 6

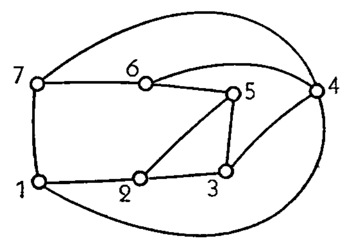



Fig. 7

Fig. 6 shows the column 1 arrangement. The mutual border 2-6 between two 4-gons may be rotated to a new position, 2-5, (dotted) to form a 5- and 3-gon: vertex 6 now lacks a necessary edge which may be supplied by the second rotation of 4-5 to 4-6 and we have the arrangement of column 2.

Note that edge 3-4 may be detached from vertex 4 and attached to vertex 1, thereby making the column 3 arrangement, of two 5-gons (Fig. 8).

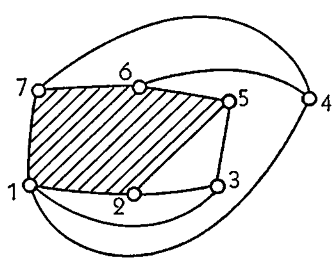



Fig. 8





However, the diagram shows clearly that the pentagons 12567 (shaded) and 14653 have *three* vertices, 1, 5, and 6, in common - an impossibility: and, actually, the arrangement of column 3 is impossible to construct without warping. Hence, only *two* of the arrangements found analytically are capable of construction.

Proceeding to the third solid of the list, the F6 V6 e20, we analyze it as before, using again the digits 5, 4, and 3,

| (1) | (2) |
|---|---|
| 4 | 5 |
| 4 | 3 |
| 3 | 3 |
| 3 | 3 |
| 3 | 3 |
| 3 | 3 |
| 20 | 20 |

and find that both columns are constructable (Fig. 9).

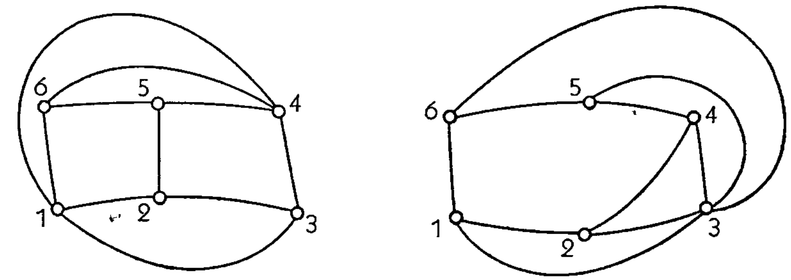



Fig. 9

Treating the F6 V5 e18 in the same manner, we see that all the faces are triangular, and that this member of the F6 family exists only in this one form.

| |
|---|
| 3 |
| 3 |
| 3 |
| 3 |
| 3 |
| 3 |
| 18 |

Fig. 10 shows the diagram of this member.

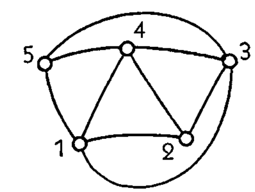
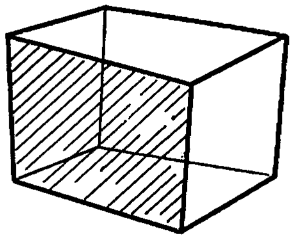
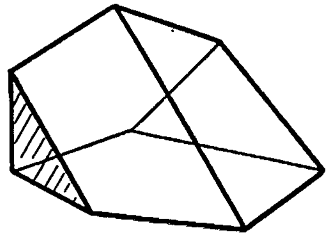


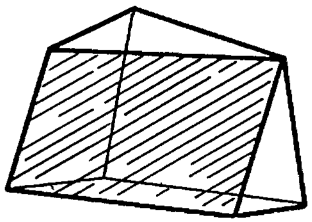

Fig. 10

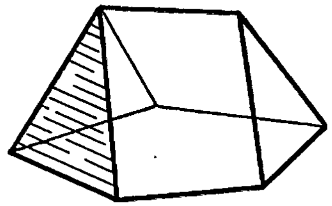
A recapitulation of the foregoing shows seven constructable convex hexahedra (there are three more concave hexahedra) - but despite this interesting fact, our dice-rolling friends will continue to use the cube for the very good reason that it is still the best roller!



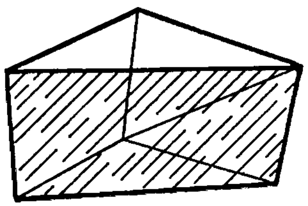


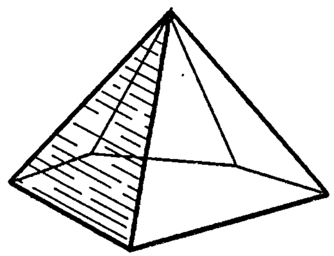
F6 V8 e24



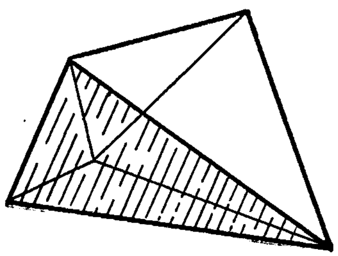


F6 V7 e22





F6 V6 e20



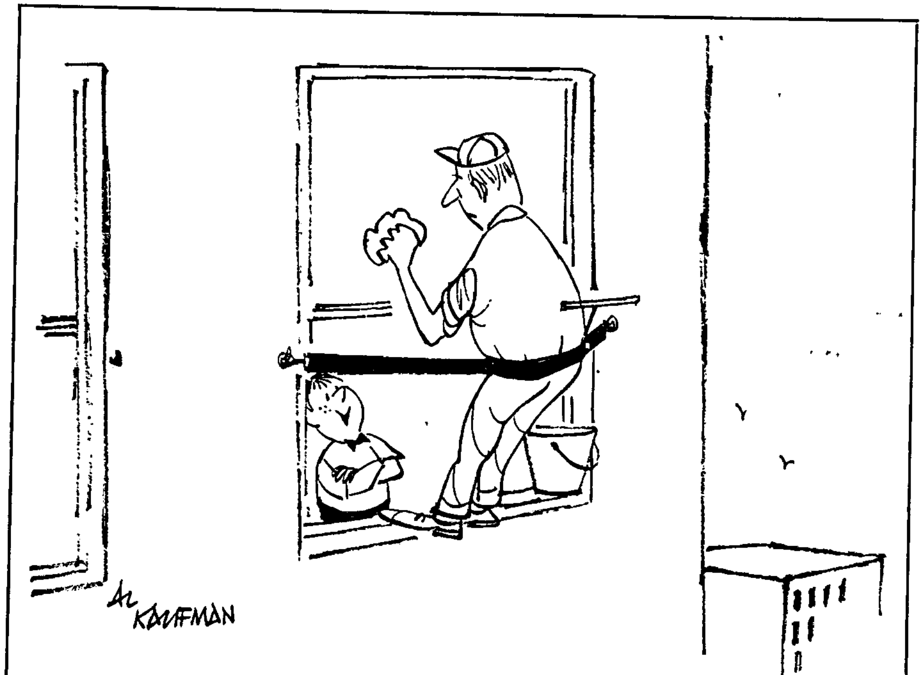
F6 V5 e18





## NOTES

1) Published as a problem in SCIENTIFIC AMERICAN Magazine, June 1961, in Martin Gardner's column, "Mathematical Games."
2) See "INTRODUCTION TO GEOMETRY," by H.S.M. Coxeter, p. 152: and "AN INTRODUCTION TO THE GEOMETRY OF N-DIMENSIONS," by D.M.Y. Sommerville, p. 100.
3) When F is odd, the range of V is from $F/2 + 2.5$ at minimum, to $2F-4$ at maximum, and there are $(3F-11)/2$ solids with the same number of faces, but differing as to number of vertices.
4) Based on the well-known formula of Euler, $V+F-E=2$.
5) See "INTRODUCTION TO GEOMETRY," by H.S.M. Coxeter, p. 153.
6) A term used by D.M.Y. Sommerville in "AN INTRODUCTION TO THE GEOMETRY OF N-DIMENSIONS," p. 101.
7) For $V-1$ points may lie on a plane, and $F=V$ in this configuration.




