RECREATIONAL MATHEMATICS magazine

ISSUE NO. 3 JUNE 1961

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NUMBER SYSTEMS - - - A series of articles by F. Emerson Andrews, J. A. H. Hunter, Charles W. Trigg and Brother Alfred

THE HAUNTED CHECKERBOARDS by Maxey Brooke

THE MATHEMATICS OF MUSIC by Ali R. Amir-Moéz

And Other Articles, Brainteasers, Problems, Number Phenomena, Word Games, Etc.

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RECREATIONAL MATHEMATICS magazine

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PUBLISHED AND EDITED BY JOSEPH S. MADACHY

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From the Editor

There is one bit of business the editor is obliged to take care of immediately.

RMM is a BIMONTHLY.

The implication that RMM is a monthly was perpetrated by no less than RMM itself! The Table of Contents page of the April issue indicated that RMM was published monthly - this is an error. *RMM is a Bimonthly - once every two months - six times a year.*

* * * *

The demand for the February (#1) Issue of RMM has long outstripped the supply. If there is enough demand the editor will undertake to have that issue reprinted *and* revised (to protect the collector-item value of the original.) The sale price of the revised reprint will have to be a full $0.65. If you are interested drop a postcard in the mail box directed to the Editor, Recreational Mathematics Magazine, Box 1876, Idaho Falls, Idaho. Acknowledgement of all these requests cannot be guaranteed *but* they will be filed away and when the total number of requests reaches a figure demanding reprinting, a notice will be placed in an issue of RMM.

* * * *

The editor would like to have the readers send in their answers to the various puzzles and problems posed in RMM. All letters are answered or acknowledged as soon as possible and, of course, all who submit correct answers may have their names listed in the answer section when a problem is held over from one issue to the next.

* * * *

Now let's see what's coming for the future: Brother Alfred will lead us into the World of Large Numbers in the August issue. Numbers do not just get bigger - they often get more interesting! Sidney Kravitz, who told us how to solve Alphametics in the April issue, will enlighten all of us with some mathematical comparisons of the Christian, Mohammedan and Jewish Calendars. The editor will do his best to publish the full, correct values of the eighteen Perfect Numbers.

1 June 1961 J.S.M.

Editor's Note: *It is indeed fortunate that RMM readers have an opportunity to read four articles by well-known writers - each presenting some aspect of number systems. J. A. H. Hunter introduces the series with comments about number systems in general; F. Emerson Andrews gives us a chapter from his forthcoming book,* Numbers, Please, *on the dozen system of counting; Brother Alfred, of St. Mary's College, throws some light on the interesting Base-7; and C. W. Trigg, of Los Angeles City College, goes a bit deeper into certain properties of Base-7 numbers. We know you'll enjoy the following bit of pure recreational mathematics!*

Number Systems for Fun

J. A. H. Hunter

There's endless fun in the study of number systems, and perhaps more so in the comparison of different systems. But, before enlarging on this theme, it may be as well to mention two very vital points which are sometimes overlooked.

A number retains its particular identity and meaning, irrespective of how we name it or denote it in symbols. For example, what we call "eleven" and denote in our base-10 system as 11, might be called "kinkob kra" and denoted as 15 in Kalota where they use a base-6 system.

The other point is that our decimal-system words, and endings such as "teen", cannot be used meaningfully when describing numbers written in the notations of other systems. We write the actual number "seventeen" as 17, and we call it "seventeen". In a base-15 system we might denote that same quantity as 12, but it would be very wrong for us to describe what we had written as being "twelve": we should describe it as "one two" or even as "one two in base fifteen".

These two points may seem trivial, but no harm will have been done by emphasizing them. And now let's look into some of the leads this whole field provides for the "fun" I promised.

Say we have a number, written in a base-m system as abc - a, b, and c being digits written in that order. Then, the actual number so represented has the value

$$am^2 + bm + c$$

From this it will be seen that numbers denoted in such ways as 100 (one zero zero), 121 (one two one), 144, 169, 196, etc., are always perfect squares irrespective of the number system used.

Similarly, numbers denoted in such ways as 1000, 1331, 1728, etc. are always perfect cubes in all number systems.

Conversion from one system to another raises many very interesting points. Say we have a number, written in a base-m system as ab_m, and the same actual number written in a base-n system as ba_n, the appropriate base being indicated for clarity by the subscript. Then we have, using normal algebraical forms,

$$am + b = bn + a$$

the general solution of which is:

$$\left.\begin{array}{l} m = kb + 1 \\ n = ka + 1 \end{array}\right| \text{ k being any suitable constant.}$$

For example, $35_{11} = 53_7$

This gives us such curiosities as:

$$12_7 = 21_4$$
$$23_{10} = 32_7$$
$$34_{13} = 43_{10}$$
$$45_{16} = 54_{13}$$

Other amusing relations appear when we consider selected types of squares. For example,

$$(44_6)^2 + (45_6)^2 = (54_{13})^2 - (44_{13})^2$$

The ancient Babylonians used what was essentially a base-60 system! Not very practical for calculating, when you remember that in terms of more enlightened arithmetical operation it would entail not less than 59 different digit symbols as well as the necessary zero! But, if we accept the theoretical possibility of systems based on very large numbers, the following are examples of intriguing comparisons:

$$123_{20} = 321_{17}$$
$$234_{62} = 432_{44}$$
$$345_{107} = 543_{83}$$
$$456_{164} = 654_{134}$$
$$567_{233} = 765_{197}$$
$$678_{314} = 876_{272}$$
$$789_{407} = 987_{359}$$

And, on similar lines,

$$135_{17} = 531_8$$
$$246_{29} = 642_{17}$$
$$357_{44} = 753_{29}$$
$$468_{62} = 864_{44}$$
$$579_{83} = 975_{62}$$

Also,

$$(12 \times 34)_{259} = (43 \times 21)_{159}$$
$$(23 \times 45)_{1143} = (54 \times 32)_{835}$$

etc.

Well, these brief comments may have given some hints as to the possibilities for "fun" in this field. And, in the articles that follow, you will find many more.

Counting By Dozens*

By F. Emerson Andrews

In baseball, a base is a safe place one strives to reach, and from which one tries to advance; it is a "stepping off place." In mathematics the base is "the stepping off place" from which the whole system is built up. Because fingers were used for practically all early counting, we count up to ten, and then begin over again. Ten is our stepping off place; we have a base-10 number system. It is so universal that it is actually one of the few things in this world that has seldom been questioned.

However, the ten system is not the only one tried out by primitive peoples. Some tribes used two - the pair. Their counting ran like this: one, a pair; a pair and one, pair and pair; and probably not much farther.

Twenty - all the fingers and all the toes - was an obvious base in warm climates where people did not wear shoes or moccasins. The Mayas in Yucatan had a highly developed counting system using twenties. Fragments of it exist in our own language in words like score, meaning 20.

But because a man always had his ten fingers and thumbs literally "on hand" as a counting machine, counting by tens became the accepted way. True, it did not work out well for some special purposes, and for these we have adopted other bases. For time, we use 12 for the half-day and the number of months; five times twelve (60) for seconds and minutes; and thirty times twelve (360) for degrees in the circle.

In recent years we have invented gigantic calculating machines which, because they run on electric current which is either positive or negative, are actually geared for counting by 2's. They have reminded us again that 10 is not the only possible base for a system of arithmetic; in fact, it is a rather poor one.

Probably the most practical base we could have selected, if we had thought of it in time, is 12. Indeed, 12 is such a handy number that we use it in many of our measures and other practical matters even now. We divide the foot into 12 inches; as already noted, the day into two 12-hour parts and the year into 12 months; we sell many things by the dozen because they pack so well that way. Indeed, we learn 12 multiplication tables, even though our number system is based on 10.

We do this because 12 has certain advantages which no other low number possesses to such a degree. Consider its factors. Twelve divides evenly into halves, thirds, quarters, sixths, and twelfths. (Ten

This article has been adapted from the book *Numbers, Please* by Mr. Andrews to be published in July by Little, Brown & Company.

divides evenly only into halves, fifths, and tenths). In addition to mathematical advantages, 12 has a number of practical ones. Consider, for example, in how many different ways 12 eggs can be packed, as compared with 10, or 7, or 13, or any other low number.

It may be interesting to explore 12 as a possible number base.

The Base Twelve

To use 12 as a number base requires inventing two new number symbols. For simplicity let us use Χ for ten, and call it *dek;* let us use Ɛ for eleven, and call it *el.* Here is the number system, with base-12 numbers in bold face.

1 = **1**	13 = **11**	30 = **26**
2 = **2**	14 = **12**	36 = **30**
3 = **3**	15 = **13**	48 = **40**
4 = **4**	16 = **14**	60 = **50**
5 = **5**	17 = **15**	100 = **84**
6 = **6**	18 = **16**	130 = **ΧΧ**
7 = **7**	19 = **17**	143 = **ƐƐ**
8 = **8**	20 = **18**	144 = **100**
9 = **9**	21 = **19**	365 = **265**
10 = **Χ**	22 = **1Χ**	1,000 = **6Ɛ4**
11 = **Ɛ**	23 = **1Ɛ**	1,728 = **1,000**
12 = **10**	24 = **20**	10,000 = **5,954**

The only secret to handling the dozen system (*duodecimal* system is the fancy name) is remembering that the second column to the left represents, not tens, but dozens; the third column, not tens-of-tens, but dozens-of-dozens; and so on.

Take the number of days in the year (365 or **265**) which accidentally looks almost like the regular number:

	5 days =	5
265 days =	6 dozen days =	72
	2 dozen-dozen days =	288
		365

It is even simpler to change ordinary numbers into duodecimals. Simply divide by 12 over and over again, and the remainders are the duodecimal number:

```
12 ) 365              12 ) 1848                12 ) 1961
   12 ) 30+5             12 ) 154+0               12 ) 163+5
         2+6                 12 ) 12+Χ                12 ) 13+7
                                   1+0                      1+1

Answer, 265           Answer, 10Χ0             Answer, 1175
```


Addition involves no new steps by duodecimals. It is simply necessary to remember that we add to a "dozen" before we carry **1**:

36	**300**	**3121**	**3.4** = 3 feet, 4 inches
49	**412**	**4996**	**2.8** = 2 feet, 8 inches
20	**110**	**5Ɛ3Χ**	**5.3** = 5 feet, 3 inches
Χ3	**822**	**11Χ35**	**Ɛ.3** = 11 feet, 3 inches

Multiplication is actually simpler than by the 10-system, because more of the products come out 0, and the duodecimal multiplication tables are easier to learn. So that we can try out a few examples, here is a base-12 multiplication table:

	2	3	4	5	6	7	8	9	Χ	Ɛ	10
2	4	6	8	Χ	10	12	14	16	18	1Χ	20
3	6	9	10	13	16	19	20	23	26	29	30
4	8	10	14	18	20	24	28	30	34	38	40
5	Χ	13	18	21	26	2Ɛ	34	39	42	47	50
6	10	16	20	26	30	36	40	46	50	56	60
7	12	19	24	2Ɛ	36	41	48	53	5Χ	65	70
8	14	20	28	34	40	48	54	60	68	74	80
9	16	23	30	39	46	53	60	69	76	83	90
Χ	18	26	34	42	50	5Χ	68	76	84	92	Χ0
Ɛ	1Χ	29	38	47	56	65	74	83	92	Χ1	Ɛ0
10	20	30	40	50	60	70	80	90	Χ0	Ɛ0	100

Now we are ready to multiply:

```
  22          231          Χ89
  19           32          3Ɛ7
 176          462         6313
 22           693        9Χ03
 396         7192       2823
                        366643
```

For some kinds of problems, especially those with feet or inches or other "twelve" units, the work is much easier by base-12 arithmetic than by our present methods:

How many square feet of carpeting are needed for a hall 14 feet 2 inches long and 3 feet 4 inches wide?

14 feet 2 inches = 12.2 (1 dozen 2 feet and two-twelfths)
3 feet 4 inches = 3.4 (3 feet and four-twelfths)

488
366

3Ɛ.28 (which is 3 dozen 11 (47) square feet, 2 dozen 8 (32) square inches)

By ordinary arithmetic it would have been necessary to turn both lengths into inches ($14 \times 12 = 168$, $168 + 2 = 170$ inches; and $3 \times 12 = 36$, $36 + 4 = 40$ inches), multiply 170 by 40 to get 6,800 square inches, and then divide by 144 to get square feet.

About Duodecimals

Because the *decimal* is used both for decimal-form fractions and the whole system of counting by tens, people sometimes jump to the conclusion that the handy "decimal" way to express fractions is a special advantage of the "decimal" system of counting.

This is not true. Provided a zero symbol is used, any number base can express fractions by using the same numbers to the right of the point. In the multiplication example above we have seen twelfths so expressed. In fact, *duodecimals* (twelfths) are much more efficient than *decimals* (tenths) for the expression of many low and much-used fractions, and permit a more accurate expression of nearly all fractional quantities with the use of the same number of places after the point.

Consider the low fractions in this table:

	Decimal	Duodecimal
One-half	.5	.6
One-third	.333333 . . .	.4
One-fourth	.25	.3
One-fifth	.2	.249724 . . .
One-sixth	.166666 . . .	.2
One-seventh	.142857 . . .	.186X35 . . .
One-eighth	.125	.16
One-ninth	.111111 . . .	.14
One-tenth	.1	.124972 . . .
One-eleventh	.090909 . . .	.111111 . . .
One-twelfth	.083333 . . .	.1

In this table, fractions are carried out to six places where not terminating. All the six-place figures keep on repeating, producing over and over again the first dotted figure, or the several figures included between dots.

A glance reveals that the 10-system has in this sample one and a half as many endlessly repeating numbers as the 12-system. Moreover, in the cases of both one-fourth and one-eighth it requires an additional figure to express accurately the same fraction.

What about *per cent*? We have got used to using percentage (one-hundredths) as a general approximation. We say "about 25 per cent" of the pupils had a perfect attendance, which says, literally, 25 out of every hundred; but really means a proportion - 1 out of every 4, 2 out of every 8, and so on. If we used duodecimals, what would happen to percentage?

We would lose the name, that is all. We might call it *per gross*, because now it would be so many out of each 100 (144). And it would be a good bit more accurate than percentage, for now with only two figures we could express the nearest 144th.

Indeed, the advantages of duodecimals are so great that a society, The Duodecimal Society of America, was formed some years ago just to promote further exploration and use of the duodecimal system - counting by dozens instead of by tens.

It is doubtful that the world will again change its system of numbers, though a few centuries ago Europe changed from Roman to our present Arabic numerals. But it is fun to experiment with a different system that in many respects is better, and to realize that the final word has by no means been said even in arithmetic. Important discoveries are still possible.

Fun, Counting by Sevens

Brother Alfred

Is your taste for numbers becoming jaded? Is your pristine enthusiasm waning from meeting the same old familiar relations? Does it pall on you that 2 plus 2 always equals 4? Then, why not try working number systems with bases other than ten? For example, in base 4, 2 plus 2 equals 10, while in base 3, the answer is 11.

These few notes will deal mainly with base 7, but for every fact mentioned here, a line of inquiry is opened up in number systems to other bases as well. To one and all we say: Happy Hunting!

FUNDAMENTALS OF BASE 7

As a start, we shall mention a few basic ideas which will be sufficient to enable any one to follow the developments in these notes. First of all, we have to learn to count. To show how this goes, we list the first numbers in base ten and base seven.

