Reconciling Alternative Definitions of Graded Cohen-Macaulay Rings

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Introduction

What does it mean for a graded ring to be Cohen-Macaulay? In the literature one finds many different answers to this question, and it can be rather difficult to prove (or even to find proofs) that all the different answers are equivalent. The purpose of this note is to rectify this situation by collecting all the different definitions and references together in one place. The reader is advised *not* to skip over the list of definitions below since the whole point of this note is to address subtle differences in terminology.

Basic definitions

Our main reference for basic definitions and terminology is [7]. For terms used but not defined below, consult either [1] or [7]. All rings in this paper will be commutative with identity.

Let A be a ring. A *chain* of ideals in A is a nested sequence of ideals $I_0 \subseteq \cdots \subseteq I_n$; the integer n here is the *length* of the chain. The *height* of a prime ideal P (denoted $\operatorname{height}(P)$) is the maximum length of a chain of prime ideals contained in P. The *Krull dimension* or simply the *dimension* of A (denoted $\dim(A)$) is the supremum of the heights of all its prime (or, equivalently, maximal) ideals. If M is an A-module then the *dimension* of M (denoted $\dim(M)$) is the dimension of the ring $A/\big(\operatorname{ann}(M)\big)$, where $\operatorname{ann}(M)$ is the annihilator of M.

Let I be an ideal in A, and let M be an A-module. A sequence of elements $a_1, \ldots, a_n$ in I is called an *M-sequence in I* if a_i is not a zero divisor of $M/(a_1, \ldots, a_{i-1})M$ for each $i \in \{1, \ldots, n\}$, and $M/(a_1, \ldots, a_n)M \neq 0$. The maximum length of an M-sequence in I is called the *I-depth* of M. In the case where $M = A$, which is the main case that we will consider, the I-depth of A is simply called the *depth* of I (denoted $\operatorname{depth}(I)$). If A is a local ring and I is the maximal ideal of A then the I-depth of M is simply called the *depth* of M.

If P is a prime ideal of A, we denote the localization of A at P by A_P and the localization of an ideal $I \subseteq A$ by I_P.

Graded rings

We say that a ring A is *graded* if it is supplied with a family $(A_n)_{n\geq 0}$ of subgroups of the additive group of A, such that $A = \bigoplus_{n=0}^{\infty} A_n$ and $A_m A_n \subseteq A_{m+n}$ for all m and n.

An element $a \in A$ is said to be *homogeneous* if $a \in A_n$ for some $n \geq 0$; n is said to be the *degree* of a. An ideal that is generated by homogeneous elements is called a *homogeneous ideal.* The notions of height, dimension, M-sequences and depth all have their obvious graded analogues heightgr, dimgr, homogeneous M-sequences and depthgr. The *irrelevant ideal* of A is defined to be the ideal $M = \bigoplus_{n \geq 1} A_n$.

In this paper the graded rings that we shall be mainly concerned with will be *finitely generated graded k-algebras,* i.e., graded rings A such that A_0 is (isomorphic to) a field k and such that A is finitely generated as a k-algebra. Note that finitely generated graded k-algebras are Noetherian ([7, remark after Theorem 3.3] or [1, Corollary 7.7]).

If A is a finitely generated graded k-algebra, then clearly each graded part A_n is a vector space over k; denote its dimension as a vector space by $H(A, n)$. Because A is finitely generated over k, $H(A, n)$ is finite for all n. The *Hilbert series* or *Poincaré series* $F(A, \lambda)$ of A is defined to be the formal power series

$$F(A, \lambda) = \sum_{n=0}^{\infty} H(A, n)\lambda^n.$$

The following definition does not appear in [7] but is important in the theory of graded rings. If A is a finitely generated graded k-algebra of Krull dimension d, then a *homogeneous system of parameters* or *hsop* of A is a set $\{h_1, \ldots, h_d\}$ of homogeneous elements of A such that A is finitely generated as a module over the subalgebra $k[h_1, \ldots, h_d]$. (In other words, $A/(h_1, \ldots, h_d)$ is a finite-dimensional vector space over k.)