"DID YOU KNOW THAT IF YOUR BELT SNAPPED YOU'D FALL AT THE RATE OF 32 FEET PER SECOND PER SECOND?"

# *Alphametics*

Here's another collection of *Alphametics* for your enjoyment — answers to these will be in the October RMM. The answers to the June Alphametics follow this months selection.

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  S E N D
  M O R E
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M O N E Y
```

Alphametics have been around for quite awhile and the Editor has had the two given here sent to him several times. The origin of each is unknown — but many RMM readers may not have seen them, or may have forgotten the answers if they have seen them.

$$\frac{\text{E V E}}{\text{D I D}} = .\text{TALKTALKTALK} \ldots\ldots$$

Now the English may not be perfect, but this little Alphametic has a unique answer. (Alan Gold)

```
      C A T S
        E A T
  -----------
    x x x x x
  x x x x x
x x x x x
-------------
x M O U S E S
```

```
 W H E A T
   F I E L D
-----------
F A R M E R
```

Alphametics do not necessarily have to be based on our usual decimal number system. So what is the value of FARMER expressed in the lowest-base system applicable here?

(Margaret M. Rohe)

Dan and Edna may have had a good time at Aden, but what kind of a time will you have trying to solve this Alphametic? (A .G. Bradbury)

```
  D A N
  A N D
E D N A
    A T
-------
A D E N
```

Here's an Alphametic in a rather different form. It's a bit difficult so be wary. (J. A. H. Hunter)

$$\sqrt{\text{CAREER}} = \text{NOW}$$

---

Here are the answers to the Alphametics from the June issue of RMM. RMM readers who solved them correctly are listed.

```
(1)  M O O N        9552
       M E N         902
       C A N    =    382
   ---------       -----
   R E A C H       10836
```

```
(2)  (V)(V E X A T I O N) = E E E E E E E E E
     (9)(9 8 7 6 5 4 3 2) = 8 8 8 8 8 8 8 8 8
```





```
(3)  A L L S        9332         9332
     W E L L        8433         8433
     T H A T   =    6596   or    6096
     E N D S        4072         4572
   ---------       -----        -----
   S W E L L       28433        28433
```