BASE 10. 1,2,3,4,5,6, 7, 8, 9,10,11,12,13,14,15,16,17,18,19,20,21,22

BASE 7. 1,2,3,4,5,6,10,11,12,13,14,15,16,20,21,22,23,24,25,26,30,31

The interpretation of a number to base 7 is entirely similar to that of a number to base 10. Thus, in base 10, the number 746 means:

$$(7\times10^2)+(4\times10)+6$$

In base 7, the number 352 means:

$$(3\times10^2)+(5\times10)+2$$

but we have to understand that 10 here means 7 in our number system. Hence to find the value of this number 352_7 in base 10, we make the following calculation: $(3\times7^2)+(5\times7)+2$

One other thing is needed in order to proceed with calculations in a number system other than that to base 10, namely, tables of the basic addition and multiplication facts for the digits. For base 7, these tables follow.

ADDITION COMBINATIONS

BASE 7

	1	2	3	4	5	6
1	2	3	4	5	6	10
2	3	4	5	6	10	11
3	4	5	6	10	11	12
4	5	6	10	11	12	13
5	6	10	11	12	13	14
6	10	11	12	13	14	15

MULTIPLICATION COMBINATIONS

BASE 7

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	11	13	15
3	3	6	12	15	21	24
4	4	11	15	22	26	33
5	5	13	21	26	34	42
6	6	15	24	33	42	51

The arithmetical processes in base 7 are carried out precisely as those of base 10 except that the above tables are used instead of the corresponding addition and multiplication tables for base 10.

We may now proceed to our number facts and curiosities for base 7.

(1) In base 10, we have a curious relation

$$12345679\times9=111111111$$

Is there a corresponding fact for base 7? Yes.

$$12346\times6=111111$$

(2) For base 10, $88^2=7744$. The corresponding relation in base 7 is still more interesting, being $55^2=4444$.

(3) In our ordinary number system, we have a process called casting out nines. If we are adding columns of figures, for example, we add the digits and keep eliminating nine from the sum, eventually arriving at a number less than nine. This should be the same sum as the sum of the digits in the answer with the nines eliminated. Such an operation provides a check on the work inasmuch as a discrepancy means there must be an error.

We illustrate casting out the nines for multiplication. Consider the product $8435\times9376=79086560$. The corresponding multipliers after eliminating the nines give $2\times7=14$, the sum of the digits in this product being 5. Casting out 9's in the product 79086560 likewise gives 5.

For base 7, we have the corresponding process of casting out sixes. This will be illustrated for addition.

$$\begin{array}{r}342\\153\\206\\144\\320\\\hline 1531\end{array}$$

The practical way to eliminate sixes is to go by combinations adding up to six. Thus, in the first line 42 is dropped; in the second, 15; between the first and second 3 and 3; and so on. The net result is 4 which agrees with the sum after the six in 15 has been eliminated and 3 is added to 1.

The procedure for multiplication is the same as shown for base 10.

(4) In base 10, if we take any number, such as 4821 and put the digits in any other order, such as 8124, the difference 8124−4821=3303 is always a multiple of 9. In other words, after casting out 9's from the difference, we obtain zero.

Similarly, in base 7, take any number, such as 3524 and place the digits in another order, such as 2435. The difference 3524−2435= 1056 is seen to be zero after casting out 6's; in other words, it is divisible by 6.

(5) In the February 1961 issue of RMM, the editor pointed out several examples of reversible primes and permutable primes. In base 7, for reversible primes we have the pairs: 14, 41; 16, 61; 23, 32; 25, 52; 56, 65: Only four two-digit primes are not reversible.

The highest number of permutable three-digit primes in base 10 is found to be four*; for base 7, three examples of four permutable three-digit primes and one example of five permutable three-digit primes were found. These examples are: 245, 254, 452, 524; 326, 362, 623, 632; 346, 364, 436, 463. The five permutable three-digit primes are: 124, 142, 214, 241, 421.

Another interesting curiosity: Two consecutive prime numbers, namely, 245 and 254, have the same digits!

(6) In base 10, the only fractions having a terminating decimal when evaluated are those with denominators of the form 2^a5^b. If there is any other prime factor in the denominator besides 2 or 5, a repeating, non-terminating decimal results.

What is the situation in base 7? Only those denominators of the form 10^a give rise to terminating decimals. All others produce non-terminating decimal expressions. For example, $^1/_2$=.3333....

$$^1/_{12}=.053053053.....$$

The process of going from a repeating decimal to the equivalent fraction in base 10 consists in putting the periodic numbers over a corresponding number of 9's. Thus for .142857142857....., the equival-

---

* 179, 197, 719, 971; 379, 397, 739, 937.

In Base-11 there are three digits with all six permutations giving primes. If we represent the tenth digit in the base-11 system by X the six numbers are: 3X6, 36X, 63X, 6X3, X36, X63. The corresponding values in the base-10 system are 379, 439, 769, 839, 1249, 1279 - all primes, of course.

ent fraction is:

$$\frac{142857}{999999} = \frac{1}{7}$$

The equivalent procedure in base 7 is to place the periodic numbers over a corresponding number of 6's. Thus the fraction that equals the periodic decimal .014301430143..... is

$$\frac{143}{6666} = \frac{1}{42}$$

The fraction equalling the decimal .033033033.... is

$$\frac{33}{666} = \frac{4}{111}$$

(7) We come now to a major curiosity. In base 10 we determine whether a number is odd or even by examining the last digit. Not so in base 7. Here, a number is even, if the sum of its digits is even; odd, if the sum of its digits is odd. The result is that no matter how we permute the digits of a number in base 7, we continue to get an even or odd number depending on what we started with.* As an example, we work out the corresponding numbers in base 10 of the six permuations of the base 7 number 134:

$$\begin{aligned}
134_7 &= 49+21+4=74\\
143_7 &= 49+28+3=80\\
314_7 &= 147+7+4=158\\
341_7 &= 147+28+1=176\\
413_7 &= 196+7+3=206\\
431_7 &= 196+21+1=218
\end{aligned}$$

A corresponding example for odd numbers is 234.

$$\begin{aligned}
234_7 &= 98+21+4=123\\
243_7 &= 98+28+3=129\\
324_7 &= 147+14+4=165\\
342_7 &= 147+28+2=177\\
423_7 &= 196+14+3=213\\
432_7 &= 196+21+2=219
\end{aligned}$$

---

* Simple Simon touring through Blunderland met a π-man casting up a sum.

"344 and 526 is 1203," said the π-man.

"Really?" asked Simple Simon, "Do two even numbers always add up to an odd number in Blunderland?"

The π-man replied, "Not always. We like variety in our arthmetic processes and so we use base-7"

"But how did you get two even numbers to add up to an odd number?" asked Simple Simon.

"It was really quite simple, Simon," said the π-man. "The truth is that I added two odd numbers and got an even number."

This leads to some interesting speculation. What number systems act like 10? What systems behave like 7? Do some number systems have a still different pattern for determining odd and even numbers?

(8) In the February 1961 issue of RMM (pp. 38-42), the editor gave over a hundred ways of arranging the digits 1 to 9 in order so as to give a value of 100. The same will be done for base 7 using entirely similar agreements and understandings. Warning to the reader: From here on in this section, everything will be to the base 7. Helpful hint: If you find yourself confused, look to the postscript of this list.

Speaking then in base 7, the intention is to arrange the digits 1 to 6 in order with appropriate signs interspersed so as to give a total of 100. We shall also offer 100 examples of this type.

(1) $-1+2+3+4+56$

(2) $12+3-4+56$

(3) $1+2+\sqrt{34}+56$

(4) $(1)(2)(34)+5-6$

(5) $1^2+3+4+56$

(6) $-(1)(2)+3^4-(5)(6)$

(10) $-1(2+3)+4!+(5)(6)$

(11) $1+2+3+\sqrt{4}+56$

(12) $1-(2)(3)+4!+(5)(6)$

(13) $1-23+4!+56$

(14) $12+3!+4+(5)(6)$

(15) $1(2+3)+4!+(5!\div 6)$

(16) $-1+(2)(3)+4!+(5!\div 6)$

(20) $1-2+3!+4!+(5!\div 6)$

(21) $-1+2-3!+4!+(5)(6)$

(22) $\sqrt{-1+23}+4+56$

(23) $\sqrt{12}+\sqrt{34}+56$

(24) $1(2+3)+4(5+6)$

(25) $1-2+3!+4(5+6)$

(26) $(1)(2)+3+4(5+6)$

(30) $1-(2)(3)+(4+5)(6)$

(31) $-1(2+3)+(4+5)(6)$

(32) $-1+2-3!+(4+5)(6)$

(33) $-(1)(2)-3+(4+5)(6)$

(34) $(-1+2^3)(-4+5+6)$

(35) $[1+(2)(3)][-4+5+6]$

(36) $1+[(2^3)^{-4+5}][6]$

(40) $1+(2)(3^4)-5!+6$

(41) $1+2^3+\Sigma 4+(5)(6)$

(42) $(1)(2^3)+(4)(5)+\Sigma 6$

(43) $1-2+(3)(\Sigma 4)+(5!\div 6)$

(44) $-1^2+(3)(\Sigma 4)+(5!\div 6)$

(45) $(12)(\Sigma 3)+4-\Sigma 5+6$

(46) $1-2^3+(\Sigma 4)(5)+6$

(50) $-1(2+3)+(4)(\Sigma 5)-6$

(51) $1-(2)(3)+(4)(\Sigma 5)-6$

(52) $1-(\Sigma 2)(\Sigma 3)+(4)(\Sigma 5)+6$

(53) $-1(23)+(4)(\Sigma 5)+6$

(54) $(1+2+3)(\Sigma 4)-5-6$

(55) $-(\Sigma 1)(\Sigma 2)+\Sigma 3+\Sigma 4+\Sigma 5+\Sigma 6$

(56) $-1-2+\Sigma 3+\Sigma 4+\Sigma 5+\Sigma 6$

(60) $-1!-2!+3!+\Sigma 4+\Sigma 5+\Sigma 6$

(61) $-1!+2-3!+4!+(5)(6)$

(62) $1!-2!+3!+4!+(5!\div 6)$

(63) $(12+3)(4)-5+6$

(64) $123+4-\Sigma 5-6$

(65) $\sqrt{(12)^3}-4+5+\Sigma 6$

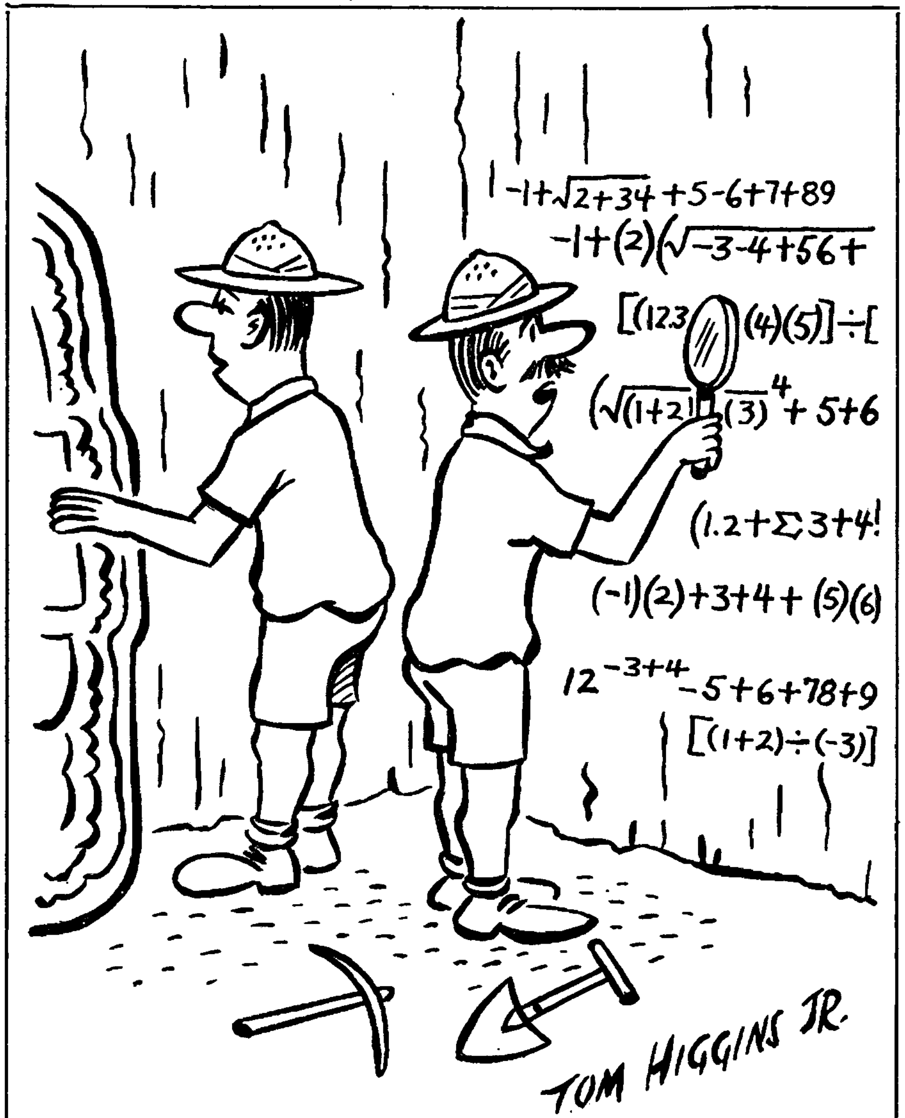
(66) $(1+2)^3-4+5+\Sigma 6$

(100) $(-1+2+3)(\Sigma 4)+\Sigma 5-6$

Postscript

The sum of each of these expressions is 100_7 or 49_{10}. This explains likewise the relative shortness of the list and demonstrates one of the advantages of base 7: To get a series of 100 items, we need only go up to 49!

Here's hoping you have fun discovering more number facts in base 7. And remember, as Mr. Hunter has indicated, there are many other number worlds to conquer.



"SAY PROFESSOR - WOULD YOU LIKE TO SEE SOMETHING INTERESTING?"

Two pints make one cavort.

Terminal Digits of MN (M²–N²) In The Scale of Seven

By Charles W. Trigg

If M and N are integers, the unit's digit of

$$P = MN(M^2 - N^2) = MN(M+N)(M-N)$$

is dependent upon the unit's digits of its four factors. Represent the unit's digits of M, N, P by m, n, p, respectively. These terminal digits, treated as signless numbers, fall into two square arrays, one for $M \geqq N$, the other for $M \leqq N$.

In the scale of notation with base seven*, p is zero if m, n, or (m—n) is zero, or if $m+n=10$. E.g., if $m+n=5+2=\text{seven}=10$. These zeros constitute the diagonals of the square arrays of the values of p for non-zero m and n. Thus:

$M \geqq N$

n \ m	1	2	3	4	5	6
1	0	6	3	4	0	0
2	1	0	2	5	0	6
3	4	5	0	0	2	3
4	3	2	0	0	5	4
5	6	0	5	2	0	1
6	0	1	4	3	6	0

$M \leqq N$

n \ m	1	2	3	4	5	6
1	0	1	4	3	6	0
2	6	0	5	2	0	1
3	3	2	0	0	5	4
4	4	5	0	0	2	3
5	1	0	2	5	0	6
6	0	6	3	4	1	0

These two arrays are intimately related in that they are mirror images, one goes into the other by rotation about a main cross-diagonal (upper right to lower left), and the corresponding elements of the two arrays are complementary, e.g., $4+3=\text{seven}=10$.
Each array has the following properties:

a) All seven digits in this scale of notation appear, each non-zero digit appearing 4 times.

b) Elements symmetrical to the main diagonals and those symmetrical to the perpendicular bisectors of the sides are complementary.