Cohen-Macaulay rings

The definition of a Cohen-Macaulay ring that we will adopt is the one given in [7]. We say that a nonzero module over a Noetherian local ring is *Cohen-Macaulay* if its depth equals its dimension. (By convention the zero module is defined to be Cohen-Macaulay.) A Noetherian local ring is said to be *Cohen-Macaulay* if it is Cohen-Macaulay as a module over itself. An arbitrary Noetherian ring is said to be *Cohen-Macaulay* if, for every maximal ideal M, its localization at M is Cohen-Macaulay. Notice in particular that this definition makes sense for both graded and non-graded rings.

We can now state the main theorem.

Theorem 1. *Let A be a finitely generated graded k-algebra with irrelevant ideal M and Krull dimension d. The following conditions on A are equivalent.*

1. *A is Cohen-Macaulay.*
2. *For every maximal ideal I,* $\operatorname{depth}(I) = \operatorname{height}(I)$.
3. *A_M is Cohen-Macaulay.*
4. $\operatorname{depth}(M) = d$.

5. *There exists an A-sequence in M consisting of d elements.*
6. *There exists a homogeneous A-sequence in M consisting of d elements.*
7. $\operatorname{depthgr}(M) = d$.
8. *Some hsop of A is an A-sequence in M.*
9. *Every hsop of A is an A-sequence in M.*
10. *For some hsop $\{h_1, \ldots, h_d\}$, A is a free module over $k[h_1, \ldots, h_d]$.*
11. *For every hsop $\{h_1, \ldots, h_d\}$, A is a free module over $k[h_1, \ldots, h_d]$.*
12. *For some hsop $\{h_1, \ldots, h_d\}$ with degree(h_i) $= f_i$,*

$$F(A, \lambda) = \frac{F\big(A/(h_1, \ldots, h_d), \lambda\big)}{\prod_{i=1}^{d}(1 - \lambda^{f_i})}.$$

13. *For every hsop $\{h_1, \ldots, h_d\}$ with degree(h_i) $= f_i$,*

$$F(A, \lambda) = \frac{F\big(A/(h_1, \ldots, h_d), \lambda\big)}{\prod_{i=1}^{d}(1 - \lambda^{f_i})}.$$

The proof of Theorem 1 will require results from [3], [5], [6], [7], [8], [10] and [11]. We first state a few of these as lemmas.

Lemma 1. *If A is a Noetherian graded ring then* $\operatorname{dimgr}(A) = \dim(A)$.

Proof. [5, chapter VIII, section 6, no. 2, Theorem 1]

Lemma 2. *Let A be a Noetherian ring and let I be a proper ideal of A. Then there exists a maximal ideal M containing I such that* $\operatorname{depth}(I) = \operatorname{depth}(I_M)$.

Proof. Except for the condition that $I \subseteq M$, this is [6, Theorem 135]. (Note that Kaplansky uses "grade" for depth and "rank" for height.) But if we look at the proof given by Kaplansky we see that that the maximal ideal he constructs contains I, so our lemma follows.

Lemma 3. *Let A be a ring with prime ideal P. Then* $\operatorname{depth}(P) \leq \operatorname{height}(P) \leq \dim(A)$.

Proof. This follows immediately from [6, Theorem 132].

Lemma 4. *Let A be a finitely generated graded k-algebra with irrelevant ideal M. If every homogeneous element in M is a zero-divisor, then there exists a nonzero homogeneous element $u \in M$ such that $uM = 0$.*

Proof. [3, Lemma 2.2]

Lemma 5. *Let A be a finitely generated graded k-algebra. Then any hsop of A is algebraically independent.*

Proof. [10, Prop 6.2]

We are now ready for the proof of the main theorem.

Proof of Theorem 1. First let us make a few preliminary observations. Clearly $\dim(A_M) = \text{height}(M_M)$. Furthermore by well-known properties of localization (e.g., [7, section 4, example 2] or [1, Corollary 3.13]), $\text{height}(M_M) = \text{height}(M)$. Now, since A is Noetherian, by Lemma 1 $\text{dimgr}(A) = d$. Since all proper homogeneous ideals of A are contained in M, we have $\text{dimgr}(A) = \text{heightgr}(M)$. Thus

$$d \geq \text{height}(M) \geq \text{heightgr}(M) = \text{dimgr}(A) = d,$$

so $\text{height}(M) = d$. Also, from Lemma 2 it follows at once that $\text{depth}(A_M) = \text{depth}(M)$, since M is maximal. Summarizing,

$$\dim(A_M) = \text{height}(M) = d \qquad \text{and} \qquad \text{depth}(A_M) = \text{depth}(M). \tag{$*$}$$

Now let us proceed to the main part of the proof.