```
(4)      U N          35           34
         U N          35           34
     D E U X   =    8230   or    8632
   D O U Z E       84372        87316
   ---------       -----        -----
   S E I Z E       92672        96016
```

(5) The First Doggerel:

PINT plus a PINT makes a QUART = 6390 plus 6390 equals 12780 and QUAINT equals 127390. URN = 289.

(6) The Second Doggerel:

PINT plus a PINT is still a QUART = 7920 plus 7920 equals 15840 and DIET equals 3960. Then DAIQUIRI is 38915949.

Here are the RMM readers who sent in correct answers to the June Alphametics. The correctly solved puzzles are in parentheses after their names.

Alan Gold of Downsview, Ontario (1, 2, 3, 4); Norvin Pallas of Cleveland, Ohio (3); George Propper of New York, New York (1, 2, 3, 4, 6) (Mr. Propper was the only one to find the alternate solution to No. 4. It was thought that this Alphametic had a unique answer.); Don Singleton of Pasadena, Texas (2, 5); Anneliese Zimmerman of Montreal, Quebec (1, 2, 3, 4, 5, 6).

---

# *Word Games*

*by S. Baker*

Mr. Baker is vacationing and will resume his Word Game duties in the October issue of RMM. In the meantime, RMM Word Game enthusiasts will not lack for puzzles. Here are a few that will challenge the most avid Word Fans.

| B | I | S | H | O | P |
|---|---|---|---|---|---|
| I | | | | | |
| S | | | | | |
| H | | | | | |
| O | | | | | |
| P | | | | | |

You are invited to make a word square which reads the same Down as Across using the word BISHOP as the key word as shown in the diagram.

(N. A. Longmore - Kent, England)

Lloyd Jim Steiger asks for an example of a word, in common English usage, which exists in the negative form and also in the double negative form but not in the simple positive form.

W. A. Robb would like RMM readers to try to find out if there are other words with the vowels in their proper order (two are ABSTEMIOUS, FACETIOUS) and if there is at least another word with the vowels in their reverse order other than SUBCONTINENTAL.

The Answers to the June Issue *Word Games.*

"7" LETTER SCRAMBLE

| | | | | | | |
|---|---|---|---|---|---|---|
| R | I | N | G | E | N | T |
| G | A | H | N | I | T | E |
| I | N | G | E | S | T | A |
| E | N | I | G | M | A | S |
| L | I | N | K | A | G | E |
| R | A | N | G | I | E | R |
| G | A | T | I | N | G | S |

The proper answer, supplying the clue *teasers* and with none of the words ending in "G", is shown to the right. A further clue, for those who couldn't find all the words ending in "G", is that the other solution to this puzzle has all the words ending in "ING".

CHANGE A LETTER

Mr. Baker's answer to this one is: PRETEND, PRINTED, PERIDOT, CORDITE, CHOIRED, HEROICS, ECHOISM.

Puzzle-solvers for the "7" LETTER SCRAMBLE include: W. A. Robb of Ottawa, Ontario who supplied both of the required answers, using ANGRIER instead of RANGIER; David Kaplan of Bronx, New York who likewise solved both parts, using GRANIER instead of RANGIER; Joseph D. E. Konhauser of State College, Pa. who found both sets of answers (again GRANIER instead of RANGIER); Corine Bickley of St. Louis, Missouri who found both solutions (with GRANIER instead of RANGIER); Robert S. Johnson of Montreal, Quebec who found all the "ING" words and used GAINETH instead of GAHNITE and GAMINES instead of ENIGMAS for the other part of this puzzle; Donald V. Trueblood of Bellevue, Washington who found both parts, using the same substitutions as Mr. Johnson in the other part.

The CHANGE A LETTER solvers: W. A. Robb of Ottawa, Ontario supplied the greatest number of solutions to this puzzle and his answers are listed in the next paragraph; Robert S. Johnson used PRETEND, PORTEND, SNORTED, SHORTED, HOISTED, ECHOIST, ECHOISM; Edith Marsh of Montreal West, Quebec used PRETEND, PORTEND, DEPORTS, STORMED, MOISTER, HEROISM, ECHOISM; Joseph D. E. Konhauser used PRETEND, SERPENT (or PRESENT), RESPECT, PRECISE, MERCIES, CHEMISE, ECHOISM; Corine Bickley used PRETEND, TENDERS, RODENTS, MONSTER, MOISTER, HEROISM, ECHOISM;





Donald V. Trueblood used PRETEND, PRINTED, POINTER, RIPOSTE, IMPOSER, HEROISM, ECHOISM.

Now here is the list of solutions found by W. A. Robb of Ottawa, Ontario:

| | | | | | | |
|---|---|---|---|---|---|---|
| PRETEND | SERPENT | RESPECT | PRECISE | COPIERS | COHEIRS | ECHOISM |
| PRETEND | REPENTS | SPECTRE | PERCHES | SPHERIC | HOSPICE | ECHOISM |
| PRETEND | TENDERS | TINDERS | RESCIND | CRONIES | INCOMES | ECHOISM |
| PRETEND | SPENDER | PONDERS | DONSIER | MERSION | HEROISM | ECHOISM |
| PRETEND | PRINTED | POINTED | DEPOSIT | IMPOSED | MEDICOS | ECHOISM |
| PRETEND | DENTERS | RODENTS | SCORNED | ORCEINS | HEROICS | ECHOISM |
| PRETEND | PRESENT | TRENISE | RESHINE | CHINESE | CHEMISE | ECHOISM |
| PRETEND | PRESTED | CRESTED | DIRECTS | CHIDERS | CHIMERS | ECHOISM |

Errata for the June 1961 RMM

Page 34: Line 5 of section 7 should read "the sum by the number of them; for the geometric mean of a few".

Page 50: $3^{10}$ divided by 10 leaves a remainder of 9 (not 10).

Page 55: Reverse equations: $001 = 10^0$.

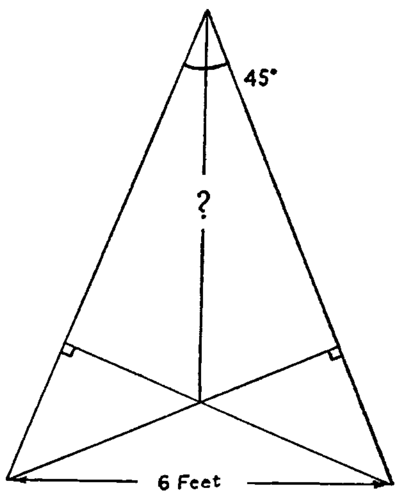
Page 58: In Mr. Hunter's figure, the p and q should be interchanged and the last denominator on this page should read $(b+c)^2$.

*Look at the advertising in RMM — it's to your advantage — and when you answer an ad, say you saw it in RECREATIONAL MATHEMATICS MAGAZINE.*

# Puzzles and Problems

It appears that RMM readers object to being able to peek at the answers, so the answers to all of the following puzzles will be published in the *October* issue of RMM.

The Editor wishes to thank all the puzzle-solvers for their answers and comments to the June puzzles (pages 48-49) and hopes that many more will try their hand at solving the puzzles and getting their names published.




1. Ladder Legs and Cross-Stays

The angle between the equal legs of this step-ladder is 45°. From each leg there is a cross-stay perpendicular to the opposite leg, their lengths being equal. The feet of the ladder are six feet apart. How far is the crossing-point of the two stays from the top of the ladder?
(Sinclair Grant - Perth, Scotland)

2. River-Crossing Dilemma

Three explorers, each with h's native bearer, were hacking their way through the jungle when a river interrupted their progress. Having been foresighted enough to bring a two-man rubber boat, they inflated it, and immediately encountered two more difficulties:

While each explorer knew how to maneuver across the river, only one of the natives was able to manipulate the boat. None of the explorers was willing to remain on either side of the river, even momentarily, while outnumbered by natives.

How was the party of six able to cross the river using the rubber boat? (E. A. Beyer - Novato, Calif.)

3. Tricky Window

A sash window with a half round top is five feet wide. If the top sash is lowered one foot, what is the area of the moon-shaped opening at the top? ( H. V. Gosling)

4. Ladder and Barrel