*In the discussion of the arrays to the base seven, numbers in that scale are expressed as numerals. Those expressed by names are to the base ten.

c) Thus the array contains four rectangles, congruent in pairs, with sides parallel to the diagonals and vertices on the sides of the array. In each case, the sum of the vertices is 20. The sum of the elements on each of two opposite sides is 10, and on each of the other two sides is 20. Hence, the sum of the elements on each perimeter is 40.

d) The array consists of three nested squares with sides totaling 20, 10 and 0, respectively.

e) The sum of the elements in each column and each row of the array equals 20.

f) The array is symmetrical to its center, and like elements lie at the vertices of nine squares. The 2's at the vertices of one square may be traversed by knight's moves. The same may be done with the 5's.

g) Each 2 is connected by knight's moves to a 1 and a 4 in such manner that the three elements whose sum is 10 lie on a straight line. Likewise each 5 is connected by knight's moves to a 6 and a 3 in such a way that the three elements whose sum is 20 lie on a straight line.

h) The diagonals divide the array into four triangles each of which contains the six positive digits. In each group, the path of the joins of the digits in order makes a symmetrical "knot." Otherwise, the path consists of knight's move, three sides of a square, and another knight's move.

i) The perpendicular bisectors of the sides divide the array into four 3x3 sub-arrays each of which contains all the digits arranged in a pattern consisting of a diagonal of zeros and three different complementary pairs symmetrical to this diagonal. The sum of the elements of the sub-array accordingly is 30.

j) Proceeding around the array, each sub-array goes into the next sub-array by a 90°-rotation. This is equivalent to stating that the array goes into itself by a 90°-rotation.

k) In each sub-array, the path of the joins of the digits in order forms a three loop "knot" of isosceles right triangles.

l) Each sub-array considered as a determinant has an absolute value of 120. The values of the cornered minors form a determinant, e.g., $\begin{vmatrix} -6 & 15 \\ 5 & -13 \end{vmatrix}$ with value zero.

m) The value of the sixth order array as a determinant is zero.

n) The array** may be viewed as the composite of twenty-five overlapping second order arrays. These second order arrays may be evaluated as determinants, e.g.,

$$\begin{vmatrix} 0 & 6 \\ 1 & 0 \end{vmatrix} = -6, \quad \begin{vmatrix} 6 & 3 \\ 0 & 2 \end{vmatrix} = 15, \quad \begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} = 10, \text{ etc.}$$

These values in order constitute a fifth order array, which vanishes when evaluated as a determinant. That is,

$$\begin{vmatrix} -6 & 15 & 10 & -5 & 6 \\ 5 & -13 & 0 & 13 & -15 \\ -10 & 0 & 0 & 0 & -10 \\ -15 & 13 & 0 & -13 & 5 \\ 6 & -5 & 10 & 15 & -6 \end{vmatrix} = 0.$$

When the same procedure is applied to this array to produce a fourth order array, again to form a third order array, and again to form a second order array, the determinants of each of these arrays vanish.

o) The original array may be treated as the composite of sixteen overlapping third order arrays, and each of these evaluated as a determinant. The absolute value of the determinant of the fourth order array thus obtained,

$$\begin{vmatrix} 120 & 50 & 20 & -120 \\ -20 & 0 & 0 & -50 \\ -50 & 0 & 0 & -20 \\ -120 & 20 & 50 & 120 \end{vmatrix} \text{ is } 12 \times 10^6.$$

When the same procedure is applied to this array, the second order determinant obtained vanishes.

p) The determinant of the third order array composed of the determinant values of the nine overlapping fourth order arrays is

$$\begin{vmatrix} -100 & 300 & -100 \\ 300 & 0 & 300 \\ -100 & 300 & -100 \end{vmatrix} = 0.$$

q) Each determinant of the four overlapping fifth order arrays has the value 12×10^3. Hence the second order determinant

**The derived determinantal arrays from the $M \geqq N$ array go into the arrays derived from the $M \leqq N$ array by interchanging columns and rows. Hence the conclusions hold.

formed from these values vanishes. Note that 12×10^3 as well as the non-zero values of the fourth order determinants in p) exactly divide the determinant in o).

OTHER SCALES OF NOTATION.

The values of p which appear in the arrays for various scales of notation are:

Number base	p	Number base	p
two	0	seven	0 1 2 3 4 5 6
three	0	eight	0 2 4 6
four	0 2	nine	0 3 6
five	0 1 4	ten	0 4 6
six	0	eleven	0 1 2 3 4 5 6 7 8 9 X
		twelve	0 6

So seven is the smallest base for a scale of notation in which all of the digits in the scale appear in the arrays. Only zeros appear in the arrays for the scales of two, three, and six. Only even digits appear when the number base is even. In the decimal scale and when the number base is odd, the non-zero digits which appear may be grouped into complementary pairs. In the scales of four and twelve, the only non-zero element is half the base, while in the scale of eight a complementary pair also appears.

The arrays for the decimal scale have been discussed in *Mathematics Magazine*, Vol. 34, pp. 159-160, 233-235, (1961). The array for the duodecimal scale appears in the *Duodecimal Bulletin*, Vol. 14, No. 2, p. 4X (Dec. 1960).

Word Games

by S. Baker

"7" LETTER SCRAMBLE

Mr. J. S. Madachy, the editor of RMM, recently wrote to me and said that the "7" *Letter Scramble* games are fine, but more *teasers* are needed. I sent him the list of seven words shown here, scrambled, and told him that when properly unscrambled they really belong in the class of *teasers*.

E I N N G R T
A E N G T I H
I A N S T E G
I M A N G E S
N I K L E G A
I R R G A N E
A G G I N S T

Editor Madachy immediately wrote back and said the solution is much too easy - the words are similar and all end with the same letter. Of course, Mr. Madachy was wrong and with the right solution none of the words will end with "G".

Let's see how many readers can find *both* of the solutions hinted at.

CHANGE A LETTER

Another teaser. You must go from PRETEND to ECHOISM in exactly six changes, forming five new words between the two given. Each change must be of one letter only. If we had started with START we could change "R" to "E" to make TASTE, then change "S" to D" to make DATED, and so one, making only one letter change at a time.

1.	P R E T E N D
2.	
3.	
4.	
5.	
6.	
7.	E C H O I S M

Mr. Baker would like to have the readers of RMM send their answers to the word games directly to him at 265 Vitre Street, West; Montreal 1, Quebec; Canada. He will reply to all letters.

The answers to the April issue *Word Games.*

Words All Ways: See Figure 1 below.

Words Between: The other eight-letter word formed by rescrambling MOLDIEST is MELODIST. To go from MOLDIEST to I, Mr.

	1	2	3
1	T	U	G
2	A	N	A
3	P	A	R
4	S	U	B

Figure 1

G	L	I	S	T	*E*	N
T	H	R	E	A	*A*	T
S	U	T	L	E	*R*	S
P	A	D	R	O	*N*	E
R	E	G	I	M	*E*	S
R	E	A	L	I	*S*	T
I	N	G	R	A	*T*	E

Figure 2

Baker gives MOLDIEST, MODISTE, DEMITS, TIDES, TIES, TIE, IT, I.

Edith Marsh of Montreal West, Quebec, gives MOLDIEST, MILDEST, SMILED, MILES(or LIMES), MILE, MIL, MI, I. W. A. Robb of Ottawa, Ontario gives MOLDIEST, MILDEST, MISTED, TIMES, EMIT, TIE, IT, I.

"7" *Letter Scramble*: See Figure 2 on page 21.

Change a Letter: To go from CLAIMED to RATABLE, Mr. Baker gives CLAIMED, DIALECT, ARTICLE, LATICES, BESTIAL, LABIATE, RATABLE or CLAIMED, CLIMATE, RECLAIM, REALISM, REALIST, BLASTER, RATABLE.

W. A. Robb of Ottawa, Ontario gives CLAIMED, MIRACLE, CARLINE, CARIOLE, ARTICLE, TRIABLE, RATABLE.

The other words possible for line 1: TINGLES, SINGLET. For Line 2: THEATRE. For Line 3: RUSTLES, RESULTS to which W. A. Robb adds LUSTRES, LUSTERS, TUSSLER. For Line 4: APRONED. For Line 5: EMIGRES, REMIGES. For Line 6: RETAILS, SALTIER to which W. A. Robb adds TAILERS. For Line 7: TEARING, GRANITE to which W. A. Robb adds TANGIER.

Mathematical Permutacrostic

Charles W. Trigg

When permutations of the letters of an ordered series of words or phrases form in order a series of words in which a specified set of letters, one from each word, form an acrostic, i.e. a word or phrase, the resulting array appropriately may be called a "permutacrostic." Each of the following 23 phrases is a permutation of the letters of a mathematical term. The first letters of these terms in order spell out an interesting activity.

(1) Large Cent
(2) Sell Pie
(3) So Nice
(4) No Lariat
(5) Ox In Aspens
(6) Gave Ear
(7) I Rest In Cot
(8) Deer Meat In Tin
(9) Do Not Reach
(10) A Rum Tenor
(11) A Girl Tenant
(12) Are Tall
(13) Same True Men
(14) I Chart Time
(15) Rent Tan Candles
(16) They Nose Up
(17) To Taxi Or Plane
(18) Main Loom
(19) Pity Mascot
(20) Met Hero
(21) I Get On Train
(22) Tomato In Cup
(23) Lips Reach

Solution is on page 27.

The Game is HOT

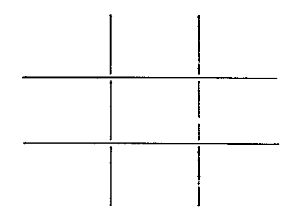
by Leo Moser

The following simple game was invented for the purpose of helping illustrate for beginners, the significance of analytical geometry, i.e. the importance of establishing a correspondence between concepts and results of geometry and those of algebra.

The game is called HOT and is played by two players using a set of 9 cards, each bearing one of the words - hot, hear, tied, form, wasp, brim, tank, ship, woes. The cards are placed face up and the players take turns picking up cards, one at each turn. The first player who can announce that he holds three cards having a common letter wins. If, after all the cards have been picked up neither player holds such a set of three cards the game is a draw.

Although the game is at first unfamiliar it is clearly not very complicated and one should be able to master it after an hour or so of study. It will be found that with correct play the game should end in a draw but an 'expert' player will have little difficulty in beating a beginner, particularly if he has the first move.

In spite of the fact that this game is in some sense a new one, it is essentially the same as the well-known game of Naughts and Crosses or Tic-Tac-Toe. This latter game is played on a 'board' which looks like this



Here the players take turns entering zeros and crosses into the squares. A player wins if he can get three of his symbols entered in a line, i.e. a row, column or diagonal. The theory of this game is, of course, very well known and even many small children are masters of it.

Now we can establish a correspondence between HOT and Tic-Tac-Toe as follows:

hot	form	woes
tank	hear	wasp
tied	brim	ship

Make the picking up of a word in HOT correspond to placing your mark on that square of Tic-Tac-Toe in which the word lies in the above diagram. It is easily verified that following this prescription you will win, lose or draw the game of HOT under the same circumstances that will cause you to win, lose or draw respectively in the game of Tic-Tac-Toe. Thus the correspondence established by the diagram makes any expert in Tic-Tac-Toe an expert in HOT and vice-versa. Any knowledge of one game can be translated into knowledge about the other. The two games are, in a very real sense, the same game. Similarly, the discovery of Descartes amounted to discovering the fact that the games of algebra and geometry, which had been played for two thousand years, and which are even today by no means fully understood, are in fact two aspects of the same game.

GOOF - *A Game to Begin All Games*

William Bunge

Goof is a game that generates games. The game that generates games operates on certain basic "principles" which govern the generation of "rules" which determine the character of the game. One principle places certain physical limits on the games to be generated. For instance, the games may be confined to a card table and the use of playing cards. This is to prevent players from generating rules such as "run quickly around the block." Another principle starts the game, similar to the instructions necessary to start computers. The start might be given to the player who draws the highest card from a deck of cards. The start consists of giving that particular player the opportunity to make the first rule. Another principle is that no rule can be made which makes it impossible to win the game. Another principle is that no rule can be made that contradicts any other rule. Another principle is that anyone who violates a principle or an established rule has "goofed" and the first person to recognize the mistake by saying "goof" has the right to the offender's turn. A "turn" consists of the actual playing of the game and the generation of another rule to develop the game. If a player says "goof" and there has been no goof, then another player can say "goof" on the person who originally said "goof." The false goof sayer looses his turn when it comes his turn to the second goof sayer. The second goof sayer might in turn be mistaken since the first might be correct. If so he suffers the same treatment as the first, etc. Disputes are settled by votes of the players after discussion. Another principle is that no rule can be implemented that does not give all players an equal chance.

The games are fascinating. Unheard of card game rules are invented and mixed with standard bridge, poker, etc., forms. The sense of the game's direction, i.e., what it takes to win, is constantly shifting as the game develops.

A sample game might be generated as follows:

1. The player who at random drew the highest card starts the game with the following rule: "The game will be played in alphabetical last name order."

2. The second player, that is, the one whose last name is nearest the beginning of the alphabet, makes the following rule: "All the cards will be dealt counterclockwise around the table."

3. The third player's rule is, "All players will pick up their hands face down so that everyone except the holder can see the cards."

4. Etc.

It seems that serious purposes might be served by Goof, especially since Goof seems to bear close resemblance to the manner in which mathematics itself grows.

PI PARADOX by Maxey Brooke

The reciprocal of any odd number can be expressed as a repeating decimal.

The sum or difference of two repeating decimals is a repeating decimal.

$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11 + \dots$ (Liebnitz, 1674, but known earlier)

Thus $\pi/4$ can be expressed as the sums and differences of a series of reciprocals of odd numbers. Consequently π can be expressed as a repeating decimal. Therefore π is rational.

But π is transcendental. (Maxey Brooke)

Interested in obtaining the first (February 1961) RMM? See Editorial on page 2.

Alphametics

Here are several *Alphametics* for your fun and enjoyment - answers to the April puzzles follow on the next page.

The answers to this month's *Alphametics* will be given in the August issue of RMM.

```
 M O O N
   M E N
   C A N
---------
R E A C H
```

This certainly won't be easy! But it can be done. (J. A. H. Hunter)

A vexing problem, we're sure! (V) (VEXATION = EEEEEEEEE
(G. Mosler)

As Shakespeare very nearly wrote:

But, we ask, what's SWELL if you have no one? (D. Murdoch)

```
  A L L S
  W E L L
  T H A T
  E N D S
---------
S W E L L
```

```
      U N
      U N
  D E U X
D O U Z E
---------
S E I Z E
```

Et voila une alphametique! Every French student will be able to agree that the addition of one, one, two and twelve amounts to sixteen - but what different figure does each letter represent if SEIZE is itself very properly divisible by 16? (D. Murdoch)

SOME ALPHAMETICAL DOGGEREL — By Derrick Murdoch

A PINT plus a PINT makes a QUART;
Of that fact you must first be aware.
Now we ask you to find just for sport
What is QUAINT when your URN is a square?

Or, since it's easy to vary solutions when playing around with liquids:

A PINT plus a PINT's still a QUART,
Though a half PINT now gives you at DIET.
That should let you decide what you ought
To make of a DAIQUIRI. Try it!