$(1 \Leftrightarrow 2)$

This follows since localization at a maximal ideal preserves its height (as noted above) and its depth (Lemma 2).

$(1 \Rightarrow 3)$ Trivial since M is maximal.

$(3 \Rightarrow 1)$ This is a special case of [8, Theorem 1.1].

$(3 \Leftrightarrow 4)$ Immediate from $(*)$.

$(4 \Leftrightarrow 5)$ Clearly $4 \Rightarrow 5$ trivially, and $5 \Rightarrow 4$ from Lemma 3.

$(5 \Rightarrow 6)$

By assumption there exists an A-sequence $a_1, \ldots, a_d$ in M of length d. If $d = 0$ there is nothing to prove, so assume $d > 0$. Then a_1 is a non-zero divisor in M, so by Lemma 4 there exists a homogeneous non-zero divisor $h_1 \in M$. Extend h_1 to a maximal A-sequence $h_1, \ldots, h_d$ in M (since A is Noetherian, all maximal A-sequences in M have the same length, by [7, Theorem 16.7]). So we have effectively replaced the first element of the A-sequence with a homogeneous element. Now note that $h_2, \ldots, h_d$ is an $A/(h_1)$-sequence in $M/(h_1)$; an easy induction completes the proof.

$(6 \Rightarrow 5)$ Trivial.

$(6 \Leftrightarrow 7)$ Same argument as $4 \Leftrightarrow 5$.

$(7 \Leftrightarrow 9 \Leftrightarrow 11)$ This is [10, Proposition 6.8]. Note that Smoke takes 7 as his definition of Cohen-Macaulay.

$(9 \Rightarrow 8 \Rightarrow 6)$ Trivial.

$(11 \Rightarrow 10 \Rightarrow 8)$ Clearly $11 \Rightarrow 10$ trivially, and $10 \Rightarrow 8$ by the same argument Smoke gives for $11 \Rightarrow 9$ in his proof of [10, Proposition 6.8]: since an hsop is algebraically independent (Lemma 5), $S = k[h_1, \ldots, h_d]$ is a polynomial ring. Considering S as an A-module, we see that $\{h_1, \ldots, h_d\}$ is an S-sequence in M and hence an A-sequence in M since A is free over S.

(8 $\Leftrightarrow$ 12, 9 $\Leftrightarrow$ 13) This is proved in [11] in the discussion following Corollary 3.2. Corollary 3.2 is slightly misstated; in the notation of [11], one needs the condition that θ_i is nonzero in $R/(\theta_1, \ldots, \theta_r)$. However, for the purpose at hand this causes no problems, since by Lemma 5 every hsop satisfies this condition.

This completes the proof of Theorem 1.

Remarks

1. Condition 2 in Theorem 1 is the definition of Cohen-Macaulay given in [6]. As may be seen from the proof, the equivalence of 1 and 2 holds for any Noetherian ring, not just the special graded rings of Theorem 1.
2. Conditions 6–13 tend to be the ones used in the literature of graded ring theory and combinatorics, and it is not too hard to locate alternative proofs of these cases, e.g., in [2] or [3] or [9]. The hard part is building the bridge between the usual definition of Cohen-Macaulay found in texts on commutative algebra and algebraic geometry (where the focus tends to be on local rings much more than on graded rings) and the conditions 6–13. The crucial result is of course the main theorem of [8].
3. Reference [9] states Lemma 1 but its definition of Krull dimension appears to involve chains of arbitrary ideals instead of just prime ideals.
4. To get a sense of how confusing the literature can be without knowledge of Theorem 1, note that condition 7 is used as the basic definition in [10], condition 8 is used in [11], [12] and [14], conditions 10 and 11 are used in [4], and condition 12 is used in [13] and [15]. Hopefully Theorem 1 will put to rest any fears that the different definitions are inconsistent.

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