A ladder is leaning against a wall at on angle steeper than 45°. Under the ladder a large barrel has been rolled which just touches both the ladder and the wall. If the diameter of the barrel, in feet, is $^1/_5$ that of the vertical distance from the top of the ladder to the ground, what is the smallest integral number of feet the ladder can be?
(N. A. Longmore - Kent, England)





5. The Oracle of Three Gods

A certain oracle is presided over by three gods, who take turns in answering the questions put to it by pilgrims. The three gods are the God of Truth, who always tells the truth; the God of Falsehood, who always lies; and the God of Equivocation, who alternately tells the truth and lies.

One day a pilgrim arrives and wishes to know whether or not his wife is faithful. Unfortunately, the poor pilgrim does not know which God will answer his question. Moreover, the priests of the oracle allow only one question from a pilgrim.

How should the pilgrim state his question, so as to be sure of his wife's fidelity or otherwise?

(N. A. Longmore - Kent, England)

6. The Prolific Author

A famous author made a present of his collected works. His friend received the books and found that one-half of the volumes were novels, one-quarter were poetry books, one-seventh were books of plays, and three volumes were collections of miscellaneous items. Can you tell how many volumes were given away? (G. Mosler)

7. A Quick Answer on This Puzzle - Sharing the Ride

A friend of yours has hired a cab to take him back and forth to work at a flat rate of \$3.00 for the round trip. Since you live exactly half-way between his house and his office, and you happen to work in the same building, he offers to take you to the building and back each day. Quickly now: what would be your fair share in the ride?

(G. Mosler)

8. The Stamp-Collecting Kid

Peter was quite excited when he came home from school. "See what I bought from one of the kids," he told his father, putting three stamps on the table. He pointed to a Cape triangle. "I got that for less than 50 cents, but the other two cost more than that for each one."

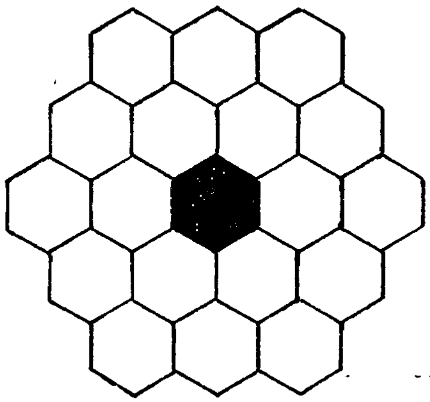
His father examined the specimans. "Not bad," he said, "but you should have made an offer for the three as one lot."

Peter shook his head. "Not with Tom. I think I did better by beating him down cent by cent on each."

"What did you pay?" his father asked, interested in one of the stamps for his own collection.

The boy wasn't giving a straight answer to that question. "If you multiply the three differences between the prices," he replied, "you get what I paid altogether in cents."

Well, what did Peter pay? (J. A. H. Hunter)



9. A Magic Hexagon

Fill in the hexagons in the figure so that each ring of hexagons will have equal whole numbers in the hexagons making up that ring and so that the sum *and* the product of a straight line of hexagons will equal the sum and product of all the other straight lines of hexagons.

(Lloyd Jim Steiger)

## ANSWERS TO THE PUZZLES AND PROBLEMS IN THE JUNE ISSUE OF RMM (Pages 41-43)

The answers to the puzzles are given first and the RMM readers who submitted correct answers are given afterwards with the correctly solved puzzles in parentheses after their names.

2. High Stakes: Mike won the first, second, fifth, seventh and eighth games of the ten games played. The comments from readers necessitate the full workings of the answer to this little gem.

$2^9 < 601 < 2^{10}$, so they played 10 games. The gross total of stakes was $2^0 + 2^1 + 2^3 \dots + 2^9 = 1023$ cents. Let's say that the gross winnings were:

Steve x cents and Mike y cents

Then $x+y=1023$ and $x-y=601$; whence $x=812$ and $y=211$. In binary notation 211 appears as 11010011. So Mike won the 8th, 7th, 5th, 2nd and 1st games.

4. Breakfast Mathematics: Mr. Smith started with a full cup of coffee (6 swallows) and successively took 1, 2 and 3 swallows before taking the full 6 swallows. Therefore he had added 6 swallows of cream - as much cream as coffee.

5. Squares at the Round Table: Mr. Smith's son took away 46 blocks from the original square arrangement of 144.

6. No Problem for an Accountant: Mr. Smith lived 2 miles from work, took 6 minutes to drive the first day, 4 minutes the second day, and should go 24 miles per hour every day to arrive on time.

7. The Doctor's Dilemma: Dr. Nimbus was born in 1897, has 12 granddaughters (3 from each of his 4 daughters), has 20 grandsons (4 from each of his 5 sons), and his license number is 77777.





8. Some Extracurricular Activity: All the clubs met once during the quarter on March 2. There were 24 days they did not meet at all: January 2, 8, 12, 14, 18, 20, 24, 30; February 1, 7, 11, 13, 17, 19, 23; March 1, 3, 9, 13, 15, 19, 21, 25, 31.

9. An Airport Problem

The young man was heading toward his destination exactly halfway around the world and so he could have gone in any direction without going to much out of his way. Credit must be given to Don Singleton for finding both answers. The young man was also go'ng to fly completely around the world, back to the airport.

Correctly solved by: Jack Abad of Central Michigan University (4, 5, 7, 8, 9,); Merrill Barnebey of Grand Forks, North Dakota (2,4,5,6,7); Richard H. Beck of Mt. Vernon, New York (9); Jeanette Bickley of St. Louis, Missouri (4,5,6,7, first part of 8,9); Micky Earnshaw of Los Angeles, California (4,5,6); Alan Gold of Downsview, Ontario (4,6,8); Joe Haseman of Lakeland, Florida (2,4,5,6,7,8,9); B. C. Kimmons of Rosedale, Ontario (4,9); John Lewis of Los Angeles, California (2,4, 5,6); Norvin Pallas of Cleveland, Ohio (2,5,6, half of 8, 9); George Propper of New York, New York (2,4,5,7,8); Don Singleton of Pasadena, Texas (4,5,6,9 including the second possible answer for 9); Spencer Stopa of Chicago, Illinois (2,4,8); Anneliese Zimmermann of Montreal, Quebec (2,4,5,6,7,8,9); Bill Watson of Macon, Georgia (2,4,6,7).

A Cross Number Puzzle (Page 40 of the June Issue of RMM)

The answer is shown in the diagram to the right. Correctly answered by: Anneliese Zimmermann of Montreal, Quebec; Don Singleton of Pasadena, Texas; George Propper of New York, New York; John Lewis of Los Angeles, California; Alan Gold of Downsview, Ontario; Jeanette Bickley of St. Louis, Missouri.

| | | | | | |
|---|---|---|---|---|---|
| ■ | [1] 1 | [2] 3 | ■ | [3] 5 | [4] 5 |
| [5] 1 | 9 | 4 | [6] 3 | ■ | 6 |
| [7] 7 | 5 | ■ | 2 | ■ | ■ |
| ■ | ■ | [8] 6 | ■ | [9] 2 | [10] 6 |
| [11] 3 | ■ | [12] 4 | [13] 2 | 2 | 5 |
| [14] 9 | 6 | ■ | [15] 1 | 5 | ■ |

## *Book Review* *by J. A. H. Hunter*

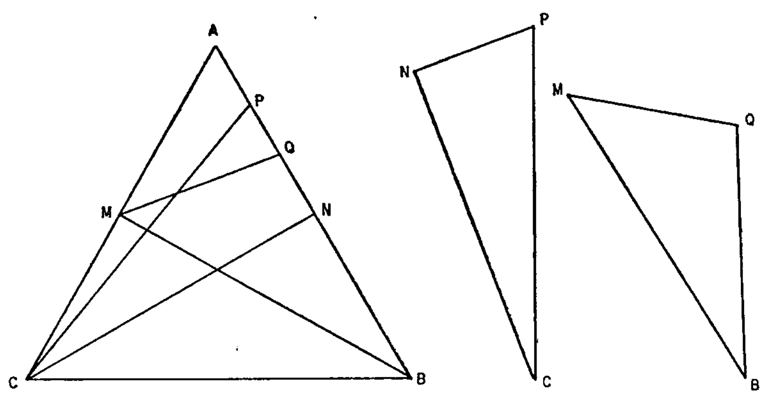
INTRODUCTION TO GEOMETRY
by H. S. M. Coxeter, F.R.S.
Published by John Wiley & Sons, Inc.; $9.95.

The title of H. S. M. Coxeter's new book - INTRODUCTION TO GEOMETRY - hardly suggests my own personal reaction immediately after a first quick skim-through of its more than 400 pages. Moments later I was in my kitchen stabbing madly with a knife at at lot of dry rice in a small jar! Figures never lie, and the truth of some very abstruse theorizing was amply demonstrated by this homely little experiment.