Here are the April issue answers:

```
A L L ) F O O L S ) D A Y        388 ) 91180 ( 235
        G x x                          776
        -------                        ----
        x A x x                        1358
        x x M x            =           1164
        -------                        ----
          x x E x                       1940
          x x x S                       1940
          -------                       ----

          R M M                          633
          S L Y                          127
        ---------                       ----
          x x x x                       4431
        x x R x            =           1226
        x x M                          633
        ---------                      -----
        G A M E S                      80391
```

(A)(SPADE) = FLUSH = (5)(13582) = 67910

Since the statement given says that the winning hand (a FLUSH in this case) was filled by a card represented by A, then the tatol of 6 letters in A FLUSH must represent only 5 cards and, therefore, one of the letters in FLUSH must be a one (A cannot be a one). The only other solution fulfilling this condition is (4)(17453) = 69812, but this would mean indicating a Queen by the notation 12 which is seldom done.

SOLUTION OF PERMUTACROSTIC FROM PAGE 22.

(1)	Rectangle	(13)	Measurement
(2)	Ellipse	(14)	Arithmetic
(3)	Cosine	(15)	Transcendental
(4)	Rational	(16)	Hypotenuse
(5)	Expansions	(17)	Extrapolation
(6)	Average	(18)	Monomial
(7)	Trisection	(19)	Asymptotic
(8)	Indeterminate	(20)	Theorem
(9)	Octahedron	(21)	Integration
(10)	Numerator	(22)	Computation
(11)	Alternating	(23)	Spherical
(12)	Lateral		

The Haunted Checkerboards

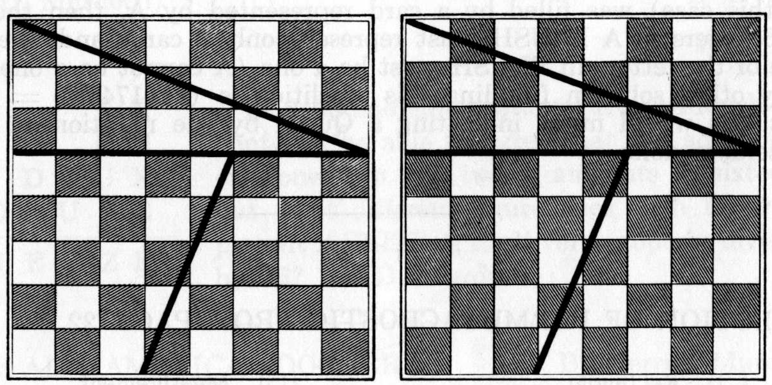
Maxey Brooke

My son, Jamie, is at the age of jig saw puzzles. He is too young to handle the three-hundred piece store-boughten variety. My wife handles the problem by pasting pictures on cardboard and cutting them into large uncomplicated shapes.

You might think the market would eventually become saturated. But then, you don't know Jamie's capacity for losing one piece of a puzzle and needing a new puzzle . . . desperately.

The other evening I was trapped into becoming a puzzle-maker. I needn't go into the mechanics of being trapped. Married men will understand and bachelors won't care.

Anyway, with my usual cunning, I invented a new puzzle. Rather than go to the picture pasting routine, I found some old checkerboards and cut them up. The results looked something like this:

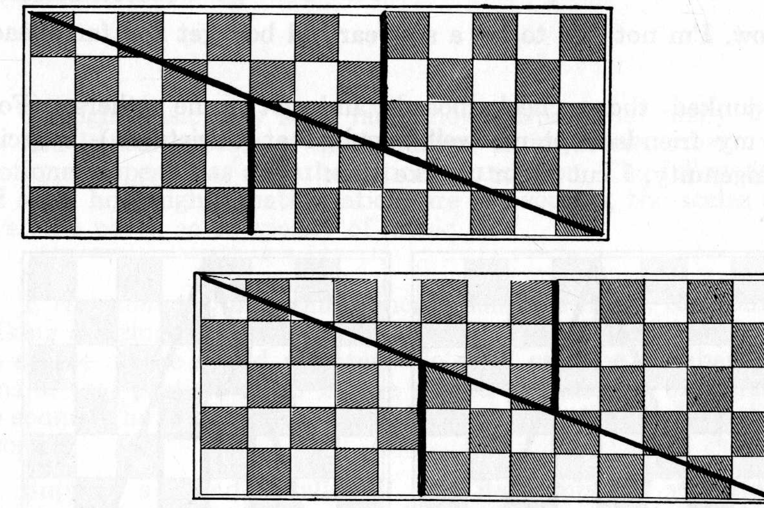


"Now, put them back together."

I resumed my interrupted reading. But not for long.

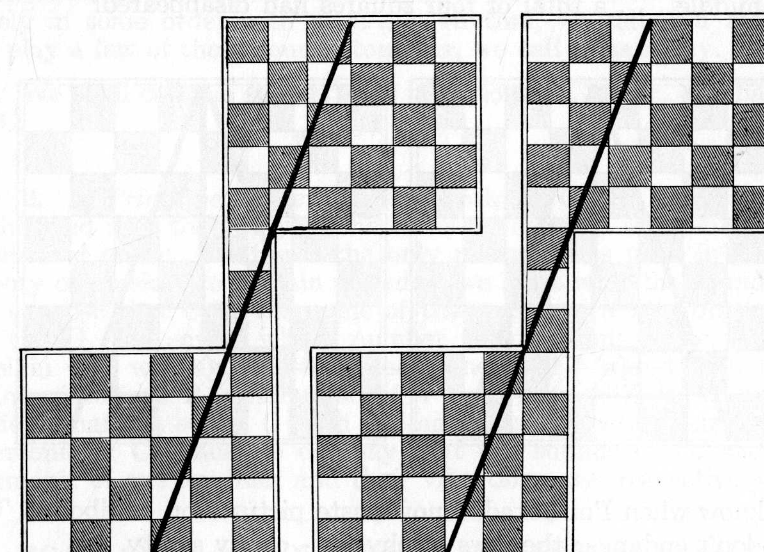
"Daddy, is this right?"

I looked over my paper ready to smile indulgently. But the smile died in mid lip. Out of two eight-by-eight checkerboards containing sixty-four squares. Jamie had produced two five-by-thirteen rectangles containing sixty-five squares each!



I got down on my hand and knees beside him and examined them carefully. Then I scrambled the pieces up and said, "Let me see you do that again."

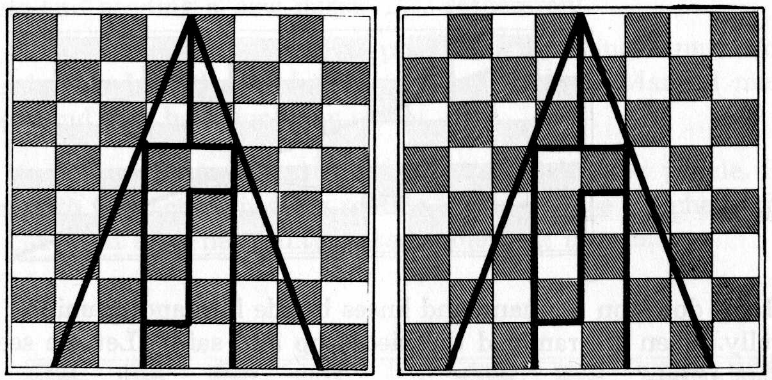
This time I didn't take my eyes off him. I was going to see where the extra squares came from. But instead of rectanges he came up with these odd-looking shapes.



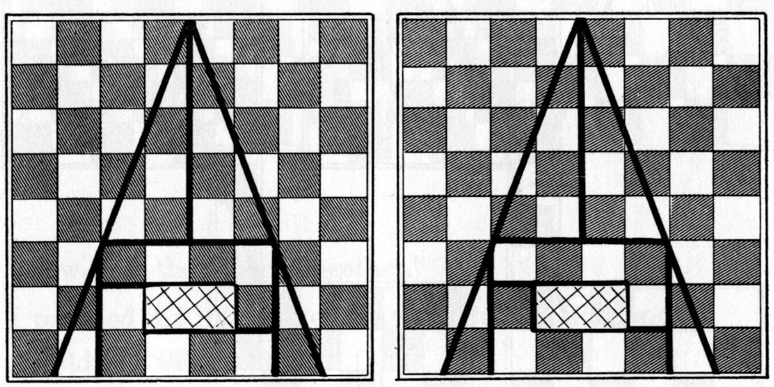
And when I counted the squares, there were only sixty-three in each set-up!

Now, I'm not one to let a six-year old boy get *too* far ahead of him.

I junked those checkerboards and got some others. (Fortunately my friends kept me well supplied at Christmas.) Exercising much ingenuity, I cut them up like this:



Then I sat down and awaited results. I got them. Jamie ended up with eight-by-eight squares, alright. But there were gaping holes in the middle . . . a total of four squares had disappeared!



I know when I'm licked. I now paste pictures on cardboard. That way I don't endanger the laws of physics - or my sanity.

Mathematics of Music

by Ali R. Amir-Moéz

Though music may seem far removed from what many think are the cold logical aspects of mathematics, nevertheless, music, with its emotional appeal, has a mathematical foundation. The following article will show how highly mathematical are the sounds, the scales and the keys (the parts, so to speak) of music.

1. *Harmonics of a Sound*: When a sound is made, for example, by striking a string of a musical instrument, each particle of air next to the source of the sound vibrates. We shall call the number of vibrations of that particle of air in one second the number of vibrations of the sound. The larger this number is, the higher the pitch of the sound becomes.

Suppose a sound is called C, and its number of vibrations is c. That is, if, for example, the sound C makes a particle of air vibrate five hundred times in one second, we say $c=500$. It was discovered by Greek mathematicians that if after the sound C is heard we make another sound S whose number of vibrations is twice the number of vibrations of C, i.e., 2c, then S will be pleasant to hear. As far as the history of mathematics shows, this idea is due to Pythagoras. The sound T with three times as many vibrations, i.e., with 3c vibrations, is also pleasant to hear after C. This fact is true for sounds with vibrations c, 2c, 3c, 4c, 5c, etc. Usually, if we play these sounds successively in some order with a certain rhythm, we call it a melody. If we play a few of these sounds together, we call it harmony.

We shall call the sounds with vibrations 2c, 3c, 4c, etc. harmonics of C.

2. *A Primitive Scale*: In the work of Omar Khayyam*, it is mentioned that the study of the ratios of integers is essential for the science of music. That was the only mathematics used in the Greek theory of music. To explain the idea, we start with the sound C, and we suppose that C_1 is the name of the sound with 2c vibrations. Let us call G_1 the sound whose number of vibrations is 3c. (We shall explain why we have chosen these names.) If a sound with twice as many vibrations is a harmonic of a given sound, it is reasonable to believe that the sound G with one-half as many vibrations as G_1 is a harmonic of C. Thus we can say that the sounds C, G, and C_1 are harmonic of one another, and their vibrations are, respectively, c, $3c/2$,

*Omar Khayyam, "Discussion of Difficulties in Euclid," *Scripta Mathematica* V. 24, pp. 275 - 303 (1959).

and 2c. We can compare these sounds and their vibrations by constructing the following table.

Sound	C	G	C_1
c	1	$\frac{3}{2}$	2

The first line of the table shows the name of each sound, and the second line shows the corresponding number of vibrations. For example, under G we see $^3/_2$, which means that G has $^{3c}/_2$ vibrations in a second.

The names chosen here are actually those chosen in the scale. If C is the natural C of the scale, then G has $^3/_2$ as many vibrations as C. C_1 is the next so-called C, which is usually called the octave of C.

In this scale we have only three sounds. If we play C, G, and the octave of C on the piano, we can almost see how they sound. Of course, we cannot make much music with three sounds.

3. *Oriental Scale*: Let us extend the idea of section 2 further. We take the fifth and seventh harmonics of C, *i.e.*, the sounds whose numbers of vibrations are 5c and 7c. We call these sounds, respectively, E_2 and K_2. We shall explain the choice of the subscripts shortly. Let us compare these sounds with C_1, the octave of C, and with C_2, the octave of C_1. Note that 2c is the number of vibrations of C_1, and 4c is the number of vibrations of C_2. Thus, if E_1 is a sound with half as many vibrations as E_2, then we see that $^{5c}/_2$ is the number of vibrations of E_1. Similarly, we can choose a sound K_1 whose number of vibrations is $^{7c}/_2$. If we compare these sounds according to their pitch, we get them in the order C_1, E_1, K_1, C_2. This is clear because

$$2 < \frac{5}{2} < \frac{7}{2} < 4.$$

Since these sounds are all harmonics of C, the sounds E and K, which have half as many vibrations as E_1 and K_1, respectively, *i.e.*, $^{5c}/_4$ and $^{7c}/_4$, are also harmonics of C. As in section 2, we can make a table as follows:

Sound	C	E	G	K	C_1
c	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2

These five sounds together approximately constitute the oriental scale.

4. *Middle-East Scale*: If we proceed with what was done in section 3, we get more sounds in the scale. Since the sounds with vibrations 2c, 4c, 8c, 16c, *etc.* do not contribute to the scale, we choose the sounds between them. In particular, let us call D_3 the sound with 9c vibrations. We also choose P_3, H_3, and B_3 with vibrations, respectively,

11c, 13c, and 15c. As before, we may choose D_2, P_2, H_2, and B_2 with vibrations $^{9c}/_2$, $^{11c}/_2$, $^{13c}/_2$, and $^{15c}/_2$, respectively. Then we choose D, P, H, and B with vibrations $^{9c}/_8$, $^{11c}/_8$, $^{13c}/_8$, and $^{15c}/_8$, respectively. We shall construct a table as before.

Sound	C	D	E	P	G	H	K	B	C_1
c	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{11}{8}$	$\frac{3}{2}$	$\frac{13}{8}$	$\frac{7}{4}$	$\frac{15}{8}$	2

A scale may be made out of these sounds with eight names in the scale instead of seven. Before we discuss this set of sounds, we make a table using the theoretical (physical) sounds of the scale.

Sound	C	D	E	F	G	A	B	C_1
c	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

If we compare P and F, we see that the ratio of the number of vibrations of P to the number of vibrations of F denoted by

$$\frac{P}{F} = \frac{11}{8} : \frac{4}{3} = \frac{33}{32}$$

This shows that P is sharper than F. This is where the middle-east music is different from the physical scale. The sound H with $^{13c}/_8$ vibrations is not used in the middle-east music. Thus, C, D, E, P, G, K, B, C_1 approximately constitute the sounds of the middle-east scale. We see that

$$\frac{K}{A} = \frac{7}{4} : \frac{5}{3} = \frac{21}{20}$$

Therefore, K is also sharper than A.

5 *Tones and half-tones*: If we study the physical scale, we observe that

$$\frac{D}{C} = \frac{9}{8},\ \frac{E}{D} = \frac{10}{9},\ \frac{F}{E} = \frac{16}{15},\ \frac{G}{F} = \frac{9}{8},\ \frac{A}{G} = \frac{10}{9},\ \frac{B}{A} = \frac{9}{8},\ \frac{C_1}{B} = \frac{16}{15}.$$

This suggests the idea of small and large intervals or tones and half-tones. We shall write this as follows:

Sound	C		D		E		F		G		A		B		C
Tone		1		1		$\frac{1}{2}$		1		1		1		$\frac{1}{2}$	

The above table indicates which interval is a tone and which is a half-tone. For example, between E and F is a half-tone. But, we really should say large and small intervals.

6. *Geometric Progression*: An ordered set of numbers is called a geometric progression when the ratio of each one to its predecessor is always the same. For example, the set

$$5, \quad 10, \quad 20, \quad 40, \quad 80, \quad . \; . \; .$$

is a geometric progression. The ratio is 2, that is, the ratio of each number to the one before it is two. Indeed, we can produce as many members of this set as we desire.