This is an amazing and quite fascinating tome. As a university text-book it is clearly one of the more important works of this decade. It covers an unusually wide field starting with Euclidian geometry, and thence by easy stages through affine and projective geometries to topology and four-dimensional concepts. And throughout its four Parts, Coxeter has contrived to leaven much very solid mathematical "meat" with frequent touches of lightness and humor.

But there is also much to intrigue the ordinary dabbler who is interested only in the recreational aspects of mathematics. This was to be expected, of course, from Professor H. S. M. Coxeter. In the first rank of mathematicians today, he is also known for his work in revising and bringing up to date Ball's popular *Mathematical Recreations and Essays.*

There has been some discussion in RMM of the famous Theorem of Lehmus - "If the internal bisectors of a triangle are equal, then the triangle is isosceles." Many proofs of this have been evolved, some of them valid, and one running to 40 pages! But Coxeter poses this as an exercise for the student, throwing out a hint which leads to his own delightfully simple and brief proof which is well within the scope of mathematically inclined High School students!




If the base angles are not equal, say $\angle ABC < \angle ACB$. Then, the equal angle bisectors being BM and CN, there is a point P on AN such that $\angle PCN = \frac{1}{2}\angle ABC$, and a point Q on PN such that BQ=CP.





For clarification, the triangles CNP and BMQ are shown separately and re-orientated. Since $\angle PCN = \angle QBM$, and CN=BM, and BQ=CP, these two triangles are congruent. Hence, $\angle NPC = \angle MQB$.

But this is already one step beyond the given hint, and many RMM readers will prefer to complete the proof on their own.

Did someone ask about trisecting an angle? Coxeter touches on this classic problem too, and even gives an outline of a simple proof of its impossibility. And in the same section we find a neat little theorem in elementary Number Theory that will amuse some of the Number addicts amongst our RMM readers.

This is an introduction to geometry in its widest sense, and it includes consideration of many things which might seen far removed from the field of mathematics. There is the quick method of forming a regular pentagon merely by folding a strip of paper. That has an obvious mathematical implication. But what about the luscious pineapple and its manifest connection with the pentagon and with the Fibonacci series?

We see how phyllotaxis - meaning "leaf arrangement" literally - follows precise mathematical laws, and that these also apply to such diverse arrangements as the seeds of a sunflower, the scales of a fir cone, and the external cells of our pineapple. And in the vegetable world, as also in the fields of architecture and design, that same Fibonacci series turns up again and again.

What vision of curved space did Shakespeare have, two full centuries before its "discovery" by Bolyai and others? Was he peering beyond the confines of Euclidean geometry when he made Hamlet say "I could be bounded in a nutshell and count myself a king of infinite space"? Two centuries later the first formal work was started on the concepts of hyperbolic geometry, and here Coxeter outlines an almost simple proof that the area of a triangle remains finite when all its sides are infinite!

But these have been only a few of the many diverting items which make this book so intriguing even for those who look only for "fun" in mathematics.

It is indeed a most notable production.

# Readers' Research Department

Several analyses of the color problem were received, but the Editor would like to see a few more. Those submitted did not agree and it would be interesting to see the analyses of others. The problem was: What is the least number of colors with which one can color a plane in such a way that no pair of points unit distance apart are colored the same It was shown that seven colors are at least sufficient - but what is the least number?

Thomas S. Briggs of San Francisco, California has submitted an analysis of the Dots and Squares game in the April issue of RMM. Commentaries, or extensions of the basic ideas involved, are welcomed by Mr. Briggs and the Editor.

An Analysis of "Square It"

The game of Square It, having players play alternately except that a player completing a square must take another stroke, is subject to the following analysis:

Let S = number of squares in the Square It array and P = the perimeter of the array in strokes. At one point in the game it becomes impossible to take another stroke without creating a region in which the opponent can complete squares. Let us call this point the "saturation" point. At saturation $M_p$ = number of strokes on the perimeter of the array, $M_i$ = number of strokes in the interior of the array, M = total number of strokes and r = maximum number of regions which can be squared in one turn after saturation. For the saturated array in figure 1 we have $S = 9$, $P = 12$, $M_p = 4$, $M_i = 7$, $r = 4$.

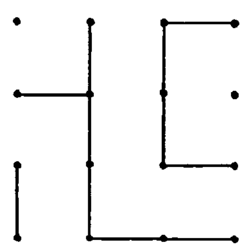


Figure 1.

Now, every region at saturation which is not closed must have at least two strokes missing or it would be "squareable". If we consider all arrays having unclosed regions with the two strokes missing at the perimeter, $M_p = P - 2r$.

Each square of these arrays must be bordered by 2 strokes if the array is to be saturated. Since each stroke in the interior borders 2 squares, we have:

$$S = (M_p + 2M_i)/2$$
$$M = M_p + M_i$$





By substituting we find:

$$M = (P - 2r) + (2S - M_p)/2$$
$$M = (P - 2r) + (2S - P + 2r)/2$$
$$M = (P + 2S - 2r)/2$$

For T = total number of turns in a complete game, notice that one move is needed after saturation to make a region squareable. Thereafter, each region corresponds to a move. The total number of moves will then be:

$$T = M + r + 1$$
$$T = (P + 2S + 2)/2$$

This formula will hold for any array of squares regardless of shape or size. However, in using the formula to determine a winning strategy it is necessary to also account for all regions which close on themselves. These regions have the effect of adding an extra turn to the total. Let $r_c$ = number of closed regions. Then for arrays having open regions open at the perimeter:

$$M_p = P - 2(r - r_c)$$
$$M = [P + 2S - 2(r - r_c)]/2$$
$$T = \frac{P + 2S + 2}{2} + r_c$$

In a 3x3 array we can have no more than one closed region at saturation. (Figure 2)

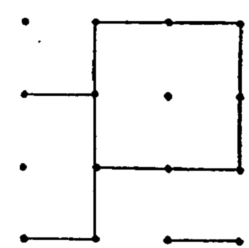


Figure 2.

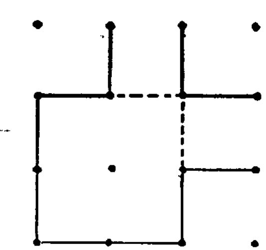


Figure 3.

Thus, T = 16 for a 3x3 array with no closed regions and 17 for one closed region.

Regions which open in the interior of an array are another complication this game can have. Figure 3 shows how this can occur in a 3x3 array. Each such opening leaves a cell with one less side at saturation. Also, there is one less opening at the perimeter per internal opening. Let i = number of internal openings:

$$M_p = P - 2(r - r_c) + i$$
$$M_i = (2C - i - M_p)/2$$