If one member of a set and the ratio are given, we can always produce as many members as needed. For example, if ½ is a member of the progression and the ratio is $\sqrt{2}$, then we can write some of the members of this progression, such as

$$\frac{1}{2}, \; \frac{1}{2}\sqrt{2}, \; \frac{1}{2}(\sqrt{2})(\sqrt{2}) = \frac{1}{2}(\sqrt{2})^2, \; \frac{1}{2}(\sqrt{2})^3, \; . \; . \; .$$

7. *Geometric Means*: For two numbers, the geometric mean of them is the square root of the product of them. This is a sort of average, similiar to one-half of the sum, which is called the arithmetic mean. As for the average of a few numbers, we add them and divide the sum by the average of a few numbers, we add them and divide numbers, we multiply them and take the root of order equal to the number of them. For example, the geometric mean of

$$5, \quad 7, \quad 2, \quad 6$$

is $\sqrt[4]{(5)(7)(2)(6)} = \sqrt[4]{420}.$

This idea has been used for the modern scale.

8. *Modern Scale*: Since two sounds are compared in terms of the ratio of their number of vibrations rather than the difference of the number of vibrations, in order to make all intervals equal and call each one a "half tone," we need to take the geometric mean of twelve half-tones of the scale. Thus the number of vibrations of the sounds in the modern scale form a geometric progression which has 1 as a member and $\sqrt[12]{2}$ as its ratio. Thus the modern scale can be shown in the following table:

Sound	C	D	E	F	G	A	B	C_1
c	1	$(\sqrt[12]{2})^2$	$(\sqrt[12]{2})^4$	$(\sqrt[12]{2})^5$	$(\sqrt[12]{2})^7$	$(\sqrt[12]{2})^9$	$(\sqrt[12]{2})^{11}$	2

As we observe, the power of $\sqrt[12]{2}$ increases by 2 whenever we have a tone: and it increases by one whenever we have a half-tone.

The modern scale is not really as natural to the ear as the old Greek scale; but with slight training, the ear gets used to it. The important fact is that modulation from one key to another becomes extremely easy.

There is one disadvantage in the modern scale, namely, the third harmonic of C, *i.e.*, G, becomes slightly flat. The sound G is called the dominant of the scale and, being flat, makes the music dull. We shall show this fact mathematically. In the modern scale

$$\frac{G}{C} = (\sqrt[12]{2})^7 = 1.498$$

But, in the natural scale

$$\frac{G}{C} = \frac{3}{2} = 1.5.$$

This mistake is always corrected in the violin. This is one of the reasons that an orchestra with string instruments sounds much better than a piano solo.

9. *Major Keys*: A sample of the scale of a major key is the one in section 8. This is called "C major" since it starts with C. C is also called the tonic of the scale. In any major key, the sound (notes) of the scale have the same relation to one another as the ones in C major. That is, the interval between the third and fourth elements is one half-tone; also the interval between the seventh and eighth elements is a half-tone, and the other intervals are all one tone.

The next major key is G major. This has been chosen for two reasons. One is that the note G is the third harmonic of C; the other is that this key has a higher pitch. Note that going from C to its second harmonic does not change the scale. The table of the scale of this key is as follows.:

Sound	G	A	B	C_1	D_1	E_1	$F_1^\sharp$	G_1
c	$(\sqrt[12]{2})^7$	$(\sqrt[12]{2})^9$	$(\sqrt[12]{2})^{11}$	2	$(\sqrt[12]{2})^{14}$	$(\sqrt[12]{2})^{16}$	$(\sqrt[12]{2})^{18}$	$(\sqrt[12]{2})^{19}$

We observe that in order to have the interval between the seventh and eighth, *i.e.*, subtonic and tonic, a half tone, we have to use $F_1\sharp$ (F_1 sharp) with vibrations $(\sqrt[12]{2})^{18}$ instead of F_1 with $(\sqrt[12]{2})^{17}$.

If we choose the fifth note of this scale as the first of a new scale, we get the key of D major. This key needs two sharps.

The reader may try this idea and work out tables for several major keys which come after G major.

As it was possible to get major keys with higher pitch, it is also possible to get major keys with lower pitch.

Suppose we look at the table in section 8 and consider a scale for which the fifth note is C. This will have the following table:

Sound	F_0	G_0	A_0	$B_0^\flat$	C	D	E	F
c	$(\sqrt[12]{2})^{-7}$	$(\sqrt[12]{2})^{-5}$	$(\sqrt[12]{2})^{-3}$	$(\sqrt[12]{2})^{-2}$	1	$(\sqrt[12]{2})^2$	$(\sqrt[12]{2})^4$	$(\sqrt[12]{2})^5$

Here we have to use $B_0\flat$, *i.e.*, B flat, in order to have the interval between the third and fourth notes be a half-tone.

If we proceed in this way, each lower key has an extra flat. We leave it to the reader to produce many major keys and write tables for the corresponding scales.

10. *Minor Keys*: To imitate the crying sound of middle-east music, minor keys seem to be proper. Most older pieces written in minor keys avoid the very large interval followed by a half-tone, but we find this combination of sounds in recent pieces.

Many forms of minor keys have been considered. We shall describe only one of the most recent pieces.

To obtain a new scale, instead of going to the third harmonic of C, we may go to the fifth harmonic of C. But, this key is not the simplest minor key. Thus we move down to A, whose fifth harmonic is approximately C. The table of the scale for A minor is the following:

Sound	A_0	B_0	C	D	E	F	G$^\sharp$	A
c	$(\sqrt[12]{2})^{-3}$	$(\sqrt[12]{2})^{-1}$	1	$(\sqrt[12]{2})^{2}$	$(\sqrt[12]{2})^{4}$	$(\sqrt[12]{2})^{5}$	$(\sqrt[12]{2})^{8}$	$(\sqrt[12]{2})^{9}$

As the physical scale shows, it is desirable to have a half-tone interval between the subtonic and the tonic of a scale. This brings G♯ into the scale. As we see, the interval between F and G♯ is one and a half tones.

Other minor keys are obtained from this in a manner similar to that by which the major keys are obtained from C major. We leave it to the reader to obtain them.

It would be very interesting for one to compare his knowledge of music with what has been said here.

Geometric Algebra

by C. Stanley Ogilvy

It is amusing and often instructive to interpret algebraic identities from the point of view of geometry. This was the Greek way of looking at elementary algebra: when Euclid said the square *on* the hypotenuse he meant exactly that. The Greek mathematicians were always concerned with "what the algebra means to the geometry," although they did not phrase it in just those terms.

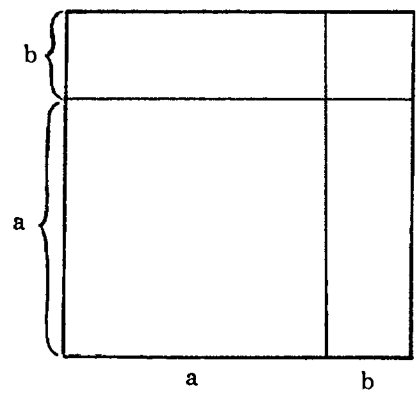


Figure 1.
$(a+b)^2=a^2+2ab+b^2$

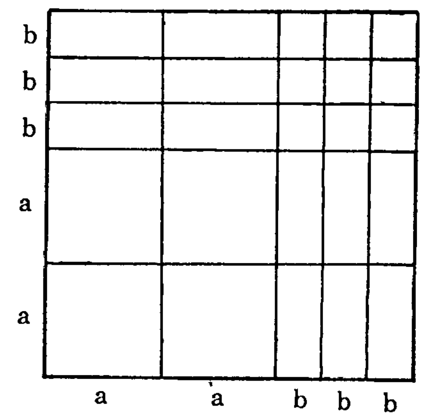


Figure 2.
$(2a+3b)^2=4a^2+12ab+9b^2$

We begin with an obvious example: Fig. 1 needs no further explanation. An easy extension is $(2a+3b)^2=4a^2+12ab+9b^2$, as pictured in Fig. 2. Perhaps not quite so familiar is the identity $(x+y+z)^2=x^2+y^2+z^2+2xy+2xz+2yz$, (Fig. 3). Only slightly more complicated is the diagram for $(a-b)^2=a^2-2ab+b^2$, (Fig. 4). Here

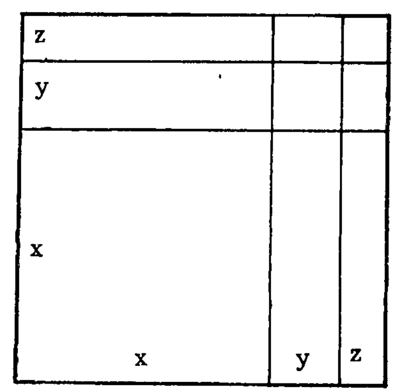


Figure 3.
$(x+y+z)^2 =$
$x^2+y^2+z^2+2xy+2xz+2yz$

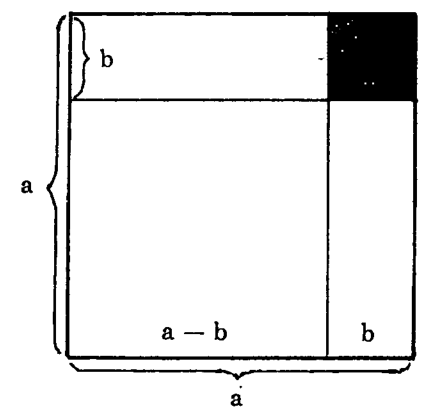


Figure 4.
$(a-b)^2=a^2-2ab+b^2$

the subtraction of 2ab removes too much: there is overlap to the extent of the shaded area, which must therefore be added back on again once, for we have subtracted it twice.

We look now at $a(b+c)=ab+ac$, an example of the distributive law. Note that Fig. 5 does not *prove* the distributive law; it merely indicates that rectangular areas as products of their sides are among those additive objects which behave in accordance with the law.

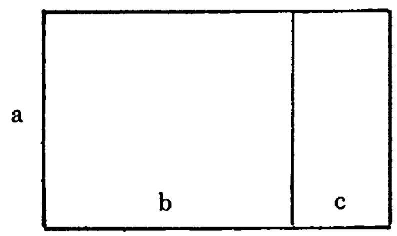
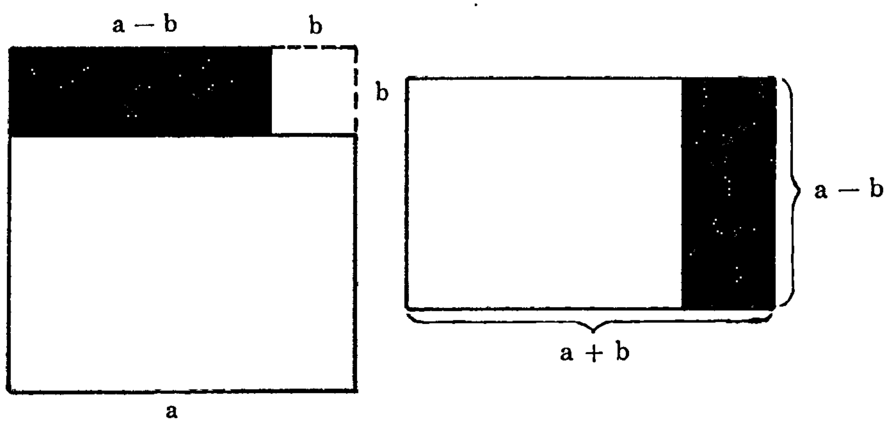


Figure 5. $a(b+c)=ab+ac$

When we come to $a^2-b^2=(a+b)(a-b)$, we have to rearrange the areas (Fig. 6). Let us do next a solid, (Fig. 7). There are many more. Can you think of any easy ones?



$a^2-b^2 \quad = \quad (a+b)(a-b)$

Figure 6.

We could also turn things around and ask algebra to do geometry for us. This is the modern mathematician's way. By means of the identity $(a+b)^2=a^2+2ab+b^2$ we can prove the pythagorean theorem:

$$a^2+2ab+b^2=(a+b)^2,$$

which, referring to Fig. 8,

$$=c^2+4 \text{ triangles}$$
$$=c^2+4(\tfrac{1}{2}ab)$$
$$=c^2+2ab.$$

Subtracting 2ab from both sides, $a^2+b^2=c^2$.

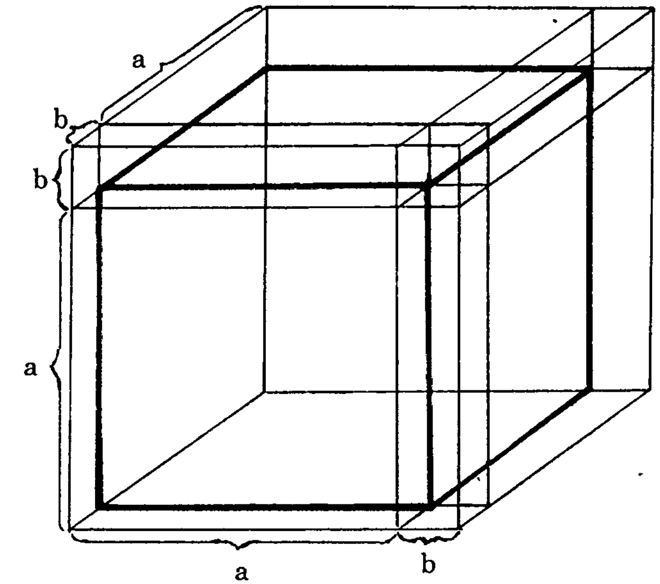


Figure 7. $(a+b)^3=a^3+3a^2b+3ab^2+b^3$
Large cube equals inner cube plus 3 slabs plus 3 columns plus small cube.

We could have done this one purely mechanically, in what we might call the engineer's way. Let the large square of Fig. 8 be a wooden frame and let the four triangles be wooden templates (draughtsman's 30-60 triangles would do nicely). Then we have only to move the triangles to the new positions of Fig. 9 to show that the open space previously allotted to c^2 has been redistributed into a^2+b^2.

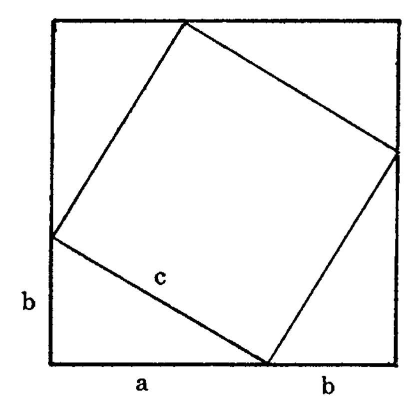


Figure 8

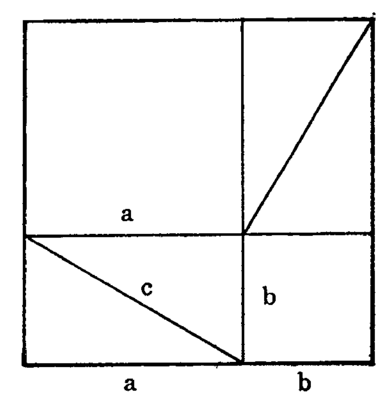


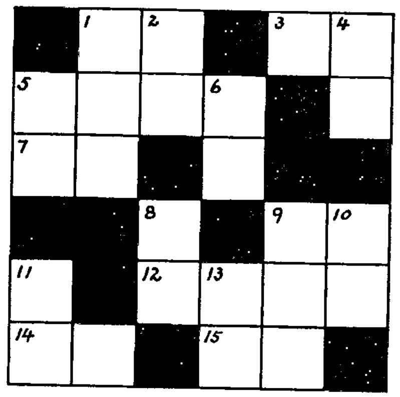
Figure 9

A Cross Number Puzzle

by J. A. H. Hunter

Bill went for a ride in his old jalopy yesterday, and here he tells us something about it and himself.

For some reason best known to Bill, he uses the term "Funny Figure" for any number which becomes nine less when its digits are reversed.



Across.

1. His average miles-per-gallon gas consumption.
3. His father's age, when Bill was eleven.
5. The year of his car.
7. Five times the number of gallons of gas he used.
9. Two-fifths of his father's age.
12. The square of his father's age.
14. Distance he drove in 3 hours at his average speed.
15. Number of gallons of gas he used.

Down.

1. The distance he drove in miles.
2. Twice his younger brother's age.
4. Reverse of "10 down".
5. His younger brother's age.
6. His average speed in miles-per-hour.
8. Distance he drove in 2 hours at that speed.
9. The square of his total gas consumption in gallons.
10. A "Funny Figure".
11. "1 Across" plus "9 Across".
13. Bill's age.

Puzzles And Problems

The puzzles and problems with an asterik, *, will be answered in the next (August) issue of RMM. The answers to the others will be found on pages 48-49.

Answers and comments may be sent to the editor.

1. Ages and Ages

Back in 1932, while talking with my grandfather, I happened to mentioned that I was old as the last two digits of the year of my birth. Grandfather promptly replied "Well, by gosh, Harry, that works out for me, too!" How old were we back in 1932? (H. V. Gosling)

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2. *High Stakes

Mike sat down and started shuffling the cards. "What stakes?" he asked.

"Let's make it a gamble," Steve replied, putting a few bills and some coins on the table. "The first game the loser pays one cent, the second two cents, and so on. Double up each time."

"Okay," laughed Mike, checking his cash. "I've got only $6.01, and I'm not playing more than ten games anyway."

So they played, and game followed game until at last Mike stood up, "That's my last cent I've just paid you," he declared, "but I'll have my revenge next week"

How many games had they played, and which did Mike win?

(J. A. H. Hunter)

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3. Planetary Daze

As astronauts are aware, the year is in not a constant among the planets of this solar system. The number of days in a year on Mercury, Venus and Earth altogether total 10 less than a Martian year. It takes 14 days more than 3 Venusian years to make a year on Mars. The number of days in a year on all four planets is 1362. How many days has a year on Mercury, Venus and Mars? (B. Newhoff)

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4. *Breakfast Mathematics

As a mathematician Mr. Smith noted everything in numbers. He knew, for example, that his coffee cup held exactly six swallows of coffee.

One morning he was hurrying to leave for work on time and didn't notice that his wife had filled his cup with coffee without putting in any cream. One swallow was enough to tell him and he promptly filled the cup with cream. But two more swallows made him decide the coffee was a bit strong and he filled it again with cream and drank half before he decided to fill the cup again with cream. He finally drank the whole cup and left for work.

While driving to work he wondered "Did I have more coffee or more cream for breakfast?" (G. Mosler)

Really Cutting Up, Now! (See inside back cover for solution).

5. *Squares at the Round Table

Mr. Smith, our accountant friend, had come home from work and found that his son left his little wooden blocks on the round table in the hall. Now the blocks were one inch on each side and Mr. Smith noted that the blocks were arranged in a square on the table. Just as he was about to count them, his little son came rushing in and grabbed about a third of the blocks for some rather important business elsewhere.. Undaunted, Smith rearranged the remaining blocks and found that he could make a rectangle 13 inches wide and have 7 blocks left over, or form a rectangle 15 inches wide and have 8 blocks left over. Noting that the round table holding the blocks was 17½ inches in diameter, Mr. Smith was able to figure out how many blocks his son took and how many were on the table when he first walked in. Can you? (W. R. Ransom)

6. *No Problem For An Accountant

Mr. Smith made a poor showing at his new job. On the first day, because of heavy traffic, he was only able to go 20 miles per hour and he arrived one minute late to work.

The next day his alarm clock failed him and he just managed to leave for work the same time as the day before. However, traffic wasn't quite so bad and by going 30 miles per hour he arrived one minute early to work.

Since Mr. Smith's new job was as an accountant he had no trouble figuring out how far from home his office was, how long it took him to drive each day, and how fast he should drive to arrive on time if he left the same time every day as on the first two days. (Edward L. Vaupel).

7. *The Doctor's Dilemma

Doctor Caput Nimbus, who generally has trouble recalling TV channels, grocery lists and a multitude of other things, was delighted with his new license plates.

"Not only are all the figures the same," he explained, "but they represent my birth year multiplied by the total of all my children and grandchildren."

Questioned about his family, he remarked that he had more sons than daughters. "My sons only have sons, each having one fewer sons than I have." he declared, "and my daughters only have daughters, each having one daughter fewer than I have."

His new license number would not have more than six figures anyway, but I'm still wondering what it was. Do you know? (D. Murdoch)

8. Some Extracurricular Activity

Our High School has five extra-curricular groups. They are the Choral, Chess, Photography, Literary and Political Clubs. The Choral group meets every other day, the Chess every third day, the Photography every fourth day, the Library every fifth day and the Political every sixth day.

The five groups first met on January 1st this year and thereafter meetings were held according to schedule.

How many times did all the Clubs meet on the same day in the first quarter, excluding January 1st?

How many days were there when none of the Clubs met in the first quarter? (H. V. Gosling)

9. An Airport Problem

A young man, about to take off in his plane, spotted a pretty girl on the concourse with lots of baggage but no prospect of a ride.

"Hi, did you miss your plane?" he asked.

"Yes, and now I'll have to wait until tomorrow for the next one."

"I'd be glad to give you a lift if you don't mind."

"But," she replied, "you don't know where I'm going."

"It doesn't matter. I can take you where you're going without going out of my way. I'll drop you off where you want and continue on my way."

Naturally the girl thought this was a fresh young man with a new line and refused his offer until he told her where he was headed. She realized he had been telling her the truth and went with him.

The questions remain, however. Where was the young man going and how far away was his destination? (Jack Halliburton)

10. The Case of The Alligator Handbags by Mel Stover

Private Eye Detective Agency,
Los Angeles, California
January 15, 1961

Operative 55,
New York City.

Dear Op.

I am investigating a kidnapping case and would like you to dig up some information for me.

The Bronx Leather Company gave a Christmas party for its 296 employees. They presented all who attended with gifts — 50 dollars to the men and 40 dollars to the women. The women received an additional gift of a genuine alligator handbag containing a twenty-five cent piece for luck.

Could you find out how many handbags were given away?

Sincerely,

Private

New York City
January 18, 1961

Dear Private:

As I am leaving town for a few days I was only able to give an hour to your case. In that time I made the acquaintance of a bookkeeper employed by the firm. I found out from him that all of the female employees attended the party but some of the men (he told me the percentage) were absent. I could not get any other information out of him but from that I was able to calculate the total amount of money the firm dispersed. But I haven't a clue as to the number of handbags. Sorry to be of so little help.

55

OP.
FROM YOUR LETTER WAS ABLE TO DEDUCE THE NUMBER OF HANDBAGS. LETTER FOLLOWING. THANKS FOR YOUR HELP.

EYE

(Private Eye's Letter is on Page 48)

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11. The 12 Coins Problem — by J. A. H. Hunter

The old problem of the counterfeit coin crops up from time to time to puzzle and infuriate new generations of solvers. It can be stated in simple terms as follows:

You have eleven perfectly identical coins, and also one counterfeit. The counterfeit coin is a perfect copy except that it is appreciably lighter or heavier than the genuine article, but you do not know whether it is lighter or heavier.

You also have a balance scale, without weights, and you are allowed to mark any of the coins as desirable.

In only three weighings, of coins against coins, you must identify the counterfeit coin and also ascertain whether it be light or heavy.

The popular solution is highly complex. There is, however a simple solution which may be of interest even to those who have dallied previously with this problem.

To avoid spoiling their fun for some, this solution is detailed on page 49. If the problem is new to you, just try it before peeking!

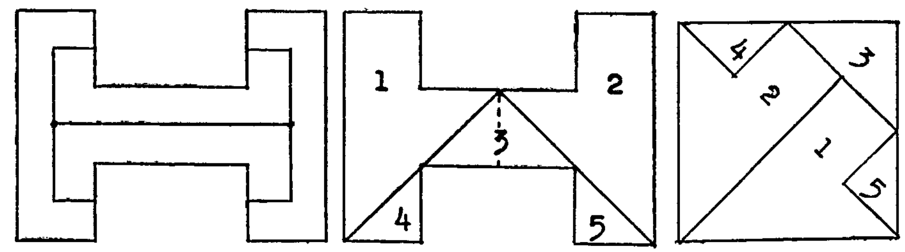
Answers to Puzzles in the April Issue

The answers to the *Word Games, Alphametics and Readers' Research Department* will be found in their respective sections in this issue. All other answers to the April issue follow.

GEOMETRIC DISSECTIONS (Page 6 in the April issue)

A 4-H Problem: The cuts are shown on the left, below, passing through the two points.

H^2: The H can be formed into a square (on the right) by two cuts along the solid lines in the middle drawing. However, by folding along the dotted line, only one cut is required to form the minimum five pieces.



A Problem in Multiple Division:

(1) Three of the many ways of forming two identical pieces are shown in Figures A, B, and C below.

(2) There are no solutions to the three identical pieces problem. Three pieces of the same shape can be cut, but the dots would not be in the same positions.

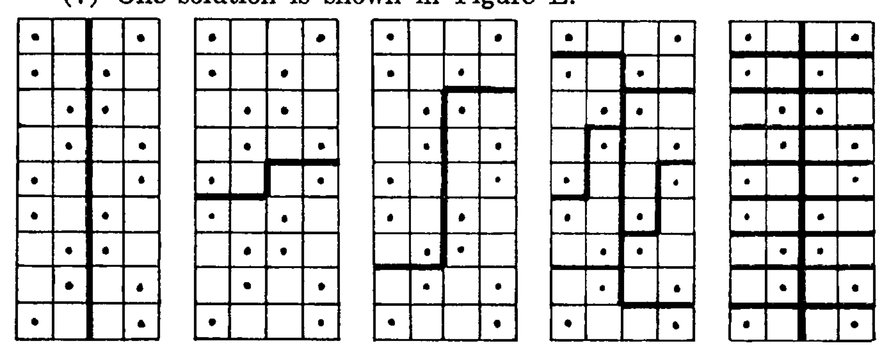
(3) Impossible - 18 dots cannot be distributed equally among 4 pieces.

(4) Figure D shows one solution.

(5) No solution - the dots in each piece would have to be on adjacent squares and two of the dots of the original diagram are in opposite corners of the figure *diagonally* related to the nearest dots.

(6) Impossible - 18 dots cannot be distributed equally among 12 pieces.

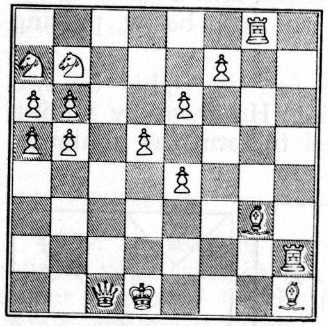
(7) One solution is shown in Figure E.



A B C D E

MAJOR PUZZLES (Pages 31-33 in the April issue)

1. Lecture Attendance: There were 3 female doctors, 27 male doctors, 3 Nurses and 2 Paratroopers.



2. Self Protection: The diagram shows a solution to the puzzle.

3. The Farmer's Financial Finagling: The oldest son sells 7 eggs for 9 cents and 3 eggs for 27 cents each; the next youngest sells 28 eggs at 9 cents per 7 and 2 eggs at 27 cents each; and the youngest sells 49 eggs at 9 cents per 7 and 1 egg at 27 cents.

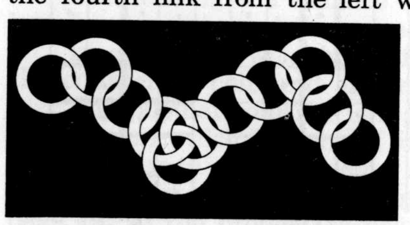
4. Moonshine Sharing: The solution to this puzzle hinges on being able to half fill (or half empty) the 10 quart container (cylindrical) by tipping until the overflow at the top is level with a point at the bottom of the can where the side and bottom meet. The solution is:

> Fill the 13 quart container, pour into the 10 quart container, leaving 3 quarts. Half empty the 10 quart container and pour the 5 quarts remaining into the 11 quart container and add the 3 quarts from the 13 quart container. There are now 8 quarts in the first moonshiner's 11 quart container. Repeat the above steps except pour the 3 quarts from the 13 quart container into the 10 quart container containing 5 quarts. There are now 8 quarts in the second moonshiner's 10 quart container. The barrel holds the remaining 8 quarts which can be poured into third moonshiner's 13 quart container.

5. The Jewel Box: The distance to the fourth corner is 7 inches (see the *Letters to the Editor* section on page 58). Solved correctly by Murray R. Falk of Calgary, Alberta)

6. Hit the Jackpot: A process of elimination by checking pot handles and visible numbers forces one to the conclusion that pot 9 held the Jackpot. (Solved correctly by Murray R. Falk)

7. The Divorcee's Dilemma: Only *one* cut is necessary. Cutting the fourth link from the left will divide the belt into two chains of 6 and 3 links each and 2 single links. She may now give one link the first day, another single link the second day, a 3-link chain in exchange for the 2 single links on the third day. This process continues on for eleven days.



3-D IN 2-D (Pages 51-53 in the April issue)

The Side View and Isometric View of Figure 5 is shown to the right.

Figure 6 corresponds to a cube from which a diagonal mass of material was removed.

Figure 7 is the internal segment of the intersection at right angles of two cylinders. Figure 8 is almost the same, being the intersection, at right angles, of two cylinders. However, the flange ends have been retained in addition to the internal segment.

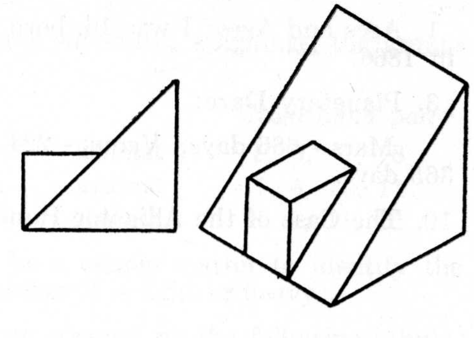


Figure 5

Figure 9 is the internal segment of the perpendicular intersection of three cylinders.

And, finally, figure 10 is the familiar Pullman car Dixie cup. The circle, square and triangle cross-sections are indicated.

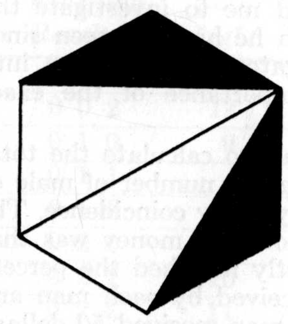


Figure 6

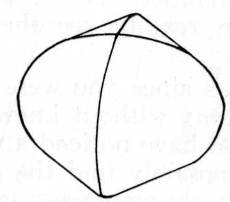


Figure 7

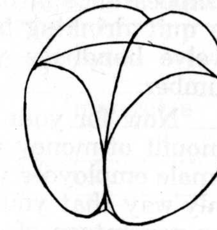


Figure 8

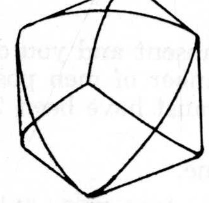


Figure 9

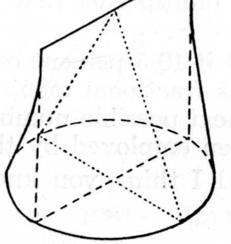


Figure 10

ANSWERS TO CERTAIN PUZZLES AND PROBLEMS ON PAGES 41-44.