In $M = M_p + M_i$ the i subtracted out; thus, internal openings have no effect on T. Internal openings are considered boundaries between two regions.

Sacrificing squares before saturation also does not affect T. Although a sacrifice creates an extra move, it also eliminates a region.

To determine the strategy needed to force a win we must assume that errorless playing will leave the deciding region to the last person to play. If this is the case, the first player wins when T is odd and the second player wins when T is even. The first player will win the games in figures 2 and 3 and the second player will win the first game. The only way the deciding region cannot be played last is when there is a small closed region joined by a long open region as in figure 4:

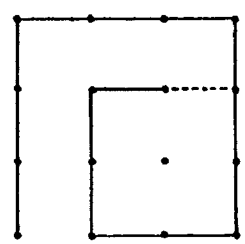


Figure 4.

Thus, although the first player plays last in this game (T= 17), the second player will win.

Now let us assume both players are aware of this analysis. Since it is not difficult to keep the opponent from changing T by closing a region, the size of the array will have the greatest bearing on who will win. For square arrays with side s:

$$T = \frac{4s + 2s^2 + 2}{2} + r_c$$

$$T = s^2 + 2s + 1 + r_c$$

$$T = (s + 1)^2 + r_c$$

When s is odd the second player wins and when s is even the first player wins unless, because of an error in tactics before saturation, $r_c$ is greater than zero and odd.

To summarize for the 3x3 or odd x odd arrays, the second player will win the game unless the first player can close a region. The second player can nullify a closed region by closing another region or extending an open region from the closed region as in figure 4.

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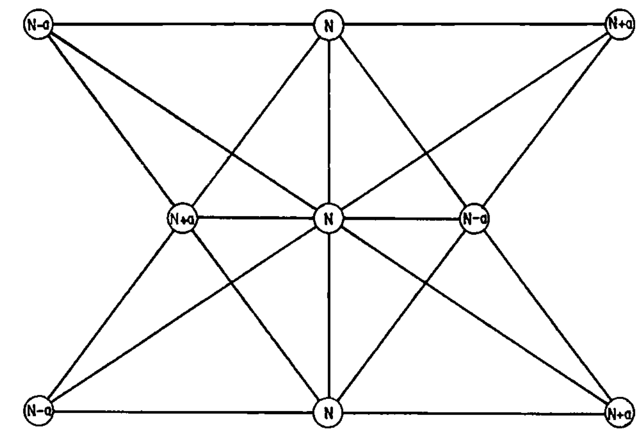
The Research Problems for this issue are short — but they may not be as easy to solve they are to state.

(1) Maxey Brooke of Sweeny, Texas states that one can form





an array as shown in the figure below so that three integers, in arithmetic order, can be arranged to equal the same along each of the indicated lines. The problem is: can a solution be found using more than three integers?




(2) The Editor has played around with this particular research problem without coming to any solution. Given a regular pentagon and any point on one of the sides: Construct a straight line through this point which bisects the area of the pentagon.

There are two points that must be cleared up first. A line through a vertex can be constructed perpendicular to the opposite side and this line will bisect the pentagon. Also a line through the mid-point of a side through the opposite vertex bisects the pentagon (actually the same case as before). The problem requires constructing the bisecting line through *any* point on a side. A line through any point throught the center of the pentagon does *not* bisect the area — except as just noted.

The problem could be generalized to include the bisection, through a point on a side, of any regular polygon with an odd number of sides.

## *Numbers, Numbers, Numbers*

### The 18 Perfect Numbers

The April and June issues of RMM featured notes about Perfect numbers in the *Numbers, Numbers, Numbers* department. The demand for the full values of all 18 Perfect numbers has forced the editor (who was willing, anyway) to print them here.

The task of proofreading this mass of digits was taken on voluntarily by Alan L. Brown (East Orange, New Jersey), Rudolf Ondrejka (Atlantic City, New Jersey), Adrian Struyk (Paterson, New Jersey) and Dmitri Thoro (San Jose State College, California).

The following Perfect numbers are of the form $2^{p-1}(2^p-1)$ where p *and* $2^p-1$ are prime. The values for p for the 18 numbers listed are, respectively, 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217.

$V_1 =$ 6

$V_2 =$ 28

$V_3 =$ 496

$V_4 =$ 8128

$V_5 =$ 335 50336

$V_6 =$ 85898 69056

$V_7 =$ 13 74386 91328

$V_8 =$ 2305 84300 81399 52128

$V_9 =$ 26 58455 99156 98317 44654 69261 59538 42176

$V_{10} =$ 1915
61942 60823 61072 94793 37808 43036 38130 99732 15481 69216

$V_{11} =$ 13164 03645 85696
48337 23975 34604 58722 91022 34723 18386 94311 77837 28128

$V_{12} =$ 14 47401 11546 64524 42794 63731
26085 98848 15736 77491 47483 58890 66354 34913 11991 52128

$V_{13} =$ 2356 27234 57267
34706 57895 48996 70990 49884 77547 85839 26007 10143 02759
75063 37283 17862 22397 30365 53960 26005 61360 25556 64625
03270 17505 28925 78043 21554 33824 98428 77715 24270 10394
49691 86640 28644 53412 80338 31439 79023 68386 24033 17143
59223 56643 21970 31017 20713 16352 74872 98747 40064 78019
39587 16593 64010 87419 37564 90579 18549 49216 05556 46976

$V_{14} =$ 1 41053 78370 67120
69063 20795 80860 63189 88148 67435 14715 66783 88386 75999
95486 77426 52380 11410 41933 29037 69025 15619 50568 70982
93271 64087 72436 63700 87116 73126 81593 13652 48745 06524
39805 87729 62072 97446 72329 51666 58228 84692 68077 86652
87018 89208 67879 45147 83645 69313 92206 03706 95064 73607
35723 78695 17647 30552 66826 25328 48863 83715 07297 43244
63835 30005 31384 29460 29657 51433 68065 57075 95373 28128





$V_{15} =$ 54162 52628 43658 47412
65446 53743 91316 14085 64905 39031 69578 46039 20818 38720
69941 58534 85919 89999 21056 71992 19190 57390 08026 36461
59280 01382 76054 39746 26278 89030 57303 44550 58270 28395
13947 52077 69044 92443 14948 61729 43511 31262 80837 90493
04627 40681 71796 04658 67348 72099 25721 90569 46554 52996
29919 82343 10310 92624 24446 35477 89635 44148 13917 19816
44160 55867 88092 14788 66773 21398 75666 16247 14551 72696

43022 17554 28178 42548 17319 61195 16598 55553 57393 77889
23405 14622 23245 06715 97919 37573 72820 86087 82143 22052
22758 45375 52897 47625 61793 95176 62442 63144 80313 44693
50852 03657 58479 82475 36021 17288 04037 83048 60287 36212
59313 78999 49003 36673 94150 37472 24966 98402 82408 06042
10869 00776 70395 25923 18946 66273 61521 27756 03535 76470
79522 50173 85830 51710 28603 02123 48966 47851 36394 99289
04973 29214 51075 05979 91145 62215 19899 34576 49842 91328

$V_{16} =$ 10 89258 35505 78293 37698 22527
35220 48981 95710 84543 02608 06731 89066 18508 47015 52986
16996 29194 09618 58901 37954 61826 85531 22005 57627 80759
34240 74990 66046 70418 20830 87124 62692 63781 64410 93145
09688 26355 20557 36716 71624 20268 66333 60807 12310 94704
52668 37153 75996 62797 48493 43590 39779 95421 36665 98820
29950 13663 80164 61908 02604 03235 22955 67305 54163 99230
30097 52651 35032 06199 30563 67369 52801 53023 04949 84686
96618 14407 20213 72831 42596 37014 60505 60637 81192 45841

38655 26001 45384 07298 33097 17141 95008 54980 85709 67138
70548 68320 47797 22990 55273 91479 84469 36214 14786 07068
87052 10731 23800 67072 60231 70094 22809 31477 47918 94700
76989 10098 18743 16930 30281 54303 29007 11993 92984 29294
02838 52217 80016 66292 29157 11026 40805 99294 01645 24830
28528 15333 11195 23441 42315 96149 34140 26555 02423 60007
85821 59367 98489 50072 71963 47516 38604 42417 21984 70655
83293 64277 99590 31022 92034 62062 80807 52342 42290 64012
83027 03464 96714 45569 32428 19468 59622 17756 66433 75489

71567 84513 11792 67593 59810 10355 56288 79719 48569 01606
00353 34607 87935 97703 71846 50765 99706 01616 99831 19838
78150 42076 33062 89490 88642 99004 81786 49953 76453 79839
36521 27254 94441 51193 27721 82768 14994 36598 49007 45724
69838 61558 26514 48231 91367 75835 03415 27780 77022 15569
45275 56650 48316 36564 85683 15025 56078 05813 30434 00055
65354 04133 13266 03463 93552 02834 00612 69054 91569 56054
24895 51023 20738 22761 37352 66571 70182 61519 60481 74171
12576 52641 05353 23991 50005 87499 96247 58083 44537 82528

$V_{17} =$ 994 97054 33708 64734 42435

20260 45228 16989 64386 35711 26408 51177 40205 75773 84932
63555 29178 68662 94981 51336 41650 25166 45641 69951 68131
40394 89794 06365 61646 54594 77532 32301 45360 35832 23268

08561 36472 33768 08164 57276 69037 39438 56965 22820 30153
58880 41815 55951 34080 36145 12387 05843 25525 81395 04871
09647 77074 38273 62571 82287 05676 43040 18472 31158 25645
59038 63133 77067 11263 81492 53171 84391 47800 65137 37344

62224 06322 95356 91247 71480 10136 31809 66448 09988 22924
53452 39542 82708 75732 53631 15392 66115 11649 07049 40164
19241 77449 19250 00089 47274 07937 22982 93005 78253 42788
44943 58459 94953 52318 19781 36144 96497 79252 94809 99098

21642 20748 55148 05768 28811 55834 09148 96987 57905 23961
87875 31249 72681 17994 42346 41016 96001 18157 88847 43661
01927 04551 63703 44725 52319 82033 65320 14561 41202 88204
92176 94041 83770 74274 38914 99243 03484 94544 61051 21267

53806 15832 99291 70797 23788 07395 01603 07654 40655 60175
91093 70564 52264 79891 56121 80427 30122 66011 78345 11022
30081 38040 19513 83582 98714 95782 29940 81818 15140 46314
81931 32063 21375 97333 67850 23565 44310 13056 33127 61023

05495 88655 60595 13323 51485 64175 75426 11227 10807 32638
89434 40959 59768 35137 41218 70253 49639 50440 40616 54653
75534 91626 80629 29055 16441 53382 76068 18622 94677 41498
90474 91922 79570 72109 20437 81113 67127 94483 49643 73559

80833 46332 95928 38140 15780 31820 55197 82170 27392 06310
97100 62603 83262 54290 00440 72533 19613 77965 52746 43905
17609 40430 08237 56411 50129 81796 01830 28081 01097 87809
02441 73368 09777 14813 54343 87525 46136 37567 51399 15776





$V_{18} =$ 33 57083 21319 86724 43701 08772 11080 38484
11380 28499 87972 54549 96241 57348 21584 50444 04288 20487
78809 43769 03884 49535 77426 08498 85573 69475 99061 73841

15743 84247 30130 80704 76236 55942 23617 48505 09108 53782
76585 90642 32548 24947 61473 19657 90746 56099 91860 07644
04702 18166 02944 69121 77873 79658 22199 90166 34780 93006

07502 23592 23201 84998 56361 44177 18592 54020 78185 07301
50450 97727 08485 94647 43635 53778 15002 84915 88024 48863
06461 78598 29560 72060 01347 49556 17851 48168 01859 88557

13660 92248 41817 87708 36089 51191 12317 48852 26416 13068
31977 10667 39235 10073 74503 75540 33525 31476 22794 35900
71651 70269 75942 41031 95552 98989 71218 00121 46417 74673

13494 44715 62560 95717 96578 81556 41912 21029 35450 29975
18133 40515 17095 61679 51095 45364 94855 76150 66010 16891
60658 01177 01932 74226 30828 05077 86835 04954 91125 76654

51011 96704 56745 93989 01942 05255 17538 44844 89909 32896
76469 88163 15598 24715 64998 19626 16327 51283 12787 95091
98074 25319 34095 80454 56248 86643 83465 37988 50027 35506

15398 88515 06645 13775 92755 53988 21942 54397 64732 39982
47124 38125 05411 75238 37438 25674 44370 55019 44105 10064
89972 34160 91179 78404 56379 49920 04873 05751 84557 48701

44495 12383 77139 62049 42879 82489 52982 72331 40637 01483
74088 56156 19951 54576 69607 96405 21269 08149 26560 17860
94447 59556 04400 59050 09176 35471 14092 25537 13974 25807

86755 43521 12542 19478 48154 94784 27620 11708 45949 27467
46329 85210 42107 55317 84918 35892 66903 95463 64972 14522
65405 71348 43880 43911 63448 54323 58638 80664 53138 26206

59113 12662 32422 00783 55773 45584 22572 03105 18698 14337
67362 19283 02111 92876 17896 14688 55848 60065 04887 63157
01088 79621 95936 40826 31162 22733 28035 60330 94756 42390

80449 94601 56797 85536 10182 46696 10125 39222 54567 24090
83153 85468 24093 18461 66962 49598 34076 07141 60125 18895
44407 00881 58747 44654 76950 72686 78051 75774 69568 91212

48545 62611 21386 66740 77111 39619 07153 09233 55823 17866
27053 74393 03504 90226 03882 47974 23347 99407 13028 01487
69298 59774 37781 93050 34874 97407 86928 09603 39062 95910

19923 81813 38557 85697 81918 60647 25620 97081 68229 11615
63009 78059 19702 68557 26877 64976 70726 84960 46345 27631
60384 09383 82922 77544 91185 78596 58328 88833 26285 25056

Here are the answers to the two problems posed by H. V. Gosling on page 56 of the June RMM.

a. An integer solution to $a^4 + b^4 + c^4 + d^4 = e^4$

$$30^4 + 120^4 + 272^4 + 315^4 = 353^4$$

b. $a^6 + b^6 + c^6 = d^6 + e^6 + f^6$

$$3^6 + 19^6 + 22^6 = 10^6 + 15^6 + 23^6$$

J. A. H. Hunter of Toronto, Ontario and Dmitri Thoro of San Jose State College, California both submitted the correct answers.

Mr. Hunter also submits a general solution to the particular equation given by H. V. Gosling on page 56 of the June issue.

$$(5/8)^2 + 3/8 = (3/8)^2 + 5/8$$

Can be generalized as:

$$(a/b)^2 + c/b = (c/b)^2 + a/b \quad \text{where } b = a + c$$

And some more by Mr. Hunter:

$$5/7 - (5/7)^3 = (8/7)^3 - 8/7$$
$$7/13 - (7/13)^3 = (15/13)^3 - 15/13$$
$$13/43 - (13/43)^3 = (48/43)^3 - 48/43$$

And:

$$(5/9)^2 + (10/9)^2 = (10/9)^3 + (5/9)^3$$
$$(5/14)^2 + (15/14)^2 = (15/14)^3 + (5/14)^3$$
$$(17/65)^2 + (68/65)^2 = (68/65)^3 + (17/65)^3$$
$$(13/63)^2 + (65/63)^2 = (65/63)^3 + (13/63)^3$$

etc.

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Malcolm H. Tallman of Brooklyn, New York gives us the following miscellanea.

*One-Arm Mathematics.* A pleasant pastime calling for only light mental calisthenics is the compiling of multidigital numbers: integers that are multiples of the sums of their digits.

For example, let's take 29 as the common factor of a group of composite numbers whose digits total 29 also. 4988 is the product of 29 and 172 while 4 + 9 + 8 + 8 = 29. Here is a short list of other numbers fulfilling this characteristic:

| | | | | |
|---|---|---|---|---|
| 4988 | 17777 | 46748 | 66845 | 84854 |
| 7598 | 29738 | 55883 | 69716 | 87464 |
| 7859 | 37874 | 58754 | 79373 | 92945 |
| 9686 | 43877 | 63974 | 81983 | 95816 |

The method is not given but should be found by the reader. When the mystery is solved a list as above can be compiled easily within an hour.





*Mnenomic for $\pi$.* The number of letters in each word gives the respective digit value.

*Let a book a month instigate an urgent quest for newer material including science novelties — all to get superior math output in school.*

$$\pi = 3.14159265358979323846 26$$

*Fibonacci Pythagorean Triangles.* The Fibonacci series of numbers is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233 and so on — each term being the sum of the preceding two terms. Let a, b, c and d represent four consecutive Fibonacci numbers. Then $c = a + b$ and $d = b + c = a + 2b$. To find integer values for the legs and hypotenuse, x, y and z, of a Pythagorean triangle the following relationship can be used:

$$x = 2ab + a^2 \qquad y = 2ab + 2b^2 \qquad z = a^2 + 2ab + 2b^2$$

where a and b are integers.

By rewriting and using the Fibonacci relationships above:

$$(a^2 + 2ab + 2b^2)^2 = (2ab + a^2)^2 + (2ab + 2b^2)^2$$
$$(a^2 + 3ab + 2b^2 - ab)^2 = a^2(2b + a)^2 + (2b)^2(a + b)^2$$
$$[(a + b)(2b + a) - (ab)]^2 = a^2(2b + a)^2 + (2b)^2(a + b)^2$$
$$(cd - ab)^2 = (ad)^2 + (2bc)^2$$

Or, in any four consecutive Fibonacci numbers a, b, c and d:

ad and 2bc are the legs of a Pythagorean triangle and cd — ab is the hypotenuse.

For example, take the terms 13, 21, 34, 55. ad = 715; 2bc = 1428; cd — ab = 1597. And, indeed $715^2 + 1428^2 = 1597^2$.

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## *Letters to the Editor*

Dear Mr. Madachy:

Maxey Brooke is substantially in error in thinking that Sherlock Holmes ever wrote a treatise on the Binomial Theorem. It was, in fact, his mortal enemy, that arch-criminal Professor Moriarty "the celebrated author of The Dynamics of an Asteriod - a book which ascends to such rarefied heights of pure mathematics that there was no man in the scientific press capable of criticizing it" (The Valley of Fear) who "at the age of twenty-one wrote a treatise upon the Binomial Theorem, which has had a European vogue." (The Final Problem - the last story in the Memoirs.)

Derrick Murdoch
Willowdale, Ontario

Dear Mr. Madachy:

Does anyone know of a formula that tells the number of digits in a given Fibonacci number? E.g. how many digits are in the 502nd number of the sequence.

We found that within four or five numbers of the sequence, the number of digits must change. It is evident that within five numbers we must change one digit; within ten numbers we change two digits. But we suspect that there must be a change of five digits within 24 numbers (not 25) and that within 67 numbers (not 68) there must be a change of 14 digits.

Gilbert W. Kessler
Stephen Raucher
Brooklyn, New York

The editor would be interested in this formula, too. If readers with helpful hints send them to the Editor (address on Contents page) he will forward them to Messrs. Kessler and Raucher. To keep the Fibonacci number reference consistent, use the series starting 1, 1, 2, 3, 5, 8, 13, . . .

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Dear Mr. Madachy:

As for that Moonshine problem given in the April issue and answered in the June issue here's my solution which does not involve tipping containers:

Fill the 13 quart container from the 24 quart container, leaving 11 quarts. Fill the 10 qt. container from the 13 qt. container and pour the remaining 3 qts. into the 11 qt. container. Now pour the 10 qts. from the 10 qt. container into the 13 qt. container and fill the remaining portion from the 11 qts. still remaining in the 24 qt. container. The 24 qt. container now holds 8 qts.

Pour the 3 qts. left in the 11 qt. container into the 10 qt. container and place the 10 qt. container in the 11 qt. container. Fill the 11 qt. container from the 13 qt. container - with the 10 qt. container (containing 3 qts of moonshine) floating in it, only 8 qts can be put into the 11 qt. container. The 3 qts. from the 10 qt. container can now be added to the 5 remaining qts. in the 13 qt. container - making up the last 8 qts.

Of course I'm assuming negligible weight for the 10 qt. container.

Lloyd Jim Steiger
Vallejo, California

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Dear Sir:

Brother Alfred, in his stimulating "Fun, Counting By Sevens" (June RMM, page 11) points out that $88^2$ (base 10) $=7744$, while $55^2$ (base 7) $=4444$. It is interesting to note that in any system to an integral base n, $(mn+m)^2$, where $m=n-2$, can be written as $(n-3)n^3+(n-3)n^2+4n+4$.

Let $mn+m$ represent a two-digit number to any integral base n greater than 1, such that $m=n-2$. Then

$$\begin{aligned}(mn+m)^2 &= m^2(n^2+2n+1)=(n-2)^2(n^2+2n+1)\\ &= n^4-2n^3-3n^2+4n+4=(n^4-3n^3+n^3-3n^2)+4n+4\\ &= (n-3)n^3+(n-3)n^2+4n+4\end{aligned}$$

So if we restrict ourselves to n greater than 4, the consecutive digits of the product desired are $(n-3)$, $(n-3)$, 4, 4. For example, by this formula:

$77^2$ (base 9) $=6644$.

Donald K. Bissonnette
Florida State University

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Dear Mr. Madachy:

In respect to my Prime Generating Polynomial note (June RMM, page 50) I would like to add that $x^2-x+41$ is never a perfect square except at $x=41$. I conjecture that it is never a perfect cube, fourth power, fifth power, etc., up to 12th power inclusive.

Some one may be able to prove this for me.

Sidney Kravitz
Dover, New Jersey

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Dear Mr. Madachy:

The claim by Mr. Kravitz in his article "Prime Generating Polynomials" (June RMM, page 50) that "No polynomial can represent primes exclusively." is punctured by the following which was written by me in 1935 and published in the March 1952 SCRIPTA MATHEMATICA.

Consider N satisfying the following conditions:

(1) N is of the form $\dfrac{P_n}{a_i a_k \ldots a_l} \pm a_i a_k \ldots a_l$ where $P_n$ is the product of the first n primes, including 1.

(2) N is less than the square of the $(n+1)$th prime. Then *every such N is a prime number.*

For example: Consider the first 4 primes 1, 2, 3, 5. We can form the following prime numbers:

$$(2)(3)(5)\pm 1=31 \text{ or } 29$$
$$(1)(3)(5)\pm 2=17 \text{ or } 13 \text{ etc.}$$

*Proof*: It is evident that no N is divisible by 2, 3, or 5 . Hence the smallest possible divisor is the next higher prime, or 7 in this case. Hence if $N<7^2$ it must be a prime.

It is evident that powers of the above factors apply also:

$(5)(3)(1) + 2^5 = 47 \qquad (1)(5)(2^3) + 3 = 43$

$(5)(2)(1) + 3^3 = 37 \qquad (5)(3)(1) + 2^3 = 23$

Malcolm H. Tallman
Brooklyn, New York

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Dear Mr. Madachy:

I am wondering if some RMM readers would know of any references to the problem of coloring the edges (not the faces) of all the regular uniform polyhedra, using three colors. E.g. all the faces of an icosahedron are triangles and every triangle must have one side red, one blue and one yellow. I have solved them all - have others done so, too?

Leigh Mercer
London, England

If RMM readers who can supply this information will forward such to the Editor at Box 1876, Idaho Falls, Idaho, he will see to it that Mr. Mercer receives the letters.

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Dear Mr. Madachy:

Does anyone know of a proof that the continued addition of a number to its reverse form will eventually yield a palindromic number? It always does, many times, but what's the proof? An example is given to the right with the resulting palindromic numbers in red type. One may start with any number and one can continue the given example. The next palindromic number reached will be 166876727678661.

H. V. Gosling
Kingston, Ontario

| |
|---:|
| 127 |
| 721 |
| 848 |
| 848 |
| 1696 |
| 6961 |
| 8657 |
| 7568 |
| 16225 |
| 52261 |
| 68486 |