1. Ages and Ages: I was 16, born in 1916; Grandfather was 66, born in 1866.

3. Planetary Daze:

Mars - 686 days; Venus - 224 days; Mercury - 87 days; Earth - 365 days.

10. The Case of the Alligator Handbags:

Private Eye Detective Agency
Los Angeles, California
January 20, 1961

Operative 55
New York City.

Dear Operative:

From the facts given in your letter I was able to deduce that exactly 96 handbags were given away by the firm.

My client who lives near the firm, asked me to investigate the disappearance of his eight pet alligators which he had not seen since he quit drinking last November. As one alligator will make up into twelve handbags you can readily see the importance of the exact number.

Now for your letter — since you were able to calculate the total amount of money given away without knowing the number of male or female employees you must have noticed a very lucky coincidence. The only way that you could possibly find the amount of money was that the percentage of absent male employees exactly matched the percentage difference between the monetary gift received by each man and the monetary gift received by each woman. A man received 50 dollars, a woman $40.25 or 19.5 percent less. The bookkeeper must have told you that 19.5 percent of the men were absent. This fact would allow you to calculate that the "average" male employee received $40.25, then by multiplying that figure by 296 you could find the firm's cash outlay.

But if 19.5 percent of the men were absent and you disallow such things as fractional men, the smallest number of men possible is 200. as the next possible number is 400 there must have been 200 men and 96 women employed by the firm.

And I think you knew it all the time.

Sincerely
Private Eye

11. The 12 Coins Problem

1. Number the coins from 1 to 12.
2. Weigh four against four, in three successive weighings, the groups being selected as follows.

	Lefhand pan		*Righthand pan*
1st weighing	1, 2, 3, 4	against	5, 6, 7, 8
2nd weighing	3, 4, 8, 9	against	1, 5, 10, 11
3rd weighing	2, 6, 8, 10	against	3, 5, 11, 12

3. From the three results it will be a simple matter to identify the counterfeit coin, and to see whether it is light or heavy.
4. For quick reference, all cases are covered by the following tabulation, the results of the three successive weighings being indicated by the code:

Even Balance	-	0
LEFT pan light	-	1
LEFT pan heavy	-	2

Weighing Results	*Coin Light*	*Coin Heavy*	*Weighing Results*	*Coin Light*	*Coin Heavy*
0 0 1		12	1 1 2	3	
0 0 2	12		1 2 0	1	
0 1 0	9		1 2 1	Impossible	
0 1 1		11	1 2 2		8
0 1 2		10	2 0 0	7	
0 2 0		9	2 0 1	6	
0 2 1	10		2 0 2		2
0 2 2	11		2 1 0		1
1 0 0		7	2 1 1	8	
1 0 1	2		2 1 2	Impossible	
1 0 2		6	2 2 0		4
1 1 0	4		2 2 1		3
1 1 1		5	2 2 2	5	

Interested obtaining the first (February 1961) RMM? See Editorial on page 2.

PRIME GENERATING POLYNOMIALS

by Sidney Kravitz

No polynomial can represent primes exclusively. However:

(1) At an unknown date Euler found that x^2+x+17 generates 16 different primes for $x = 0, 1, 2, \ldots 15$.

(2) In 1798 Legendre noted that $2x^2+29$ generates 29 different primes for $x = 0, 1, 2, \ldots 28$.

(3) In 1772 Euler discovered that x^2-x+41 generates 40 different primes for $x = 1, 2, 3, \ldots 40$.

(4) By substituting $(y-39)$ for x in x^2-x+41, Escott found, in 1899, that $y^2-79y+1601$ generates 40 different primes for $y = 0, 1, 2, \ldots 79$.

(5) By substituting $(-3w+82)$ for x in x^2-x+41 I find that $9w^2-489w+6683$ generates 40 different primes for $w = 1, 2, 3, \ldots 40$.

The following are additional facts about the prime generating polynomial x^2-x+41.

(1) Between $x=1$ and $x=2398$ inclusive, precisely one half of the numbers generated are prime.

(2) This polynomial is never divisible by a prime less than 41.

(3) The lowest positive x generating a number with two factors is $x=41$; three factors first occur at $x=421$; four factors first occur at $x=1722$. Five factors first occur somewhere between $x=10,764$ and $x=84,420$. Six factors first occur at $x=139,564$.

* * * *

The dates listed here were obtained from Dickson's HISTORY OF THE THEORY OF NUMBERS (Chelsea Publishing Co.).

It can easily be verified that when 3^{10} is divided by 10 there is a remainder of 10; when 3^{15} is divided by 15 there is a remainder of 12; and when 3^{19} is divided by 19 there is a remainder of 3. Would anyone care to disprove the fact that if $3^{18584774046020617}$ is divided by 18584774046020617 there is a remainder of 3?

(Alan L. Brown)

The answer to (A) (FLUSH) = TRUMPS is (6) (45183) = 271098. (See *Letters to the Editor page* 57).

Readers' Research Department

It appears that the first Research Problem in the April issue proved a bit tough. There were no adequate analyses of either the circle problem (what is the maximum number of regions into which a plane can be divided by M circles) or the sphere problem (what is the maximum number of regions into which space can be divided by S spheres).

Richard Body (age 14) of Calgary, Alberta; Maxey Brooke of Sweeny, Texas; James V. Ralston, of Exeter, New Hampshire; J. A. H. Hunter, of Toronto, Ontario; and the editor all agree that the formula for M circles is $N_r=M^2-M+2$ where N_r is the maximum number of regions on the plane.

There was no agreement on the formula for S spheres.

The second problem, the determination of the best strategy for playing the game of Dots and Squares (or Square it, or the French Polytechnic School's Game), also yielded no analyses.

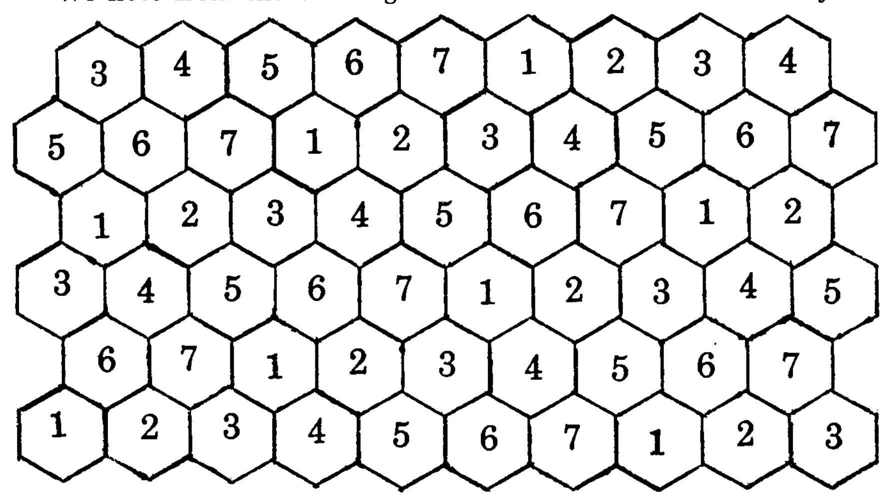
It was noted by one reader that if we adopt the first rule (the players play one stroke alternately throughout the game) the second player can easily win on the board with an odd number of squares by simply playing symmetrically opposite the stroke made by the first player. He will then win with one square more than the first player.

However, the second rule, usually adhered to, (players play alternately unless a square is completed which gives an extra stroke to the player - the extra stroke continuing as long as the player completes squares) yielded no analyses.

Since there are two problems held over, we shall present only one problem for research in this issue.

The problem was suggested by Leo Moser of the University of Alberta. What is the least number of colors with which one can color a plane in such a way that no pair of points unit distance apart are colored the same?

We note from the drawing below that seven colors certainly suf-



fice. The plane is filled with regular hexagons such that the radius of the circumscribing circle is slightly less than one, say 0.99, and the hexagons are colored with seven colors (indicated by numbers). The pattern used fulfills the general condition of the problem - but is seven the *least* number of colors required?

Naturally, we don't suggest that hexagons are the correct pattern. The solution may involve regular or irregular figures. The problem might be worked from the other end, viz., show that *at least* four, or five, or possibly six colors are required.

The next issue of RMM will have a geometry research problem and a number-array problem.

Fowl Play

by William R. Ransom

Is it fair play to ask one to solve for four unknowns when only two equations are given?

A farmer sold 22 birds, an odd number of each. He got \$3.60 each for the geese, \$1.10 for the ducks, \$1.15 for the hens, and \$0.40 for the pigeons: that made a total of \$22.35.

Represent the number of each by g, d, h, and p, respectively. Then we have the equation:

$$360\,g + 110\,d + 115\,h + 40\,p = 2235$$

which, divided by 5, gives the equation

$$72\,g + 22\,d + 23\,h + 8\,p = 447$$

Also

$$g + d + h + p = 22$$

Multiplying the second equation by 22 allows us to eliminate d, and multiplying it by 23 allows us to eliminate h. Then

$$d = 49\,g - 15\,p + 59$$
$$h = 14\,p - 50\,g - 37$$

Now g is at least 1, in which case

$$d = 49 - 15\,p + 59 = 108 - 15\,p$$
$$h = 14\,p - 50 - 37 = 14\,p - 87$$

Since d and h are positive

$$108 - 15\,p > 0,\ p < {}^{108}/_{15} = 7\,{}^{1}/_{5}$$
$$14\,p - 87 > 0,\ p > {}^{87}/_{14} = 6\,{}^{11}/_{14}$$

so if $g = 1$, $p = 7$, and that makes $d = 3$ and $h = 11$.

This is the only possible solution, for since g is odd, if it is not 1 it must be greater than 2. But if $g > 2$,

$$d > 98 - 15p + 59$$
$$h < 14p - 100 - 37$$

and so

$$157 - 15p \geqq 0,\ \text{so}\ p < 10\,{}^{7}/_{15}$$
$$14p - 137 \leqq 0,\ \text{so}\ p > 11\,{}^{11}/_{14}$$

But p cannot be less than the smaller and more than the greater.

This problem is adapted from page 141 of Vol. I of Schubert's *Mussenstunde.*

The Next 496 Prime Numbers - 10007 to 14737

A total of four errors found their way into the prime table in the April issue. 7917, 9027 and 9399 *should read* 7919, 9029 and 9397, respectively; 9049 was missing and should be included.

The primes listed below have been doubly proofread against two independently derived tables of primes, both of which agreed perfectly between themselves. One list was derived from D. N. Lehmer's Factor Table (Carnegie Institution of Washington, 1909) by Alan L. Brown of East Orange, New Jersey. The other list was computer calculated by Vernon J. Shipley of Kitchener, Ontario.

10007

10009	10429	10861	11299	11779	12161	12569	12979	13417	13841	14327
10037	10433	10867	11311	11783	12163	12577	12983	13421	13859	14341
10039	10453	10883	11317	11789	12197	12583	13001	13441	13873	14347
10061	10457	10889	11321	11801	12203	12589	13003	13451	13877	14369
10067	10459	10891	11329	11807	12211	12601	13007	13457	13879	14387
10069	10463	10903	11351	11813	12227	12611	13009	13463	13883	14389
10079	10477	10909	11353	11821	12239	12613	13033	13469	13901	14401
10091	10487	10937	11369	11827	12241	12619	13037	13477	13903	14407
10093	10499	10939	11383	11831	12251	12637	13043	13487	13907	14411
10099	10501	10949	11393	11833	12253	12641	13049	13499	13913	14419
10103	10513	10957	11399	11839	12263	12647	13063	13513	13921	14423
10111	10529	10973	11411	11863	12269	12653	13093	13523	13931	14431
10133	10531	10979	11423	11867	12277	12659	13099	13537	13933	14437
10139	10559	10987	11437	11887	12281	12671	13103	13553	13963	14447
10141	10567	10993	11443	11897	12289	12689	13109	13567	13967	14449
10151	10589	11003	11447	11903	12301	12697	13121	13577	13997	14461
10159	10597	11027	11467	11909	12323	12703	13127	13591	13999	14479
10163	10601	11047	11471	11923	12329	12713	13147	13597	14009	14489
10169	10607	11057	11483	11927	12343	12721	13151	13613	14011	14503
10177	10613	11059	11489	11933	12347	12739	13159	13619	14029	14519
10181	10627	11069	11491	11939	12373	12743	13163	13627	14033	14533
10193	10631	11071	11497	11941	12377	12757	13171	13633	14051	14537
10211	10639	11083	11503	11953	12379	12763	13177	13649	14057	14543
10223	10651	11087	11519	11959	12391	12781	13183	13669	14071	14549
10243	10657	11093	11527	11969	12401	12791	13187	13679	14081	14551
10247	10663	11113	11549	11971	12409	12799	13217	13681	14083	14557
10253	10667	11117	11551	11981	12413	12809	13219	13687	14087	14561
10259	10687	11119	11579	11987	12421	12821	13229	13691	14107	14563
10267	10691	11131	11587	12007	12433	12823	13241	13693	14143	14591
10271	10709	11149	11593	12011	12437	12829	13249	13697	14149	14593
10273	10711	11159	11597	12037	12451	12841	13259	13709	14153	14621
10289	10723	11161	11617	12041	12457	12853	13267	13711	14159	14627
10301	10729	11171	11621	12043	12473	12889	13291	13721	14173	14629
10303	10733	11173	11633	12049	12479	12893	13297	13723	14177	14633
10313	10739	11177	11657	12071	12487	12899	13309	13729	14197	14639
10321	10753	11197	11677	12073	12491	12907	13313	13751	14207	14653
10331	10771	11213	11681	12097	12497	12911	13327	13757	14221	14657
10333	10781	11239	11689	12101	12503	12917	13331	13759	14243	14669
10337	10789	11243	11699	12107	12511	12919	13337	13763	14249	14683
10343	10799	11251	11701	12109	12517	12923	13339	13781	14251	14699
10357	10831	11257	11717	12113	12527	12941	13367	13789	14281	14713
10369	10837	11261	11719	12119	12539	12953	13381	13799	14293	14717
10391	10847	11273	11731	12143	12541	12959	13397	13807	14303	14723
10399	10853	11279	11743	12149	12547	12967	13399	13829	14321	14731
10427	10859	11287	11777	12157	12553	12973	13411	13831	14323	14737

Numbers, Numbers, Numbers

The editor would like to add a few notes received from various readers pertaining to the Perfect numbers given in the April issue.

It has been observed that all Perfect numbers, greater than 6, have digital roots of 1, i.e., the ultimate sum of their digits equals 1.

$V_2=28 \quad 2+8=10 \quad 1+0=1$

$V_3=496 \quad 4+9+6=19 \quad 1+9=10 \quad 1+0=1$

$V_4=8{,}128 \quad 8+1+2+8=19 \quad 1+9=10 \quad 1+0=1$

$V_5=33{,}550{,}336 \quad 3+3+5+\text{etc.}=28 \quad 2+8=10 \quad 1+9=1$

and so on for all 18 Perfect numbers.

Every Perfect number greater than 6 is the sum of consecutive odd cubes, beginning with 1.

$V_2=28=1^3+3^3$

$V_3=496=1^3+3^3+5^3+7^3$

$V_4=8{,}128=1^3+3^3+5^3+7^3+9^3+11^3+13^3+15^3$

$V_5=33{,}550{,}336=1^3+3^3+5^3+\ \ldots\ldots\ +127^3$

and similarly for the remaining Perfect numbers.

Malcolm H. Tallman, of Brooklyn, N.Y., points out that all the Perfect numbers are the sums of successive powers of 2 from 2^{p-1} to 2^{2p-2}, where Perfect numbers are of the form $2^{p-1}(2^p-1)$ as pointed out in the April issue.

$V_1=2^1(2^2-1)=6=2^1+2^2$

$V_2=2^2(2^3-1)=28=2^2+2^3+2^4$

$V_3=2^4(2^5-1)=496=2^4+2^5+2^6+2^7+2^8$

$V_4=2^6(2^7-1)=8{,}128=2^6+2^7+2^8+2^9+2^{10}+2^{11}+2^{12}$

$V_5=2^{12}(2^{13}-1)=33{,}550{,}336=2^{12}+2^{13}+2^{14}\ \ .\ \ .\ \ .\ \ .\ \ +2^{24}$

and so on.

Alan L. Brown, of East Orange, New Jersey, observes that the Perfect numbers look much more interesting when written in Base 2 (for number systems, see the series of articles in this issue beginning on page 1).

$V=2^{p-1}(2^p-1)$	Base 10	Base 2
$V_1=2^1(2^2-1)$	6	110
$V_2=2^2(2^3-1)$	28	11100
$V_3=2^4(2^5-1)$	496	111110000
$V_4=2^6(2^7-1)$	8,128	1111111000000
$V_5=2^{12}(2^{13}-1)$	33,550,336	1111111111111000000000000

and so on. The number of ones is equal to p and the number of zeros following is equal to p—1.

In the next issue of RMM (August 1961) the editor will do his best to have the full values of the 18 Perfect numbers.

Francis L. Miksa, of Aurora, Illinois, who has compiled some fantastically large tables of Pythagorean triangles, Binomial Coefficients,

and miscellaneous other tables, culled from his tables of integer solutions to

$$x^2+y^2+z^2+w^2=t^2$$

some interesting reversed-digit solutions, some of which include:

x	y	z	w	t	x	y	z	w	t
03	12	22	42	49	12	12	58	72	94
30	21	22	24	49	21	21	85	27	94
01	28	40	72	87	11	31	17	15	98
10	82	04	27	87	11	13	71	51	98
02	27	44	70	87	13	37	55	71	98
20	72	44	07	87	31	73	55	17	98
05	12	26	82	87	15	33	57	71	98
50	21	62	28	87	51	33	75	17	98

H. V. Gosling, of Kingston, Ontario, completes the remainder of the *Numbers, Numbers, Numbers* department with some interesting miscellaneous observations.

1. Here are two series of whole numbers written in geometric progression whose sums are perfect squares:

$$1+3+9+27+81=121=11^2$$

$$1+7+49+343=400=20^2$$

Are there others?

2. Having Fun With Digits

a. Reverse Equations

$001=(0^0)^1 \qquad 36=(6)(3!)$

$2.5=5:2 \qquad 64=\sqrt{4^6}$

$24=\sqrt{(4!)^2} \qquad 71=\sqrt{1+7!}$

$25=5^2 \qquad 125=5^{2+1}$

b. Same-Order Equations

$387{,}420{,}489=3^{87+420-489} \qquad 456=(4)(5!-6)$

$46656=[-(4)(6)+(6)(5)]^6 \qquad 384=(3!)(\sqrt{8^4})$

$16384=(\sqrt{16})^{3+8-4} \qquad 360=(3!)(60)$

$3125=(3!+1-2)^5 \qquad 360=[3][(6-0!)!]$

$660=6!-60 \qquad 355=(3)(5!)-5$

c. Some Neat Equivalents

$$1 = \frac{(1)(1)}{1}$$

$$121 = \frac{(22)(22)}{1+2+1}$$

$$12321 = \frac{(333)(333)}{1+2+3+2+1}$$

$$1234321 = \frac{(4444)(4444)}{1+2+3+4+3+2+1}$$

$$123454321 = \frac{(55555)(55555)}{1+2+3+4+5+4+3+2+1}$$

$$12345654321 = \frac{(666666)(666666)}{1+2+3+4+5+6+5+4+3+2+1}$$

$$1234567654321 = \frac{(7777777)(7777777)}{1+2+3+4+5+6+7+6+5+4+3+2+1}$$

$$123456787654321 = \frac{(88888888)(88888888)}{1+2+3+4+5+6+7+8+7+6+5+4+3+2+1}$$

$$12345678987654321 = \frac{(999999999)(999999999)}{1+2+3+4+5+6+7+8+9+8+7+6+5+4+3+2+1}$$

3. Miscellanea

a. $(^5/_8)^2 + {^3/_8} = (^3/_8)^2 + {^5/_8}$

b. The editor, in the February RMM, showed 113 ways of writing 100 using the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9 *in order*. H. E. Dudeney (*Amusements in Mathematics*, Dover Publications, page 158) shows how to arrange the nine digits, not necessarily in order, as fractions to equal 100.

$96^{2148}/_{537}$ $96^{1752}/_{438}$ $96^{1428}/_{357}$ $94^{1578}/_{263}$ $91^{7524}/_{836}$

$91^{5823}/_{647}$ $91^{5742}/_{638}$ $82^{3546}/_{197}$ $81^{7524}/_{396}$ $81^{5643}/_{297}$ $3^{69258}/_{714}$

4. Problems

a. Find an integer solution for $a^4+b^4+c^4+d^4=e^4$

b. Likewise for $a^6+b^6+c^6=d^6+e^6+f^6$

Letters to the Editor

Dear Sir:

I would like to correct your statement (April RMM, page 45) that the IBM 709 found the Mersenne prime M_{3217}.

Hans Riesel announced this discovery on September 8, 1957 using the Swedish electronic digital computer BESK for 5½ hours. D. Scheffler and I calculated the 18th Perfect number using M_{3217} found by Riesel. The work was done here at NAFEC on November 17, 1959.

Rudolph Ondrejka
NAFEC
Atlantic City, N. J.

Dear Mr. Madachy:

In *Mathematical Tables and Other Aids to Computation*, Vol. XII, No. 61, page 60 (January 1859) "A New Mersenne Prime" was announced by Hans Riesel. The numerical evaluation of the 18th Perfect Number appears in *Mathematics of Computation*, Vol. 14, No. 70, pages 199-200 (April 1960) (D. Scheffler and R. Ondrepka - "The Numerical Evaluation of the 18th Perfect Number").

Sidney Kravitz
Dover, N. J.

The editor humbly admits his error and thanks both of the correspondents for the information and copies of the 18th Perfect Number.

Dear Mr. Madachy:

There appears to be some doubt as to the card Mrs. Nelson drew to fill her flush in (A) (SPADE)=FLUSH Alphametic (April RMM, page 15). Maybe it was a 2 (with SPADE 38215, 39215, 35218, 35219 and FLUSH 76430, 78430, 70436, 70438), or a 4 (with SPADE 17453 and FLUSH 69812).

As a matter of interest, a rather more intricate alphametic results with this variation: (A) (FLUSH)=TRUMPS. I cannot, however, hazard any guess as to what game is being played!

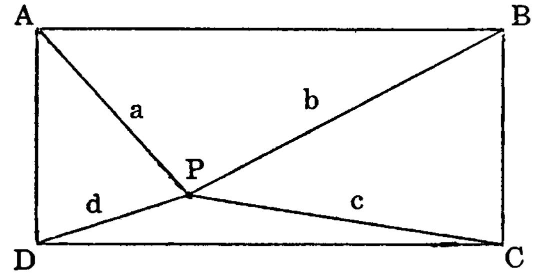
Derick Murdoch
Willowdale, Ontario

As a straight Alphametic there may be several solutions for (A) (SPADE) = FLUSH but the reader is referred to page 27 for another answer.

As for the game which makes (A) (FLUSH)=TRUMPS the reader is referred to Mr. Bunge's articles on page 24. A game *could* be devised! (Oh, yes, the answer to Mr. Murdoch's little Alphametic will be found elsewhere in this issue).

Dear Sir:

In finding the answer to the Jewel Box Problem (April RMM, page 32) I noted that there is a neat mathematical relationship between any point and the four corners of a rectangle, including a square. The sum of the squares of the distances of a point to two opposite corners of a rectangle is equal to the sum of the squares of the distances of that point to the other two corners. This works even if the point is inside or outside the rectangle, or even on a corner or side of the rectangle.



i.e. $a^2+c^2=b^2+d^2$

U. Clid
Cleveland, Ohio

The editor notes that relationship holds even if the point is not in the same plane as the rectangle. The readers are left to their own devices or must assume the correctness of the proposition.

Dear Mr. Madachy:

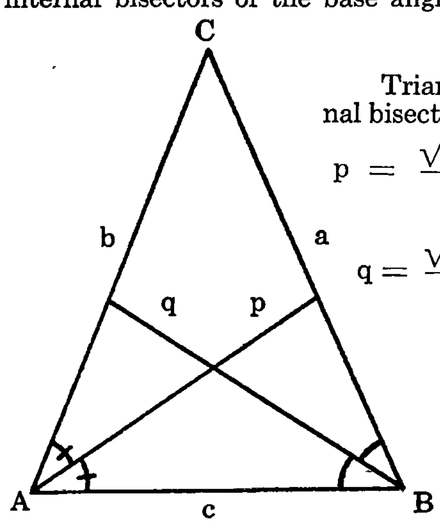
While studying the problem of the two factors of 10^9 and 10^{18} which contain no zeroes (February RMM, page 44 and the answers in the April RMM, page 29) I find that the next larger power of 10 having two factors without zeroes is 10^{33}:

$10^{33} = (2^{33})(5^{33}) = (8{,}589{,}934{,}592)(116{,}415{,}321{,}826{,}934{,}814{,}453{,}125)$

T. H. Engel
Forest Hills, L. I.

Dear Mr. Madachy:

To add to the many, many proofs of Lehmus' Theorem (If the internal bisectors of the base angles of a triangle are equal, then the triangle is isosceles), here is a non-geometrical proof.



Triangle ABC has sides a, b, c and internal bisectors p and q. Say, $2s=a+b+c$. Then,

$$p = \frac{\sqrt{ac[(a+c)^2-b^2]}}{a+c} = \frac{2\sqrt{acs(s-b)}}{a+c}$$

$$q = \frac{\sqrt{bc[(b+c)^2-a^2]}}{b+c} = \frac{2\sqrt{bcs(s-a)}}{b+c}$$

But, by definition, $p=q$, hence:

$$\frac{a(a+c-b)}{(a+c)^2} = \frac{b(b+c-a)}{(b+)^2}$$

whence, $(a-b)[c^3+(a+b)c^2+3abc+ab(a+b)]=0$. The polynomial in c cannot be zero, hence $a-b=0$, so $a=b$ which proves the theorem.

J. A. H. Hunter
Toronto, Ontario

Possibly some readers can come up with other proofs of Lehmus' theorem and also submit lists of references to proofs worked out by others. The April RMM Letters to the Editor department listed four references, but there are many more. References giving the author, publication, date, page, and title are most desirable. If an abstract of the highlights of the proof are given this would be nearly perfect. Anyone who has a collection can rest assured that the editor will return them after copies have been made. If this proves an interesting project to RMM readers, the bibliography will be published in a future issue of RMM.

This particular theorem has intrigued many mathematicians because of its apparent simplicity - but insidious difficulty. Archibald Henderson wrote a 40-page paper on the theorem for the *Journal of the Elisha Mitchell Scientific Society* in December 1937. H. S. M. Coxeter, in his new book *Introduction to Geometry* (John Wiley & Sons), discusses the theorem in considerably less space.

Dear Mr. Madachy:

It is mentioned that Sherlock Holmes wrote a treatise on the Binomial Theorem. Does anyone know in which adventure Holmes states this?

For those interested there are other fictional detectives with mathematical leanings:

August Dupin in *The Purloined Letter* (Poe) discusses probability.

Philo Vance in *The Bishop Murder Case* (van Dine) discusses non-Euclidean geometry.

Nero Wolfe used his knowledge of ancient Arabic numbers to solve a case - but I forgot which case.

There must be many more.

Maxey Brooke
912 Old Ocean Ave.
Sweeny, Texas

Dear Mr. Madachy:

One can have as many composite numbers as is desired:

$$A!+2,\ A!+3,\ A!+4,\ \ldots,\ A!+A$$

which would be $A-1$ consecutive composite numbers. But you can't have two numbers B and 2B without a prime between them.

I can't prove this. Can some RMM readers come up with the answer?

W. R. Ransom
Reading, Massachusetts

Bibliography

Some readers may wish to delve further into some of the ideas presented in some of the articles and departments in this issue of RMM. We give a brief list of suggested references below.

Base-n Series

JOHNSON, D. A., & W. H. GLENN *Understanding Numeration Systems*, Webster Publishing Co., St. Louis, 1960. One of a series of booklets called "Exploring Mathematics on Your Own." This particular booklet does a wonderful job of introducing the other number systems.

ROSSMASSLER, RICHARD Number Systems Other Than Ours *The Mathematics Student Journal*, Vol. 7, No. 4, pages 1-4 (May 1960). A high school student admirably discusses the binary and duodecimal system.

DUODECIMAL SOCIETY OF AMERICA, INC. *Manual of the Dozen System*, Duodecimal Society of America, Inc., 20 Carlton Place, Staten Island, N.Y., 1960. An authoritative manual of the methods and meaning of the duodecimal system.

Mathematics of Music

AMIR-MOEZ, ALI Numbers and the Music of the East and West *Scripta Mathematica* Vol. 22, pages 268-270 (1956)

JEANS, SIR JAMES *The Mathematics of Music* in James R. Newman's *World of Mathematics* published by Simon & Schuster, 1956, pages 2278-2309. This is an excerpt from Sir James Jeans' book *Science and Man.*

KLINE, MORRIS *Mathematics in Western Culture*, Oxford University Press, 1953, pages 287-303.

Geometric Algebra

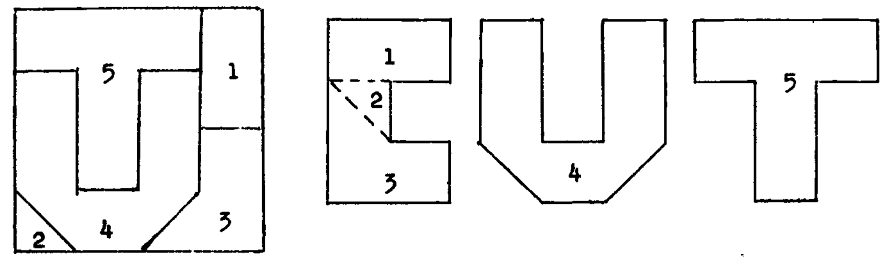
HOGBEN, LANCELOT *Mathematics for the Million*, W. W. Norton & Co., Inc., N. Y., 1951 (Third Edition). Some of the problems discussed by Mr. Ogilvy are given as exercises along with others, on pages 107-108.

The Haunted Checkerboards

GARDNER, MARTIN Mathematical Games *Scientific American* (August 1959, pages 129-130). An explanation of the apparent paradoxes of Mr. Brooke's jigsaw puzzles are shown by Mr. Gardner to be related to the Fibonacci Series and the Golden Ratio.

The 12-Coins Problem

The Amateur Scientist *Scientific American* (May 1955, pages 120-126). An account of the variations to this famous puzzle-type. It includes the instructions for constructing a slide rule to solve the problem; a different type of table than Mr. Hunter's; and some odd notes such as that with only *seven* weighings one can find the odd coin among 1092 otherwise identical coins!



ERRATA

Page 42: The phrase "*Really Cutting* . . . at the top of the page should have been at the top of page 46.

Page 43: Problems 8 and 9 will be answered in the August issue and should have been starred, *.

Page 57: For Derick Murdoch read Derrick Murdoch.